

## **$\Phi$ -CARLESON MEASURES AND MULTIPLIERS BETWEEN BERGMAN–ORLICZ SPACES OF THE UNIT BALL OF $\mathbb{C}^n$**

**BENOÎT F. SEHBA**

(Received 1 April 2016; accepted 24 January 2017; first published online 22 March 2017)

Communicated by C. Meaney

### **Abstract**

We define the notion of  $\Phi$ -Carleson measures, where  $\Phi$  is either a concave growth function or a convex growth function, and provide an equivalent definition. We then characterize  $\Phi$ -Carleson measures for Bergman–Orlicz spaces and use them to characterize multipliers between Bergman–Orlicz spaces.

*2010 Mathematics subject classification:* primary 32A36, 46E30; secondary 32A35.

*Keywords and phrases:* Bergman–Orlicz spaces, Carleson measures, multipliers.

### **1. Introduction**

The aim of this paper is to characterize Carleson measures of some weighted spaces of holomorphic functions  $A^\Phi$  on the unit ball  $\mathbb{B}^n$  of  $\mathbb{C}^n$ , that is, measures  $\mu$  on  $\mathbb{B}^n$  such that  $A^\Phi$  embeds continuously into  $L^\Psi(\mu)$ . The spaces we consider are of Orlicz type and particular cases thereof are the Bergman spaces. Those results are then applied to the characterization of multipliers between such spaces.

Let us now give a more precise description of the results. To do so, we first need to introduce some notations and to recall the classical results on Carleson measures.

Let us denote by  $d\nu$  the Lebesgue measure on the unit ball  $\mathbb{B}^n$  of  $\mathbb{C}^n$  and  $d\sigma$  the normalized measure on  $\mathbb{S}^n = \partial\mathbb{B}^n$ , the boundary of  $\mathbb{B}^n$ . By  $\mathcal{H}(\mathbb{B}^n)$ , we denote the space of holomorphic functions on  $\mathbb{B}^n$ .

For  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  in  $\mathbb{C}^n$ , we let

$$\langle z, w \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n},$$

so that  $|z|^2 = \langle z, z \rangle = |z_1|^2 + \dots + |z_n|^2$ .

We say that a function  $\Phi$  is a growth function if it is a continuous and nondecreasing function from  $[0, \infty)$  onto itself.

For  $\alpha > -1$ , we denote by  $dv_\alpha$  the normalized Lebesgue measure  $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$ ,  $c_\alpha$  being the normalization constant. For  $\Phi$  a growth function, the weighted Bergman–Orlicz space  $A_\alpha^\Phi(\mathbb{B}^n)$  is the space of all holomorphic functions  $f$  such that

$$\|f\|_{\Phi,\alpha} = \|f\|_{A_\alpha^\Phi} := \int_{\mathbb{B}^n} \Phi(|f(z)|) dv_\alpha(z) < \infty.$$

We define on  $A_\alpha^\Phi(\mathbb{B}^n)$  the following (quasi)-norm:

$$\|f\|_{\Phi,\alpha}^{\text{lux}} = \|f\|_{A_\alpha^\Phi}^{\text{lux}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{B}^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) dv_\alpha(z) \leq 1 \right\},$$

which is finite for  $f \in A_\alpha^\Phi(\mathbb{B}^n)$  (see [17]).

The usual weighted Bergman spaces  $A_\alpha^p(\mathbb{B}^n)$  correspond to  $\Phi(t) = t^p$  and are defined by

$$\|f\|_{p,\alpha}^p := \int_{\mathbb{B}^n} |f(z)|^p dv_\alpha(z) < \infty.$$

For  $0 < p < \infty$ , the usual Hardy space  $\mathcal{H}^p(\mathbb{B}^n)$  is the space of all  $f \in \mathcal{H}(\mathbb{B}^n)$  such that

$$\|f\|_p^p := \sup_{0 < r < 1} \int_{\mathbb{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty.$$

Two growth functions  $\Phi_1$  and  $\Phi_2$  are said to be equivalent if there exists some constant  $c$  such that

$$\frac{1}{c} \Phi_1\left(\frac{t}{c}\right) \leq \Phi_2(t) \leq c \Phi_1(ct).$$

Such equivalent growth functions define the same Orlicz space.

