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ON THE STABILITY OF A MIXED-TYPE LINEAR AND QUADRATIC FUNCTIONAL EQUATION

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Abstract

We give the general solution of the n-dimensional mixed-type linear and quadratic functional equation,

$$\binom{n-2}{m-2} f\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{\{i_1,\ldots,i_m\}\in P_m} f\left(\sum_{k=1}^{m} x_{i_k}\right),$$

where $P_m = \{A \subset \{1, 2, \ldots, n\} : |A| = m\}$, and $1 < m < n$ are integers.

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1. Introduction

In 1940 Ulam [6] proposed the famous Ulam stability problem of linear mappings. In 1941 Hyers [2] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \le \varepsilon$. Rassias [5] generalized the result to the case when the inequality is controlled by the sum of norms. Since then, the stability problem has been investigated for various functional equations.

Rassias [4] established the Ulam stability of the following mixed-type functional equation:

$$f\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} f(x_i) = \sum_{1\le i<j\le 3} f(x_i + x_j).$$

The present author [3] generalized the above functional equation to the following n-dimensional functional equation:

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2) \sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j).$$

In this paper, we will further generalize the above equation to

$$\binom{n-2}{m-2} f\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{\{i_1,\ldots,i_m\}\in P_m} f\left(\sum_{k=1}^{m} x_{i_k}\right),$$

where $1 < m < n$, and we will investigate its generalized stability.

Throughout the paper, we denote the dimensionality of the problem by n, and let $P_m = \{A \subset \{1, 2, \ldots, n\} : |A| = m\}$. Moreover, we use subscripts e and o to denote the *even* part and the *odd* part of a function, respectively. The even part of a function f is defined by

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

and the odd part of f is defined by

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

2. The general solution

THEOREM 1. *Let $1 < m < n$ be integers, and let X and Y be vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\binom{n-2}{m-2} f\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{\{i_1,\ldots,i_m\}\in P_m} f\left(\sum_{k=1}^{m} x_{i_k}\right), \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in X$ if and only if f_e satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in X, \tag{2}$$

and f_o satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in X. \tag{3}$$

PROOF. To prove the necessity, suppose that a function $f : X \to Y$ satisfies (1). We will show that f_e satisfies (2) and f_o satisfies (3).

Putting $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ in (1), we obtain

$$\binom{n-2}{m-2} f(0) + \binom{n-2}{m-1} n f(0) = \binom{n}{m} f(0). \tag{4}$$

It can be verified that $\binom{n-2}{m-2} + n\binom{n-2}{m-1} > \binom{n}{m}$ for all integers m and n with $1 < m < n$. Thus, $f(0) = 0$. Putting $(x_1, x_2, \ldots, x_n) = (x, y, -y, 0, 0, \ldots, 0)$ in (1) and taking

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into account the fact that $f(0) = 0$, we obtain

$$\binom{n-2}{m-2} f(x) + \binom{n-2}{m-1} (f(x) + f(y) + f(-y)) \\ = \binom{n-3}{m-3} f(x) + \binom{n-3}{m-2} (f(x+y) + f(x-y)) \\ + \binom{n-3}{m-1} (f(x) + f(y) + f(-y)),$$

which simplifies to

$$2f(x) + f(y) + f(-y) = f(x+y) + f(x-y) \quad \text{for all } x, y \in X. \tag{5}$$

Replacing x and y in (5) with $-x$ and $-y$, respectively, we obtain

$$2f(-x) + f(-y) + f(y) = f(-x-y) + f(y-x) \quad \text{for all } x, y \in X. \tag{6}$$

Taking half the sum of (5) and (6), we obtain

$$2f_e(x) + 2f_e(y) = f_e(x+y) + f_e(x-y) \quad \text{for all } x, y \in X, \tag{7}$$

which shows that f_e satisfies (2). Taking half the difference of (5) and (6), we obtain

$$2f_o(x) = f_o(x+y) + f_o(x-y) \quad \text{for all } x, y \in X, \tag{8}$$

which is recognized as the Jensen functional equation. Noting that $f_o(0) = 0$, we can verify that f_o satisfies (3).

