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MAXIMAL IDEALS AND THE STRUCTURE OF CONTRACTIBLE AND AMENABLE BANACH ALGEBRAS

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Properties of minimal idempotents in contractible and reflexive amenable Banach algebras are exploited to prove that such a kind of Banach algebra is finite dimensional if each maximal ideal is contained in a maximal left or a maximal right ideal that is complemented as a Banach subspace. This result covers several known results on this subject.

1. INTRODUCTION

Suppose that $\mathfrak{A}$ is a Banach algebra over the complex field $\mathbb{C}$ and X is a Banach $\mathfrak{A}$-bimodule. A derivation from $\mathfrak{A}$ into X is a linear operator D: $\mathfrak{A} \to X$ which satisfies $D(ab) = a \cdot D(b) + D(a) \cdot b$, $a, b \in \mathfrak{A}$. For any $x \in X$, the mapping δ_x: $\mathfrak{A} \to X$ given by $\delta_x(a) = ax - xa$, $a \in \mathfrak{A}$, is a continuous derivation, called an *inner derivation*. A Banach algebra $\mathfrak{A}$ is said to be *contractible* if for every Banach $\mathfrak{A}$-bimodule X each continuous derivation from $\mathfrak{A}$ into X is inner. If X is a Banach $\mathfrak{A}$-bimodule, then X^*, the conjugate space of X, is naturally a Banach $\mathfrak{A}$-bimodule with the module actions defined by

$$\langle x, af\rangle = \langle xa, f\rangle, \quad \langle x, fa\rangle = \langle ax, f\rangle, \quad (a \in \mathfrak{A},\ f \in X^*,\ x \in X),$$

where $\langle x, f\rangle$ denotes the evaluation of f at x. A Banach algebra $\mathfrak{A}$ is said to be *amenable* if for every Banach $\mathfrak{A}$-bimodule X each continuous derivation from $\mathfrak{A}$ into the dual module X^* is inner. It is a basic fact that a contractible Banach algebra has an identity and an amenable Banach algebra has a bounded approximate identity. We call a Banach algebra a reflexive Banach algebra if the underlying space is reflexive as a Banach space. Since a reflexive Banach algebra is weak* complemented, a reflexive amenable Banach algebra has an identity. In this paper we simply denote the identity element in a Banach algebra by 1.

The structure of contractible and reflexive amenable Banach algebras has been studied by many authors. It is a simple fact that any finite dimensional semi-simple Banach

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algebra is contractible and, of course, is (reflexive) amenable (see [5, I.3.68 and VII.1.74]). The converse is known to be true for some special cases: A contractible Banach algebra is finite dimensional if it has the bounded (compact) approximation property [9] or if it is commutative [2]. As to reflexive amenable Banach algebras, it has been conjectured that they should also be finite dimensional. Gale, Ransford and White proved in [3] that this is true if irreducible representations of $\mathfrak{A}$ are finite dimensional. This result was improved by Johnson in [6], where he showed that this is the case if each maximal left ideal of $\mathfrak{A}$ is complemented as a Banach subspace. Later, Ghahramani, Loy and Willis showed in [4] that the conjecture is true if the underlying space of $\mathfrak{A}$ is a Hilbert space. Although this result is covered by Johnson's preceding result, the method is different and will be exploited in this paper. Recently, Runde [8] obtained some new results on this problem. One of his main results is that if each maximal ideal of $\mathfrak{A}$ is of finite codimension then the conjecture is true. In this paper we shall give a theorem which improves both Johnson's and Runde's results.

2. Complemented left ideals

Suppose that $\mathfrak{A}$ is a Banach algebra and $E \subset \mathfrak{A}$. The *left annihilator* and the *right annihilator* of E are, respectively, the following sets

$$\operatorname{lan}(E) = \{a \in \mathfrak{A};\ aE = \{0\}\}, \quad \operatorname{ran}(E) = \{a \in \mathfrak{A};\ Ea = \{0\}\}.$$

For an element $a \in \mathfrak{A}$, $\operatorname{lan}(\{a\})$ and $\operatorname{ran}(\{a\})$ will be simply denoted by $\operatorname{lan}(a)$ and $\operatorname{ran}(a)$ respectively. The following lemma is trivial.

Lemma 1. *Suppose that $\mathfrak{A}$ is a Banach algebra having an identity* 1. *Then for any $a \in \mathfrak{A}$,*

(i) $\operatorname{ran}(\mathfrak{A}(1-a)) = \operatorname{ran}(1-a) = \{x \in \mathfrak{A};\ ax = x\}$,

(ii) $\operatorname{lan}((1-a)\mathfrak{A}) = \operatorname{lan}(1-a) = \{x \in \mathfrak{A};\ xa = x\}$.

Recall that a non-zero element e of $\mathfrak{A}$ is a *minimal idempotent* if e is an idempotent (that is, $e^2 = e$) and $e\mathfrak{A}e$ is a division algebra.