For any  $\xi \in \mathbb{S}^n$  and  $\delta > 0$ , the Carleson tube  $Q_\delta(\xi)$  is defined by

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n : |1 - \langle z, \xi \rangle| < \delta\}.$$

Let  $\mu$  be a positive Borel measure on  $\mathbb{B}^n$  and  $0 < s < \infty$ . We say that  $\mu$  is an  $s$ -Carleson measure on  $\mathbb{B}^n$  if there exists a constant  $C$  such that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,

$$\mu(Q_\delta(\xi)) \leq C\delta^{ns}. \tag{1.1}$$

When  $s = 1$ , the above measures are called Carleson measures. Carleson measures were first introduced in the unit disk of the complex plane  $\mathbb{C}$  by Carleson [4, 5]. These measures are pretty adapted to the studies of various questions on Hardy spaces. In his work, Carleson obtained that a measure  $\mu$  is a Carleson measure if and only if the Hardy space  $H^p$  embeds continuously into the Lebesgue space  $L^p(d\mu)$ . For  $s > 1$ , again in the unit disk, Duren [9] has proved that a measure  $\mu$  is an  $s$ -Carleson measure if and only if the Hardy space  $H^p$  embeds continuously into the Lebesgue space  $L^{ps}(d\mu)$ . The characterizations of Carleson measures for Hardy spaces of the unit ball can be found in [11, 16]. These characterizations can be summarized as follows.

**THEOREM 1.1** (Hörmander [11], Power [16]). *Let  $\mu$  be a positive measure on  $\mathbb{B}^n$  and  $s > 0$ . Then the following assertions are equivalent.*

- (a) *There exists a constant  $C_1 > 0$  such that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,*

$$\mu(Q_\delta(\xi)) \leq C_1 \delta^{ns}.$$

- (b) *There exists a constant  $C_2 > 0$  such that*

$$\int_{\mathbb{B}^n} \frac{(1 - |a|^2)^{ns}}{|1 - \langle a, z \rangle|^{2ns}} d\mu(z) \leq C_2$$

*for all  $a \in \mathbb{B}^n$ .*

*If moreover  $s \geq 1$ , then the above assertions are both equivalent to the following assertion.*

- (c) *There exists a constant  $C_3 > 0$  such that for any  $f \in H^p(\mathbb{B}^n)$ ,*

$$\int_{\mathbb{B}^n} |f(z)|^{ps} d\mu(z) \leq C_3 \|f\|_p^s. \quad (1.2)$$

The characterization of measures satisfying (1.2) with  $0 < s < 1$  in the setting of the unit disk is due to Videnskii [21]. The extension of the results of Carleson and Duren to the setting of Bergman spaces of the unit disk is due to Hastings [10] and the Bergman-space version of the result of Videnskii is due to Luecking [15]. The extensions of the latter results to the unit ball are due to Cima and Wogen [8] and Luecking [12, 13].

Theorem 1.1 translates as follows for Bergman spaces.

**THEOREM 1.2** (Cima and Wogen [8], Luecking [12]). *Let  $\mu$  be a positive measure on  $\mathbb{B}^n$ ,  $s > 0$  and  $\alpha > -1$ . Then the following assertions are equivalent.*

- (a) *There exists a constant  $C_1 > 0$  such that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,*

$$\mu(Q_\delta(\xi)) \leq C_1 \delta^{(n+1+\alpha)s}.$$

- (b) *There exists a constant  $C_2 > 0$  such that*

$$\int_{\mathbb{B}^n} \frac{(1 - |a|^2)^{(n+1+\alpha)s}}{|1 - \langle a, z \rangle|^{2(n+1+\alpha)s}} d\mu(z) \leq C_2$$

*for all  $a \in \mathbb{B}^n$ .*

*If moreover  $s \geq 1$ , then the above assertions are both equivalent to the following assertion.*

- (c) *There exists a constant  $C_3 > 0$  such that for any  $f \in A_\alpha^p(\mathbb{B}^n)$ ,*

$$\int_{\mathbb{B}^n} |f(z)|^{ps} d\mu(z) \leq C_3 \|f\|_{p,\alpha}^s.$$

Let us observe that (1.1) can be read as

$$\mu(Q_\delta(\xi)) \leq \frac{C}{\Phi\left(\frac{1}{\delta^n}\right)},$$

where  $\Phi(t) = t^s$ . Our aim is then to obtain characterizations of such measures when power functions are replaced by appropriate growth functions. We apply our characterizations to obtain some first results on the still open question of multipliers between Bergman–Orlicz spaces.