To prove the sufficiency, suppose that the even part and the odd part of a function $f : X \to Y$ satisfy (2) and (3), respectively. We need to show that f satisfies (1). It should be noted that a linear combination of two solutions of (1) yields just another solution; therefore, it is sufficient to prove that both f_e and f_o satisfy (1).

First consider the odd part, f_o, and make use of the linearity of the Cauchy functional equation. The left-hand side of (1) becomes

$$\binom{n-2}{m-2} f_o\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1} \sum_{i=1}^{n} f_o(x_i) \\ = \binom{n-2}{m-2} \sum_{i=1}^{n} f_o(x_i) + \binom{n-2}{m-1} \sum_{i=1}^{n} f_o(x_i) \\ = \binom{n-1}{m-1} \sum_{i=1}^{n} f_o(x_i),$$

and the right-hand side of (1) becomes

$$\sum_{\{i_1,\ldots,i_m\}\in P_m} f_o\left(\sum_{k=1}^{m} x_{i_k}\right) = \sum_{\{i_1,\ldots,i_m\}\in P_m} \sum_{k=1}^{m} f_o(x_{i_k}).$$

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Expanding the sum on the right-hand side and collecting the terms,

$$\sum_{\{i_1,\ldots,i_m\}\in P_m} f_o\left(\sum_{k=1}^{m} x_{i_k}\right) = \frac{m}{n}\binom{n}{m}\sum_{i=1}^{n} f_o(x_i) = \binom{n-1}{m-1}\sum_{i=1}^{n} f_o(x_i).$$

Thus, we have established (1) on the odd part of f.

For the even part, it can be proved by mathematical induction (see, for example, [3]) that

$$f_e\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f_e(x_i) = \sum_{1\le i<j\le n} f_e(x_i + x_j) \tag{9}$$

for all integers n. For any integers m and n with $1 < m < n$, the m-dimensional case of (9) with variables $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ is

$$f_e\left(\sum_{k=1}^{m} x_{i_k}\right) + (m-2)\sum_{k=1}^{m} f_e(x_{i_k}) = \sum_{1\le k<l\le m} f_e(x_{i_k} + x_{i_l}).$$

Summing the above equation for all $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \subset \{x_1, x_2, \ldots, x_n\}$,

$$\begin{aligned}\sum_{\{i_1,\ldots,i_m\}\subset P_m} f_e\left(\sum_{k=1}^{m} x_{i_k}\right) + (m-2)\binom{n-1}{m-1}\sum_{i=1}^{n} f_e(x_i)\\ = \binom{n-2}{m-2}\sum_{1\le i<j\le n} f_e(x_i + x_j).\end{aligned} \tag{10}$$

Finally, eliminating $\sum_{1\le i<j\le n} f(x_i + x_j)$ from (9) and (10),

$$\binom{n-2}{m-2} f_e\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1}\sum_{i=1}^{n} f_e(x_i) = \sum_{\{i_1,\ldots,i_m\}\in P_m} f_e\left(\sum_{k=1}^{m} x_{i_k}\right),$$

which shows that f_e satisfies (1).

Thus, f satisfies (1) and the proof is complete. □

3. The generalized stability

The following theorem provides a general condition for which a true solution discussed in Theorem 1 exists near an approximate solution. For convenience, we define

$$\begin{aligned}&D_m f(x_1, \ldots, x_n)\\ &= \binom{n-2}{m-2} f\left(\sum_{i=1}^{n} x_i\right) + \binom{n-2}{m-1}\sum_{i=1}^{n} f(x_i) - \sum_{\{i_1,\ldots,i_m\}\subset P_m} f\left(\sum_{k=1}^{m} x_{i_k}\right),\end{aligned} \tag{11}$$

for any integers m and n with $1 < m < n$.