Lemma 2. *Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. Let L be a closed proper left (right) ideal of $\mathfrak{A}$. If L is complemented in $\mathfrak{A}$, then* $\operatorname{ran}(L)$ *(respectively,* $\operatorname{lan}(L)$*) contains an idempotent e such that the following hold.*

(i) $\operatorname{ran}(L) = e\mathfrak{A}$ *(respectively,* $\operatorname{lan}(L) = \mathfrak{A}e$*);*

(ii) $L = \mathfrak{A}(1-e)$ *(respectively,* $L = (1-e)\mathfrak{A}$*);*

(iii) *If in addition, L is a maximal left (respectively, right) ideal, then e is a minimal idempotent and $\mathfrak{A}e$ (respectively, $e\mathfrak{A}$) is a minimal left (respectively, right) ideal.*

PROOF: We prove the case when L is a left ideal. To prove (i) and (ii) we can assume $L \neq \{0\}$, for otherwise $e = 1$ satisfies the requirements. First we show that L contains a right identity μ. Then it is clear that μ is an idempotent and $L = \mathfrak{A}\mu$.

If $\mathfrak{A}$ is contractible, we consider the following exact short sequence of left $\mathfrak{A}$-modules:

$$\sum: \quad 0 \longrightarrow L \xrightarrow{\imath} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/L \longrightarrow 0,$$

where $\imath$ is the inclusion mapping and q is the quotient mapping. Since L is complemented, $\sum$ is admissible. From [**2**, Theorem 6.1] $\sum$ is a splitting sequence, that is, there is a left $\mathfrak{A}$-module morphism δ: $\mathfrak{A} \to L$, such that $\delta \circ \imath = I_L$, the identity operator on L. Then $\mu = \delta(1)$ is obviously a right identity of L.

If $\mathfrak{A}$ is reflexive and amenable, then according to [**2**, Theorem 3.7], L contains a right bounded approximate identity, say (l_α). Since $\mathfrak{A}$ is reflexive, as a closed subspace of $\mathfrak{A}$, L is also reflexive. Then a weak* cluster point μ of (l_α) in L is a right identity of L.

Now let $e = 1 - \mu$. Then $e \in \operatorname{ran}(L)$, $e \neq 0$ and e is also an idempotent. Since $L = \mathfrak{A}\mu = \mathfrak{A}(1-e)$, from Lemma 1, $\operatorname{ran}(L) = \{x \in \mathfrak{A};\ ex = x\} = e\mathfrak{A}$. This proves the first two statements of this lemma.

Now suppose that L is a maximal left ideal. Then, since $\mathfrak{A}$ is the direct sum of $\mathfrak{A}(1-e)$ and $\mathfrak{A}e$, we have that $\mathfrak{A}e$ is a minimal left ideal of $\mathfrak{A}$. By [**1**, Lemma 30.2], e is a minimal idempotent. This completes the proof. □

In the following we use $\operatorname{rad}(\mathfrak{A})$ to denote the *radical* of $\mathfrak{A}$.

LEMMA 3. *Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. If $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is finite dimensional, then so is $\mathfrak{A}$ and $\mathfrak{A}$ is semi-simple.*

PROOF: If $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is finite dimensional, then $\operatorname{rad}(\mathfrak{A})$ is a complemented closed ideal of $\mathfrak{A}$. From Lemma 2 $\operatorname{rad}(\mathfrak{A})$ contains an idempotent which is non-zero if $\operatorname{rad}(\mathfrak{A}) \neq \{0\}$. But $\operatorname{rad}(\mathfrak{A})$ can never have a non-zero idempotent. So $\operatorname{rad}(\mathfrak{A}) = \{0\}$. □

It is known that if a Banach algebra is contractible or amenable, then its image under a continuous algebraic homomorphism is also contractible or amenable (see [**5**, Proposition VII.1.71] and [**7**, Proposition 5.3]). This leads to part of the following lemma.

LEMMA 4. *Suppose that $\mathfrak{A}$ is a Banach algebra. Then the following statements hold.*

(i) *If $\mathfrak{A}$ is contractible, or reflexive and amenable, then so is $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$;*

(ii) *An ideal $M \subset \mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is a maximal ideal if and only if $q^{-1}(M)$ is a maximal ideal in $\mathfrak{A}$, where q: $\mathfrak{A} \to \mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is the quotient mapping;*

(iii) *If L is a maximal left (right) ideal of $\mathfrak{A}$ containing $\operatorname{rad}(\mathfrak{A})$, then $q(L)$ is a maximal left (respectively, right) ideal of $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$. If L is complemented in $\mathfrak{A}$, then $q(L)$ is complemented in $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$.*

PROOF: The first statement is clear. Checking of the second one is also a routine. Suppose that L is a left (right) ideal of $\mathfrak{A}$ and $\operatorname{rad}(\mathfrak{A}) \subset L$. Then it is easily verified that