## 2. Statement of the results

We recall that the growth function  $\Phi$  is of upper type  $q$  if we can find  $q > 0$  and  $C > 0$  such that, for  $s > 0$  and  $t \geq 1$ ,

$$\Phi(st) \leq Ct^q \Phi(s). \quad (2.1)$$

We denote by  $\mathcal{U}^q$  the set of growth functions  $\Phi$  of upper type  $q$  (with  $q \geq 1$ ) such that the function  $t \mapsto \Phi(t)/t$  is nondecreasing. We write

$$\mathcal{U} = \bigcup_{q \geq 1} \mathcal{U}^q.$$

We also recall that  $\Phi$  is of lower type  $p$  if we can find  $p > 0$  and  $C > 0$  such that, for  $s > 0$  and  $0 < t \leq 1$ ,

$$\Phi(st) \leq Ct^p \Phi(s). \quad (2.2)$$

We denote by  $\mathcal{L}_p$  the set of growth functions  $\Phi$  of lower type  $p$  (with  $p \leq 1$ ) such that the function  $t \mapsto \Phi(t)/t$  is nonincreasing. We write

$$\mathcal{L} = \bigcup_{0 < p \leq 1} \mathcal{L}_p.$$

Note that we may always suppose that any  $\Phi \in \mathcal{L}$  (respectively  $\mathcal{U}$ ) is concave (respectively convex) and that  $\Phi$  is a  $\mathcal{C}^1$  function with derivative  $\Phi'(t) \asymp \Phi(t)/t$ .

**DEFINITION 2.1.** Let  $\mu$  be a positive measure on  $\mathbb{B}^n$  and let  $\Phi \in \mathcal{L} \cup \mathcal{U}$ . We say that  $\mu$  is a  $\Phi$ -Carleson measure if there exists a constant  $C > 0$  such that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,

$$\mu(Q_\delta(\xi)) \leq \frac{C}{\Phi\left(\frac{1}{\delta^n}\right)}.$$

The following first result provides an equivalent definition of  $\Phi$ -Carleson measures.

**THEOREM 2.2.** Let  $\mu$  be a positive measure on  $\mathbb{B}^n$  and let  $\Phi \in \mathcal{L} \cup \mathcal{U}$ . Then the following assertions are equivalent.

- (i)  $\mu$  is a  $\Phi$ -Carleson measure.
- (ii) There exists a constant  $C > 0$  such that for any  $a \in \mathbb{B}^n$ ,

$$\int_{\mathbb{B}^n} \Phi\left(\frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}}\right) d\mu(z) \leq C. \quad (2.3)$$

Our next result extends Theorem 1.2 to Bergman–Orlicz spaces.

**THEOREM 2.3.** *Let  $\mu$  be a positive measure on  $\overline{\mathbb{B}^n}$  and  $\alpha > -1$ . Let  $\Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U}$ . Then the following assertions are equivalent.*

- (a) *There exists a constant  $C_1 > 0$  such that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,*

$$\mu(Q_\delta(\xi)) \leq \frac{C_1}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right)}. \tag{2.4}$$

- (b) *There exists a constant  $C_2 > 0$  such that*

$$\int_{\mathbb{B}^n} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{(1-|a|^2)^{n+1+\alpha}}\right)\frac{(1-|a|^2)^{2(n+1+\alpha)}}{|1-\langle a, z \rangle|^{2(n+1+\alpha)}}\right) d\mu(z) \leq C_2$$

for all  $a \in \mathbb{B}^n$ .

If moreover  $\Phi_2/\Phi_1$  is nondecreasing, then the above assertions are both equivalent to the following assertion.

- (c) *There exists a constant  $C_3 > 0$  such that for any  $f \in A_\alpha^{\Phi_1}(\mathbb{B}^n)$  with  $\|f\|_{\Phi_1, \alpha}^{\text{lux}} \neq 0$ ,*

$$\int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{C_3\|f\|_{\Phi_1, \alpha}^{\text{lux}}}\right) d\mu(z) \leq 1. \tag{2.5}$$

We call a measure satisfying (2.4) a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure. If a measure  $\mu$  satisfies (2.5), then we say that  $\mu$  is a  $\Phi_2$ -Carleson measure for  $A_\alpha^{\Phi_1}(\mathbb{B}^n)$ .