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THEOREM 2. *Let $1 < m < n$ be integers, X be a real vector space, Y be a Banach space and $\phi : X^n \to [0, \infty)$ be an even function with respect to each variable. Define $\varphi(x) = \phi(x, x, -x, 0, \ldots, 0)$ for all $x \in X$. If*

$$\begin{cases} \sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x) & \text{converges for all } x \in X, \text{ and} \\ \lim_{s\to\infty} 2^{-s}\phi(2^s x_1, \ldots, 2^s x_n) = 0 & \text{for all } x_1, \ldots, x_n \in X, \end{cases} \tag{12}$$

or

$$\begin{cases} \sum_{i=1}^{\infty} 4^{i}\varphi(2^{-i} x) & \text{converges for all } x \in X, \text{ and} \\ \lim_{s\to\infty} 4^{s}\phi(2^{-s} x_1, \ldots, 2^{-s} x_n) = 0 & \text{for all } x_1, \ldots, x_n \in X, \end{cases} \tag{13}$$

and a function $f : X \to Y$ satisfies

$$\|D_m f(x_1, \ldots, x_n)\| \le \phi(x_1, \ldots, x_n) \quad \text{for all } x_1, \ldots, x_n \in X, \tag{14}$$

then there exists a unique function $T : X \to Y$ that satisfies (1) and, for all $x \in X$,

$$\|f(x) + pf(0) - T(x)\| \le \begin{cases} \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x) + \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x) & \text{if (12) holds} \\ \frac{1}{2}\sum_{i=1}^{\infty} 2^{i}\varphi(2^{-i} x) + \frac{1}{4}\sum_{i=1}^{\infty} 4^{i}\varphi(2^{-i} x) & \text{if (13) holds} \end{cases} \tag{15}$$

where $p = (((n-1)(n-2))/(3m)) - 1$. The function T is given by

$$T(x) = \begin{cases} \lim_{s\to\infty} 2^{-s} f_o(2^s x) + 4^{-s} f_e(2^s x) & \text{if (12) holds,} \\ \lim_{s\to\infty} 2^{s} f_o(2^{-s} x) + 4^{s} f_e(2^{-s} x) & \text{if (13) holds.} \end{cases} \tag{16}$$

for all $x \in X$.

PROOF. We will first prove the theorem for a function ϕ satisfying (12). Putting $(x_1, x_2, \ldots, x_n) = (x, x, -x, 0, 0, \ldots, 0)$ in (14) and simplifying,

$$\|3pf(0) + 3f(x) + f(-x) - f(2x)\| \le \varphi(x), \tag{17}$$

where p is defined as in the theorem. Replacing x in the above equation by $-x$,

$$\|3pf(0) + 3f(-x) + f(x) - f(-2x)\| \le \varphi(-x) = \varphi(x). \tag{18}$$

From (17) and (18), we infer that, for all $x \in X$,

$$\|3pf(0) + 4f_e(x) - f_e(2x)\| \le \varphi(x), \tag{19}$$

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and

$$\|2f_o(x) - f_o(2x)\| \le \varphi(x).$$

Define a function $g_e : X \to Y$ by

$$g_e(x) = f_e(x) + pf(0) \quad \text{for all } x \in X. \tag{20}$$

Then (19) becomes

$$\|4g_e(x) - g_e(2x)\| \le \varphi(x),$$

which can be rewritten as

$$\|g_e(x) - 4^{-1}g_e(2x)\| \le 4^{-1}\varphi(x).$$

For each positive integer s,

$$\begin{aligned}\|g_e(x) - 4^{-s}g_e(2^s x)\| &= \left\|\sum_{i=0}^{s-1}(4^{-i}g_e(2^i x) - 4^{-(i+1)}g_e(2^{i+1}x))\right\| \\ &\le \sum_{i=0}^{s-1} 4^{-i}\|(g_e(2^i x) - 4^{-1}g_e(2\cdot 2^i x))\| \\ &\le \frac{1}{4}\sum_{i=0}^{s-1} 4^{-i}\varphi(2^i x).\end{aligned}$$