$L = q^{-1}(q(L))$. So $q(L)$ is maximal in $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ whenever L is maximal in $\mathfrak{A}$. If L is complemented in $\mathfrak{A}$, then there is a closed complement, say J, of L in $\mathfrak{A}$. The image $q(J)$ is also closed since q is open, and

$$q(L) + q(J) = \mathfrak{A}/\operatorname{rad}(\mathfrak{A}).$$

If $m \in q(L) \cap q(J)$, then for some $j \in J$ and $l \in L$, $m = q(j) = q(l)$. Hence $j - l \in \operatorname{rad}(\mathfrak{A}) \subset L$. So $j \in L$. This shows that $j = 0$ and hence $m = 0$. Therefore $q(J)$ is a complement of $q(L)$ in $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$. Thus $q(L)$ is complemented in $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$. ∎

3. Main results

Theorem 5. *Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. If each maximal ideal of $\mathfrak{A}$ is contained in either a maximal left ideal or a maximal right ideal of $\mathfrak{A}$ which is complemented in $\mathfrak{A}$, then $\mathfrak{A}$ is finite dimensional.*

Proof: From Lemmas 3 and 4 we can assume that $\mathfrak{A}$ is semi- simple. We can also assume that $\mathfrak{A}$ is not a division algebra. Then $\mathfrak{A}$ contains at least one maximal ideal and hence has at least one maximal left or maximal right ideal which is complemented in $\mathfrak{A}$. By Lemma 2, $\mathfrak{A}$ has at least one minimal idempotent. Let E be the set of all minimal idempotents of $\mathfrak{A}$, and let J be the ideal generated by E (the socle of $\mathfrak{A}$). We prove $J = \mathfrak{A}$.

If $J \neq \mathfrak{A}$, then there is a maximal ideal M containing J, since $\mathfrak{A}$ has an identity. Then, by assumption, there is either a maximal left ideal or a maximal right ideal of $\mathfrak{A}$ which contains M and which is complemented in $\mathfrak{A}$. Assume the former is true and the corresponding left ideal is L. Then from Lemma 2, $\operatorname{ran}(L) \neq \{0\}$ and contains a minimal idempotent e such that $L = \mathfrak{A}(1-e)$. So we would have $Ee = \{0\}$ and $e \in E$. This implies that $e = e^2 = 0$, a contradiction.

Therefore $J = \mathfrak{A}$. Then the identity 1 of $\mathfrak{A}$ can be represented as

$$1 = \sum_{i=1}^{n} a_i e_i b_i,$$

where e_i, $i = 1, 2, \dots, n$, are minimal idempotents, and $a_i, b_i \in \mathfrak{A}$. We then have

$$\begin{aligned}\mathfrak{A} = 1 \cdot \mathfrak{A} \cdot 1 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i e_i b_i \mathfrak{A} a_j e_j b_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i (e_i \mathfrak{A} e_j) b_j.\end{aligned}$$

Since each space $e_i \mathfrak{A} e_j$ has dimension of at most one [**1**, Theorem 31.6], each subspace $a_i(e_i \mathfrak{A} e_j)b_j$ has dimension of at most one. It follows that $\mathfrak{A}$ is finite dimensional. This completes the proof. ∎

If every maximal ideal in $\mathfrak{A}$ is of finite codimension or every maximal left ideal of $\mathfrak{A}$ is complemented, then the condition of Theorem 5 holds automatically. Therefore Theorem 5 covers [**6**, Theorem 2.2] and [**8**, Proposition 2.3].

Recall that a *simple algebra* is an algebra which has no proper ideals other than the zero ideal. For this sort of Banach algebra we have the following.

COROLLARY 6. *Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. If $\mathfrak{A}$ is also a simple algebra and has a maximal left or right ideal which is complemented in $\mathfrak{A}$, then $\mathfrak{A}$ is of a finite dimension.*

PROOF: In this case, $\{0\}$ is the only maximal ideal and is contained in a maximal left or a maximal right ideal which is complemented in $\mathfrak{A}$ by the assumption. ∎

COROLLARY 7. *Suppose that $\mathfrak{A}$ is a contractible or a reflexive amenable Banach algebra. If every maximal ideal in $\mathfrak{A}$ is either a maximal left or a maximal right ideal, then $\mathfrak{A}$ has finite dimension.*

PROOF: For each maximal ideal M, $\mathfrak{A}/M$ is a simple Banach algebra. Since M itself is a maximal left or maximal right ideal in $\mathfrak{A}$, $\mathfrak{A}/M$ has either no non-zero proper left ideals or no non-zero proper right ideals, meaning that $\{0\}$ is either a maximal left or a maximal right ideal in $\mathfrak{A}/M$ which is certainly complemented. From the preceding corollary, $\mathfrak{A}/M$ is of finite dimension. Then any subspace containing M is finite codimensional and hence complemented in $\mathfrak{A}$. This is true for every maximal ideal M and so, from Theorem 5, $\mathfrak{A}$ is finite dimensional. ∎

REMARK 8. If $\mathfrak{A}$ is commutative, then the condition of Corollary 7 is satisfied automatically. Therefore Corollary 7 covers [**2**, Theorem 6.2].

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