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ON COMPLETE REDUCIBILITY OF MODULE BUNDLES

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We prove the local triviality of module bundles over semisimple Lie algebra bundles and using this result we establish the complete reducibility of module bundles over semisimple Lie algebra bundles.

A Lie algebra bundle, for short a Lie bundle, as introduced by Douady and Lazard [1], is a vector bundle (E, p, X) together with a morphism $\theta : E \oplus E \to E$, which induces a Lie algebra structure on each fibre E_x.

A locally trivial Lie bundle is a vector bundle (E, p, X) in which each fibre E_x is a Lie algebra and for every x in X, there exists a neighbourhood U of x, a Lie algebra L and a homeomorphism $\varphi : U \times L \to p^{-1}(U)$ such that for each y in U, $\varphi_y : L \to p^{-1}(y)$ is a Lie algebra isomorphism. Every locally trivial Lie bundle is a Lie bundle [2], but the converse need not be true [4].

In this paper we prove the complete reducibility of module bundles over semisimple Lie bundles where a module bundle $\eta = (\eta, q, X)$ over a Lie bundle E is a vector bundle together with a morphism $\rho : E \oplus \eta \to \eta$ such that for each x in X, ρ_x induces a E_x-module structure on η_x.

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of η, if each fibre η'_x is a submodule of η_x. We say an E-module η is simple if η has no proper non-zero submodule bundles.

Let us consider the trivial bundle $\eta = (X \times V, q, X)$ and the trivial Lie bundle $E = (X \times L, p, X)$. Let $\rho : L \oplus V \to V$ be an L-module structure on V. The morphism $\hat{\rho} : X \times (L \oplus V) \to X \times V$ given by $\hat{\rho}(x, 1+v) = (x, \rho(1, v))$ induces on each fibre $\eta_x = V$, the L-module structure of V. Such a module bundle is called the trivial module bundle over E.

We prove that a module bundle η over a semisimple Lie bundle is locally trivial. That is for each x in X, we find a trivial module bundle $U \times V$, where U is some open set around x such that $q^{-1}(U)$ is isomorphic to $U \times V$ as module bundles.

A representation ρ of a Lie bundle E on a vector-bundle η is a Lie bundle morphism from E to the Lie bundle $\text{End}(\eta) = \bigcup_{x \in X} \text{End}(\eta_x)$ [4]. We also establish that the concepts of a representation and a module bundle of a Lie bundle are equivalent over a suitable base space.

NOTATIONS AND TERMINOLOGY. The underlying field considered throughout is the field of real numbers. We denote the total space of the vector-bundle (E, p, X) by E itself and the fibres by E_x. All the bundles considered in this paper have the first countable space X as the base space. Further our vector spaces are finite dimensional.

1.

In proving the complete reducibility of module bundles over a semisimple Lie bundle, we need the rigidity of submodules of a module over a semisimple Lie algebra. Richardson [6, Proposition 15.3] has given the rigidity of submodules over an algebraically closed field. Here we prove the rigidity of submodules of a module over a real field.

As a first step we shall prove the following.

PROPOSITION 1. *If M is a submodule of an L-module V, where L is a semisimple Lie algebra, then every L-module homomorphism from M to V/M is induced by a member of* $\text{Hom}_L(V, V)$*, the collection of all*

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L-module homomorphisms defined on V .

Proof. Since L is semisimple and M is a submodule of V, we can find a submodule M' of V such that $V = M \oplus M'$ as modules. Further the map $h : V/M \to M'$ given by $h(v+M) = m'$ where $v = m + m'$, $m \in M$, $m' \in M'$ defines a module isomorphism. Now given the L-module homomorphism $f : M \to V/M$, let us define $g : V \to V$ by $g(v) = m + \big(hf(m)+m'\big)$. If $f(m) = v_1 + M$ where $v_1 = m_1 + m_1'$, then $hf(m) = m_1'$, and so $\pi \cdot g(m) = m_1' + M = f(m)$ where $\pi : V \to V/M$ is the canonical projection. Thus f is induced by g.