Similarly, we can show that, for every integer s,

$$\|f_o(x) - 2^{-s}f_o(2^s x)\| \le \frac{1}{2}\sum_{i=0}^{s-1} 2^{-i}\varphi(2^i x).$$

The convergence of the sequence $\{4^{-s}g_e(2^s x)\}$ can be settled as follows. For every positive integer t,

$$\begin{aligned}\|4^{-s}g_e(2^s x) - 4^{-(s+t)}g_e(2^{s+t}x)\| &= 4^{-s}\|g_e(2^s x) - 4^{-t}g_e(2^t\cdot 2^s x)\| \\ &\le 4^{-s}\cdot\frac{1}{4}\sum_{i=0}^{t-1} 4^{-i}\varphi(2^i\cdot 2^s x) \\ &\le \frac{1}{4}\sum_{i=0}^{\infty} 4^{-(i+s)}\varphi(2^{i+s}x).\end{aligned}$$

From (12), we know that $\sum_{i=0}^{\infty} 4^{-(i+s)}\varphi(2^{i+s}x) \le \sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x)$ converges; so, it follows that $\lim_{s\to\infty}(1/4)\sum_{i=0}^{\infty} 4^{-(i+s)}\varphi(2^{i+s}x) = 0$. Therefore, we have a Cauchy sequence in a Banach space. Let

$$T_e(x) = \lim_{s\to\infty} 4^{-s}g_e(2^s x) = \lim_{s\to\infty} 4^{-s}f_e(2^s x) \quad \text{for all } x \in X.$$

Thus,

$$\|g_e(x) - T_e(x)\| \le \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x).$$

Similarly, the inequality on f_e leads us to

$$T_o(x) = \lim_{s\to\infty} 2^{-s} f_o(2^s x) \quad \text{for all } x \in X,$$

and

$$\|f_o(x) - T_o(x)\| \le \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x).$$

If we define a function $T : X \to Y$ by

$$T(x) = T_o(x) + T_e(x) \quad \text{for all } x \in X,$$

then

$$\begin{aligned}\|f(x) + pf(0) - T(x)\| &\le \|f_o(x) - T_o(x)\| + \|g_e(x) - T_e(x)\| \\ &\le \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x) + \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x).\end{aligned}$$

In order to show that T satisfies (1), we will prove that the even part and the odd part of T satisfy (1). Define the even part and the odd part of $D_m f$ by

$$\begin{aligned}D_m f_e(x_1, \ldots, x_n) &= \frac{D_m f(x_1, \ldots, x_n) + D_m f(-x_1, \ldots, -x_n)}{2}, \\ D_m f_o(x_1, \ldots, x_n) &= \frac{D_m f(x_1, \ldots, x_n) - D_m f(-x_1, \ldots, -x_n)}{2}.\end{aligned}$$

For a positive integer s and for all $x_1, x_2, \ldots, x_n \in X$,

$$\begin{aligned}\|D_m f_e(2^s x_1, \ldots, 2^s x_n)\| &\le \tfrac{1}{2}\|D_m f(2^s x_1, \ldots, 2^s x_n)\| \\ &\quad + \tfrac{1}{2}\|D_m f(-2^s x_1, \ldots, -2^s x_n)\| \\ &\le \phi(2^s x_1, \ldots, 2^s x_n).\end{aligned}$$

If we divide the above inequality by 4^s and take the limit as $s \to \infty$, then the right-hand side vanishes according to (12) and we obtain from the definition of T_e that

$$\binom{n-2}{m-2} T_e\Big(\sum_{i=1}^{n} x_i\Big) + \binom{n-2}{m-1}\sum_{i=1}^{n} T_e(x_i) = \sum_{\{i_1,\ldots,i_m\}\in P_m} T_e\Big(\sum_{k=1}^{m} x_{i_k}\Big),$$

for all $x_1, x_2, \ldots, x_n \in X$. We can similarly show that T_o satisfies (1). Hence, $T = T_e + T_o$ satisfies (1).

