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An application of a theorem of Gaschütz

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A theorem of Gaschütz is used to prove:

Let τ be a homomorphism of the distributively generated near-ring R with identity element and descending chain condition for left modules, τ have finite kernel, and $U(R)$ be the group of units of R ; then $U(R\tau) = U(R)\tau$.

Furthermore it is shown that the finiteness condition for $\ker \tau$ can be dropped in the case of R being a ring.

Gaschütz [3] has proved the following theorem:

(*) *Let G be an n-generator group, and N a finite normal subgroup of G. Then in each generating set $\bar{e}_1N, \bar{e}_2N, \ldots, \bar{e}_nN$ for G/N, there exists a generating set $e_1, e_2, \ldots, e_n$ for G such that $e_i \in \bar{e}_iN$.*

There is an obvious generalization to Ω-groups, Ω being a set of operators. The theorem has then the following shape:

(**) *Let G be an Ω-group with an Ω-generating set of n elements, and N a finite Ω-admissible normal subgroup of G. Then in each Ω-generating set $\bar{e}_1N, \bar{e}_2N, \ldots, \bar{e}_nN$ for G/N, there exists an Ω-generating set $e_1, e_2, \ldots, e_n$ for G such that $e_i \in \bar{e}_iN$.*

This version of the theorem will enable us to obtain a result on distributively generated near-rings with identity element and descending chain condition for left modules by viewing these near-rings as operator groups the operators being the left-multiplications by distributive elements. We start with a few definitions.

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A near-ring R with identity is an algebra with two binary operations $+$ and $\cdot$ called addition and multiplication such that R is a (not necessarily abelian) group under addition and a monoid under multiplication, satisfying one distributive law

$$(a+b)c = ac + bc \quad \text{for all} \quad a, b, c \in R .$$

0 will denote the zero-element of the additive group, 1 the identity element of the multiplicative monoid of R.

We say that R is distributively generated if there exists a set S of elements in R such that S generates R additively and $s(a+b) = sa + sb$ for all $s \in X$, $a,b \in R$. By this definition since $a \to sa$, $a \in R$, $s \in S$, is an endomorphism of the additive group R^+ of R, we may consider R^+ as an S-group which is S-generated by a single element, namely 1. Any S-admissible subgroup of R^+ is called a left module of R. A unit of R is an element $e \in R$ such that there exists $e' \in R$ with $ee' = e'e = 1$. If R satisfies the descending chain condition for left modules then an element $e \in R$ is a unit if and only if there exists $e' \in R$ such that $e'e = 1$. For assume $e'e = 1$ and consider the descending chain of left modules of R:
$Re' \supseteq Re'^2 \supseteq \ldots \supseteq Re'^n \supseteq \ldots$. Then there exists n such that $Re'^n = Re'^{n+1}$, hence $R = Re'^n e^n = Re'^{n+1} e^n = Re'$. Thus $1 = re'$ for some $r \in R$ and so $e = re'e = r$ whence $ee' = 1$.

THEOREM. *Let R be a distributively generated near-ring with identity element, satisfying the descending chain condition for left modules. Let τ be a near-ring homomorphism of R such that* $\ker \tau$ *is finite and let $U(R)$, $U(R\tau)$ be the groups of units of R and $R\tau$ respectively. Then*

$$U(R)\tau = U(R\tau) .$$

Proof. Let S be a set of additive, distributive generators and consider R as an S-group. Since $S\tau$ is a set of distributive generators, $R\tau$ can be regarded as an $S\tau$-group. By $s\bar{r} = (s\tau)\bar{r}$, $s \in S$, $\bar{r} \in R\tau$, $R\tau$ becomes an S-group and $(sr)\tau = (s\tau)(r\tau) = s(r\tau)$, $r \in R$, $s \in S$ proves τ to be an S-homomorphism as well. Let $\bar{e} \in U(R\tau)$, then $\bar{e}$ is an S-generator for $R\tau$, for if $\bar{e}' \in R\tau$ such that $\bar{e}'\bar{e} = 1\tau$, then

$\bar{e}' = \sum_i \varepsilon_i(s_i\tau)$, $\varepsilon_i = \pm 1$, $s_i \in S$ and so $1\tau = \sum \varepsilon_i(s_i\tau)\bar{e} = \sum \varepsilon_i s_i \bar{e}$.
Hence, by the version *(**)* of Gaschütz's theorem, there exists an S-generator e of R such that $e\tau = \bar{e}$. But this means that $1 = \sum_j \varepsilon_j s_j e$, $\varepsilon_j = \pm 1$, $s_j \in S$, thus $e' = \sum_j \varepsilon_j s_j$ satisfies $e'e = 1$ proving $e \in U(R)$ since R satisfies the descending chain condition for left modules. Q.E.D.

COROLLARY. *If R is a finite distributively generated near-ring with identity element, and τ is a near-ring homomorphism of R then*

$$U(R)\tau = U(R\tau) \ .$$

REMARK. The finiteness condition for $\ker \tau$ is not indispensable as the following simple example shows:

Let R be a ring with descending chain condition for left ideals. Then, with the notation of the theorem, $U(R)\tau = U(R\tau)$. For let $I = \ker \tau$, J the Jacobson radical of R and $e + I$ a unit in R/I . Then $e + I + J$ is a unit in $R/I + J$, and since R/J is semisimple there exists a unit $e_1 + J$ of R/J such that $e_1 + I + J = e + I + J$. But then e_1 is a unit in R , for there exists $\bar{e}_1 \in R$ such that $\bar{e}_1 e_1 = 1 - j$, for some $j \in J$. Since J is nilpotent, $J^n = 0$ for some n . Thus $(1+j+j^2+\ldots+j^{n-1})\bar{e}_1 e_1 = 1$. Moreover $e + I = e_1 - j_1 + I$, $j_1 \in J$, and $e_1 - j_1$ is a unit in R , for if $e_1'e_1 = 1$, then $e_1'(e_1-j_1) = 1 - j_1'$, for some $j_1' \in J$ whence $(1+j_1'+j_1'^2+\ldots+j_1'^{n-1})e_1'(e_1-j_1) = 1$. Q.E.D.

More generally, if a distributively generated near-ring R with identity element and descending chain condition satisfies $[xy-x,y] = 0$, the brackets denoting a commutator in R^+ , then, with the notation of the theorem, $U(R)\tau = U(R\tau)$. The proof is almost the same as for rings using results by Blackett [2], and Betsch [1].

References

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