First we note that $\hat{G}$, the collection of all L-module automorphisms is a Lie subgroup of $\mathrm{Aut}(V)$ being a pseudo-algebraic subgroup and that $\mathrm{Hom}_L(V, V)$ is the Lie algebra of $\hat{G}$.

If $\Gamma_r(V)$ is the space of all r-dimensional submodules of V where $r < \dim V$, then $\hat{G}$ acts on $\Gamma_r(V)$ as follows.

Given $g \in \hat{G}$, $M \in \Gamma_r(V)$, $g \cdot M \in \Gamma_r(V)$ is given by $g \cdot M = g(M)$.

PROPOSITION 2. *Let* V *be an* L*-module and* M *an* r*-dimensional submodule of* V. *If* L *is semisimple, then* M *is rigid. That is* $\hat{G} \cdot M$ *is open in* $\Gamma_r(V)$.

Proof. Let W be a subspace of V, transversal to M and Γ_W the collection of all r-dimensional subspaces of V, transversal to W. Then Γ_W is an open submanifold of $G_r(V)$, the Grassmann variety of r-dimensional subspaces of V.

Let P be the projection operator on V with kernel M and image W and $Q = I - P$. The vector space $\mathrm{Hom}(M, W)$ of all linear transformations from M to W, is identified with

$$H = \{T \in \mathrm{End}(V) \mid T(W) = 0;\ T(V) \subseteq W\} .$$

Then the mapping $\varphi : \mathrm{Hom}(M, W) \to \Gamma_W$ given by $\varphi(T) = \mathrm{Im}(Q+T)$ is a diffeomorphism.

For each x in L, we define $\psi_x : \mathrm{Hom}(M, W) \to \mathrm{Hom}(M, W)$ by

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$\psi_x(T) = (P-T)\rho(x)(Q+T)$. It can be seen that $\varphi(T)$ is a member of $\Gamma_r(V)$ if and only if $\psi_x(T) = 0$ for all x in L . Hence

$$\varphi^{-1}\left(\Gamma_r(V)\right) = \bigcap_{x\in L} \psi_x^{-1}(0) .$$

Let $\hat{G}_1$ be the open subset of $\hat{G}$ consisting of all g such that $g(M)$ is transversal to W and $g(W)$ is transversal to M . Let us define $\beta : \hat{G}_1 \to \mathrm{Hom}(M, W)$ by $\beta(g) = PgQg^{-1}\left(P+gQg^{-1}\right)^{-1}$. Then we obtain $\varphi\left(\beta(g)\right) = g(M)$.

The differentials $d\beta_e : T\left(\hat{G}_1, e\right) = T(\hat{G}, e) \to \mathrm{Hom}(M, W)$ and $\left(d\psi_x\right)_{(0)} : \mathrm{Hom}(M, W) \to \mathrm{Hom}(M, W)$ are given by $\left(d\beta_e\right)(D) = PDQ$ and $\left(d\psi_x\right)_{(0)}(T) = P \circ \rho(x) \circ T - T \circ \rho(x) \circ Q$.

We can identify $\mathrm{Hom}(M, W)$ with $\mathrm{Hom}_L(M, V/M)$ through the isomorphism θ_0 given by $\theta_0(T) = \pi|_W \cdot T$ where $\pi : V \to V/M$ is the projection. Then we obtain $\bigcap_{x\in L} \ker\left(d\psi_x\right)_{(0)}$ is precisely $\mathrm{Hom}_L(M, V/M)$ and $\mathrm{Im}\ d\beta_{(e)}$ is the subcollection of $\mathrm{Hom}_L(M, V/M)$ consisting of elements which are induced by elements of $\mathrm{Hom}_L(V, V)$. Because of this interpretation of $\bigcap_{x\in L} \ker\left(d\psi_x\right)_{(0)}$ and $\mathrm{Im}\ d\beta_{(e)}$, we obtain $\bigcap_{x\in L} \ker\left(d\psi_x\right)_{(0)} = \mathrm{Im}\ d\beta_{(e)}$, by applying Proposition 1. Now we apply the result due to Weil [8, Lemma 1] to the spaces $\hat{G}_1$ and $\mathrm{Hom}(M, W)$ and get a neighbourhood N of zero in $\mathrm{Hom}(M, W)$ such that $\varphi^{-1}\left(\Gamma_r(V)\right) \cap N$ is a submanifold of N and $\beta\left(\hat{G}_1\right) \cap N$. So $\varphi^{-1}\left(\Gamma_r(V)\right) \cap N \subseteq \beta\left(\hat{G}_1\right)$. Hence $\varphi(N)$ is an open set in $\Gamma_r(V)$ containing the element $\{M\}$ and contained in the orbit $\hat{G} \cdot M$. Thus $\hat{G} \cdot M$ is open in $\Gamma_r(V)$.