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To prove the uniqueness of T, suppose there exists another function $T' : X \to Y$ such that T' satisfies (1) and (15). We have proved in Theorem 1 that T_e satisfies the quadratic functional equation (2) and T_o satisfies the Cauchy functional equation (3); therefore, $T_e(rx) = r^2 T_e(x)$ and $T_o(rx) = r T_o(x)$ for every rational number r and for every $x \in X$. Thus,

$$\|T(x) - T'(x)\| \le \|T_e(x) - T'_e(x)\| + \|T_o(x) + T'_o(x)\|.$$

For any positive integer s and for each $x \in X$,

$$\begin{aligned}\|T_e(x) - T'_e(x)\| &= 4^{-s}\|T_e(2^s x) - T'_e(2^s x)\| \\ &\le 4^{-s}\|g_e(2^s x) - T_e(2^s x)\| + 4^{-s}\|g_e(2^s x) - T'_e(2^s x)\| \\ &\le 2 \cdot 4^{-s} \cdot \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i \cdot 2^s x) \\ &= \frac{1}{2}\sum_{i=0}^{\infty} 4^{-(i+s)}\varphi(2^{i+s}x).\end{aligned}$$

Taking the limit as $s \to \infty$, we have $\|T_e(x) - T'_e(x)\| \le 0$. Thus $T_e(x) = T'_e(x)$ for all $x \in X$. Similarly, we can show that $T_o(x) = T'_o(x)$ for all $x \in X$. Hence, $T(x) = T'(x)$ for all $x \in X$.

The proof for the case when (13) holds can be done in a similar manner. □

In the next few corollaries, we will give the stability of (1) in various senses. The following corollary proves the Hyers–Ulam stability.

COROLLARY 3. *If a function $f : X \to Y$ satisfies*

$$\|D_m f(x_1, x_2, \ldots, x_n)\| \le \varepsilon \quad \text{for all } x_1, x_2, \ldots, x_n \in X$$

for some $\varepsilon > 0$, then there exists a unique function $T : X \to Y$ that satisfies (1) and

$$\|f(x) + pf(0) - T(x)\| \le \frac{4\varepsilon}{3} \quad \text{for all } x \in X.$$

PROOF. Let $\phi(x_1, x_2, \ldots, x_n) = \varepsilon$ for all $x_1, x_2, \ldots, x_n \in X$ in Theorem 2. Hence, $\varphi(x) = \varepsilon$ for all $x \in X$. We can see that (12) holds. Therefore, it follows from the theorem that there exists a unique function $T : X \to Y$ such that

$$\|f(x) + pf(0) - T(x)\| \le \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varepsilon + \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varepsilon = \frac{4\varepsilon}{3} \quad \text{for all } x \in X. \qquad \square$$

The following corollary proves the Hyers–Ulam–Rassias stability of (1).

COROLLARY 4. *Let p be a real number with $0 < p < 1$ or $p > 2$. If a function $f : X \to Y$ satisfies*

$$\|D_m f(x_1, x_2, \ldots, x_n)\| \le \varepsilon \sum_{i=1}^{n} \|x_i\|^p \quad \text{for all } x_1, x_2, \ldots, x_n \in X \tag{21}$$

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for some $\varepsilon > 0$, *then* $f(0) = 0$ *and there exists a unique function* $T : X \to Y$ *that satisfies* (1) *and*

$$\|f(x) - T(x)\| \le \frac{6\varepsilon|3 - 2^p|}{(2 - 2^p)(4 - 2^p)}\|x\|^p \quad \textit{for all } x \in X.$$