2.

In this section we shall show that the concepts of representation and module are equivalent. The first countability of the base space is required only in proving that a module bundle structure induces a representation. Further we prove the local triviality of a module bundle over a semisimple Lie bundle.

PROPOSITION 3. *Let* E *be a Lie bundle and* η *an* E*-module. Then the module structure induces a representation of* E *on the vector bundle* η *and conversely.*

Proof. Let $\rho : E \oplus \eta \to \eta$ induces the module structure on η. We can define $\rho_1 : E \to \mathrm{Hom}(\eta, \eta)$ by $\rho_1(a)(m) = \rho(a, m)$, $a \in E_x$, $m \in \eta_x$. Then obviously ρ_1 induces a Lie algebra homomorphism on each fibre E_x. So it is sufficient to prove the continuity of ρ_1.

We have vector bundle isomorphisms $\alpha : U \times V_1 \to \bigcup_{y \in U} E_y$ and $\beta : U \times V_2 \to \bigcup_{y \in U} \eta_y$ where V_1 and V_2 are vector spaces. Then $\mathrm{Hom}\,\beta : U \times \mathrm{Hom}(V_2, V_2) \to \bigcup_{y \in U} \mathrm{Hom}(\eta_y, \eta_y)$ given by $\mathrm{Hom}\,\beta(y, f) = \beta_y \cdot f \cdot \beta_y^{-1}$, is a vector bundle isomorphism. Now consider $\hat{\rho}_1 : U \times V_1 \to U \times \mathrm{Hom}(V_2, V_2)$ given by $\hat{\rho}_1 = (\mathrm{Hom}\,\beta)^{-1} \cdot \rho_1 \cdot \alpha$.

Let $\{(y_n, v_n)\}$ converge to (y, v) in $U \times V_1$. Then $(\hat{\rho}_1(y_n, v_n))$ converges to $\hat{\rho}_1(y, v)$ because

$$\hat{\rho}_1(y, v_1)(v_2) = (\mathrm{Hom}\,\beta)^{-1}(\rho_1(\alpha(y, v_1))(v_2))$$
$$= (\mathrm{Hom}\,\beta)^{-1}\rho(\alpha_y(v_1), v_2) \quad \text{for } v_1 \in V_1,\ v_2 \in V_2 .$$

By the first countability of X, $\hat{\rho}_1$ is continuous. Hence ρ_1 is continuous.

Conversely let $\rho_1 : E \to \mathrm{Hom}(\eta, \eta)$ be a representation of E on η. Let us define $\rho : E \oplus \eta \to \eta$ by $\rho(a, m) = \rho_1(a)(m)$, $a \in E_x$, $m \in \eta_x$. Obviously each η_x is an E_x-module, the structure being induced by ρ_x. Now we shall prove the continuity of ρ.

Consider $\rho^* : \mathrm{Hom}(\eta, \eta) \oplus \eta \to \eta$ given by $\rho^*(f, a) = f(a)$,

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$f \in \mathrm{Hom}(\eta_x, \eta_x)$ and $a \in \eta_x$. If we define

$$\hat{\rho}^* : U \times \{\mathrm{Hom}(V_2, V_2) \oplus V_2\} \to U \times V_2$$

by $\hat{\rho}^*(y, f+v) = (y, f(v))$ which is continuous, then since $\rho^*(\mathrm{Hom}\ \beta + \beta) = \beta \cdot \hat{\rho}^*$, we obtain the continuity of ρ^* . Hence $\rho = \rho^* \cdot (\rho_1 + \mathrm{id\ on\ } \eta)$ is continuous.

LEMMA 4. *Every module bundle* η *over a semisimple Lie bundle* E *is locally trivial.*

Proof. Let the module structure on η be given by $\rho : E \oplus \eta \to \eta$, which gives rise to the representation $\rho : E \to \mathrm{Hom}(\eta, \eta)$.

E is locally trivial being semisimple [3, Lemma 2.1]. Let the local triviality be given by the Lie bundle isomorphism $\varphi : U \times L \to p^{-1}(U)$. The module bundle η being a vector bundle we have a vector space V and a vector bundle isomorphism $\alpha : U \times V \to q^{-1}(U)$. Let $\hat{\rho} : U \times L \to U \times \mathrm{Hom}(V, V)$ be the map $\hat{\rho} = (\mathrm{Hom}\ \alpha)^{-1}\rho\varphi$. If Γ denotes the collection of all Lie algebra homomorphisms from L to $\mathrm{Hom}(V, V)$, then $\hat{\rho}_y \in \Gamma$ for each y in U . The Lie group $G = \mathrm{Aut}(V)$ acts on Γ in the following way.

Given $g \in G$, $\tilde{\rho} \in \Gamma$, $g \cdot \tilde{\rho} \in \Gamma$ is given by

$$(g \cdot \tilde{\rho})(l) = g \cdot \tilde{\rho}(l) \cdot g^{-1} \quad \text{for all } l \in L .$$

Since L is semisimple, $\tilde{\rho}_x$ is rigid [5, Theorem A]. Hence the orbit $G(\hat{\rho}_x) = G \cdot \hat{\rho}_x$ is open in Γ . The mapping $y \to \hat{\rho}_y$ is continuous from U to Γ . So the set $U_1 = \{y \in U \mid \hat{\rho}_y \in G(\hat{\rho}_x)\}$ is open in U , being the inverse image of $G(\hat{\rho}_x)$ in U under the mapping $y \to \hat{\rho}_y$. If $y \in U_1$, then there exists a g_y in G such that $g_y \cdot \hat{\rho}_x = \hat{\rho}_y$.

We can apply Aren's theorem to G and $G(\hat{\rho}_x)$ and proceed in a similar manner as in the proof of Theorem 3 [2], we get a neighbourhood U of x in X and a module bundle isomorphism $\beta : U \times V_x \to \bigcup_{y \in U} \eta_y$ where $V_x = (V, \hat{\rho}_x)$ and β is given by $\beta(y, v) = \alpha_y g_y(v)$, for all v in V_x

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and y in U. Hence the result.

3.

Now we prove the theorem on complete reducibility of module bundles of semisimple Lie bundles.

THEOREM 5. *Let E be a semisimple Lie bundle and η an E-module bundle. Then η can be written as a direct sum of simple module bundles.*

Proof. Let η' be a submodule of η and let us consider the quotient vector bundle $\eta/\eta' = \eta''$. For any $a \in E_x$, $m + \eta'_x \in \eta_x$, we define $\rho'' : E \oplus \eta'' \to \eta''$ by $\rho''(a, m+\eta'_x) = \rho(a, m) + \eta'_x$. Thus η'' is a module bundle. We get the exact sequence

$$0 \to \eta' \xrightarrow{\mu} \eta \xrightarrow{\pi} \eta'' \to 0$$

of module bundles where π is the projection and μ is the inclusion map.

Since E_x is semisimple we obtain $\eta_x = \eta'_x \oplus \tilde{\eta}_x$ where $\tilde{\eta}_x$ is a submodule of η_x isomorphic to η''_x. We can define $f_x : \eta_x \to \eta'_x$ by $f_x(m'+\tilde{m}) = m'$, $m' \in \eta'_x$, $\tilde{m} \in \tilde{\eta}_x$. Let $f : \eta \to \eta'$ be given by $f/\eta_x = f_x$ we have $f \circ \mu$ equals the identity on η'. The splitting of the exact sequence follows if the function f is continuous. Now we shall show the continuity of f.