PROOF. Substituting $x_1 = x_2 = \cdots = x_n = 0$ into (21), we obtain

$$\binom{n-2}{m-2} f(0) + \binom{n-2}{m-1} n f(0) = \binom{n}{m} f(0),$$

as in (4). Thus, $f(0) = 0$. Let $\phi(x_1, x_2, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, x_2, \ldots, x_n \in X$. Then $\varphi(x) = 3\varepsilon\|x\|^p$ for all $x \in X$. If $0 < p < 1$, then (12) holds and it follows from Theorem 2 that

$$\begin{aligned}
\|f(x) - T(x)\| &\le \frac{1}{2}\sum_{i=0}^{\infty}(2^{-i} \cdot 3\varepsilon\|2^i x\|^p) + \frac{1}{4}\sum_{i=0}^{\infty}(4^{-i} \cdot 3\varepsilon\|2^i x\|^p) \\
&= \frac{3\varepsilon}{2 - 2^p}\|x\|^p + \frac{3\varepsilon}{4 - 2^p}\|x\|^p \\
&= \frac{6\varepsilon(3 - 2^p)}{(2 - 2^p)(4 - 2^p)}\|x\|^p \quad \text{for all } x \in X.
\end{aligned}$$

If $p > 1$, then (13) holds, and we get a similar result. □

For the generalized stability in the sense of Gavruta [1], we get a superstability of (1) when $n > 3$ as stated in the following corollary.

COROLLARY 5. *Let* $p_1, p_2, \ldots, p_n \ge 0$ *and* $r = \sum_{i=1}^{n} p_i$ *with* $0 < r < 1$ *or* $r > 2$. *If a function* $f : X \to Y$ *satisfies*

$$\|D_m f(x_1, x_2, \ldots, x_n)\| \le \varepsilon \prod_{i=1}^{n} \|x_i\|^{p_i} \quad \textit{for all } x_1, x_2, \ldots, x_n \in X.$$

for some $\varepsilon > 0$, *then:*

(1) *if* $n > 3$, *then* f *satisfies equation* (1); *and*

(2) *if* $n = 3$, *then there exists a unique function* $T : X \to Y$ *that satisfies equation* (1) *and*

$$\|f(x) - T(x)\| \le \frac{\varepsilon|3 - 2^r|}{(2 - 2^r)(4 - 2^r)}\|x\|^r \quad \textit{for all } x \in X.$$

PROOF. We can show that $f(0) = 0$ by the same substitution used in the proof of Corollary 4. Let $\phi(x_1, x_2, \ldots, x_n) = \varepsilon \prod_{i=1}^{n} \|x_i\|^{p_i}$ for all $x_1, x_2, \ldots, x_n \in X$. Then, for all $x \in X$,

$$\varphi(x) = \begin{cases} 0 & \text{if } n > 3, \\ \varepsilon\|x\|^r & \text{if } n = 3. \end{cases}$$

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If $n > 3$, then we can see that f satisfies (1). If $n = 3$, then we consider two cases: $0 < r < 1$ and $r > 2$. If $0 < r < 1$, then (12) holds and for all $x \in X$, by Theorem 2,

$$\begin{aligned} \|f(x) - T(x)\| &\leq \frac{1}{2}\sum_{i=0}^{\infty}(2^{-i}\cdot\varepsilon\|2^i x\|^r) + \frac{1}{4}\sum_{i=0}^{\infty}(4^{-i}\cdot\varepsilon\|2^i x\|^r) \\ &= \frac{\varepsilon}{2-2^r}\|x\|^r + \frac{\varepsilon}{4-2^r}\|x\|^r \\ &= \frac{2\varepsilon(3-2^r)}{(2-2^r)(4-2^p)}\|x\|^r. \end{aligned}$$

If $r > 2$, then (13) holds and we get a similar result. □

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