Since E is semisimple, η and η' are locally trivial module bundles by Lemma 3. So we obtain L-module bundle isomorphisms $\alpha : U \times V \to \bigcup_{y \in U} \eta_y$ and $\alpha' : U \times W \to \bigcup_{y \in U} \eta'_y$. Let $\psi : U \times W \to U \times V$ be given by $\psi(y, w) = \alpha^{-1}\alpha'(y, w)$.

For each y, $\psi_y(W)$ is a submodule of V. Our aim is to find a submodule V_1 of V and a module bundle isomorphism between $U \times V_1$ and $\bigcup_{y \in U} \eta'_y$.

Consider $G_r(V)$ the Grassmann variety of all r-dimensional subspaces of V, where $r = \dim W$. Let h be any hermitian metric on V. Then we define the metric $M : U \to U \times \text{Herm } V$ on the bundle $U \times V$ by $H(y) = (y, h)$, where Herm V is the collection of all hermitian metrics defined on V. Then the subbundle $F = \psi(U \times W)$ has an orthogonal complement $F^{\perp}$ in $U \times V$. If $P : U \times V \to F^{\perp}$ is the orthogonal projection, then the mapping $y \to \ker P_y = \psi_y(W)$ is continuous from U to $G_r(V)$. Consequently the mapping $y \to \psi_y(W)$ is continuous from U to $\Gamma_r(V)$. Let $\hat{G}(x)$ denote the orbit $\hat{G}\left(\psi_x(W)\right)$. By the rigidity of submodules, $\hat{G}(x)$ is an open subset of $\Gamma_r(V)$. The subset $\Gamma_r(V)$ is locally compact being a closed subset of the compact space $G_r(V)$. Since $G_r(V)$ is second countable, $\hat{G}(x)$ is also second countable. Hence by [7, Lemma 2.9.1] we obtain that $\hat{G}/\hat{G}_x$ is homeomorphic to $\hat{G}(x)$ where $\hat{G}_x$ is the stability subgroup of $\hat{G}$, corresponding to $\psi_x(W)$.

If $U_1 = \{y \in U \mid \psi_y(W) \in \hat{G}(x)\}$ then for each y in U_1, there exists a g_y in $\hat{G}$ such that $\psi_y(W) = g_y\psi_x(W)$. Now by applying the fact that $\hat{G} \to \hat{G}/\hat{G}_x$ is a principal bundle, we obtain the continuity of the mapping $y \to g_y$ from U_1 to $\hat{G}$.

Let $V' = \psi_x(W)$ and define $\alpha_1 : U_1 \times V' \to \bigcup_{y \in U_1} \eta'_y$ by $\alpha_1(y, v') = \alpha_y g_y(v')$. Given v in V, there exists a unique v_1 in V such that $v = g_y\left(v_1\right)$. We define $\hat{\alpha} : U_1 \times V \to \bigcup_{y \in U_1} \eta_y$ by $\hat{\alpha}(y, v) = \alpha_y g_y\left(v_1\right)$. The maps $\hat{\alpha}$ and α_1 are module bundle isomorphisms.

Now we define $\hat{f} : U_1 \times V \to U_1 \times V'$ by $\hat{f}(y, v) = \left(y, v_1'\right)$ where v_1' is the component of $v_1 = g_y^{-1}(v)$ in V'. That is $\hat{f}(y, v) = \left(y, \pi g_y^{-1}(v)\right)$ where $\pi : V \to V'$ is the projection operator on V'. It can be verified that $\hat{f}$ is continuous and the following diagram is commutative:

$$\begin{array}{ccc} U_1 \times V & \xrightarrow{\hat{\alpha}} & \bigcup_{y \in U_1} \eta_y \\ \Big\downarrow \hat{f} & & \Big\downarrow f \\ U_1 \times V' & \xrightarrow{\alpha_1} & \bigcup_{y \in U_1} \eta'_y \end{array}$$

Then f becomes a continuous function. Hence the result.

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