Note that in [6] and [7], it is proved that (2.5) holds if and only if there exists  $\delta_0$  such that for any  $\delta \in (0, \delta_0)$ ,

$$\mu(Q_\delta(\xi)) \leq \frac{C_1}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right)}.$$

Moreover, the proof in both papers uses, among others, a maximal function characterization of Bergman–Orlicz spaces. Here, we provide a more direct proof that generalizes the classical proof for the power-function case (see for example [20]).

Let  $X$  and  $Y$  be two analytic function spaces which are metric spaces, with respective metrics  $d_X$  and  $d_Y$ . We say that an analytic function  $g$  is a multiplier from  $X$  to  $Y$  if there exists a constant  $C > 0$  such that for any  $f \in X$ ,

$$d_Y(fg, 0) \leq Cd_X(f, 0).$$

We denote by  $\mathcal{M}(X, Y)$  the set of multipliers from  $X$  to  $Y$ . The question of multipliers between Bergman spaces has been considered in [1–3, 14, 22, 23]. In particular, Attele obtained the characterization of multipliers between unweighted Bergman spaces of the unit disk of the complex plane in [1], while the case of weighted Bergman spaces of the same setting was handled by Zhao in [23]. The proofs in [23] heavily make use of Carleson measures for Bergman spaces. We also use here our characterization of Carleson measures to extend the result of [23] on multipliers between the Bergman spaces  $A_\alpha^p(\mathbb{B}^n)$  and  $A_\beta^q(\mathbb{B}^n)$  with  $0 < p \leq q < \infty$  to a corresponding situation for Bergman–Orlicz spaces.

Let us introduce two subsets of growth functions. We say that a growth function  $\Phi \in \mathcal{U}^q$  belongs to  $\mathcal{U}^q$  if:

(a<sub>1</sub>) there exists a constant  $C_1 > 0$  such that for any  $0 < s < \infty$  and any  $t \geq 1$ ,

$$\Phi(st) \leq C_1 \Phi(s)\Phi(t); \quad (2.6)$$

(a<sub>2</sub>) there exists a constant  $C_2 > 0$  such that for any  $a, b \geq 1$ ,

$$\Phi\left(\frac{a}{b}\right) \leq C_2 \frac{\Phi(a)}{b^q}. \quad (2.7)$$

As examples of functions in  $\tilde{\mathcal{U}}$ , we have power functions and, for nontrivial examples, we have the functions  $t \mapsto t^m \log^\gamma(C + t)$ , where  $m \geq 1$ ,  $\gamma > 0$  and the constant  $C > 0$  is large enough.

We say that a growth function  $\Phi \in \mathcal{L}_p$  belongs to  $\tilde{\mathcal{L}}$  if  $\Phi$  satisfies condition (2.6) and if there exists a constant  $C_3 > 0$  such that for any  $a, b \geq 1$ ,

$$\Phi\left(\frac{a}{b}\right) \leq C_3 \frac{a^p}{\Phi(b)}. \quad (2.8)$$

Clearly, power functions are in  $\tilde{\mathcal{L}}$ . For nontrivial examples, we have the functions  $t \mapsto t^m \log^\gamma(C + t)$ , where  $0 < m \leq 1$ ,  $\gamma < 0$  and the constant  $C > 0$  is large enough. To see that the latter satisfies (2.6), use that if  $\Phi \in \mathcal{L}_p$ , then for any  $t \geq 1$ ,  $t^p \leq C\Phi(t)$ , with  $C$  the constant in (2.2).

Before stating our result on multipliers of Bergman–Orlicz spaces, let us introduce another space of analytic functions. Let  $\omega : (0, 1] \rightarrow (0, \infty)$ . An analytic function  $f$  in  $\mathbb{B}^n$  is said to be in  $\mathcal{H}_\omega^\infty(\mathbb{B}^n)$  if

$$\|f\|_{\mathcal{H}_\omega^\infty} := \sup_{z \in \mathbb{B}^n} \frac{|f(z)|}{\omega(1 - |z|)} < \infty.$$

Clearly,  $\mathcal{H}_\omega^\infty(\mathbb{B}^n)$  is a Banach space.

Here is our result on pointwise multipliers between Bergman–Orlicz spaces for the families of growth functions introduced above.

**THEOREM 2.4.** *Let  $\Phi_1 \in \mathcal{L} \cup \mathcal{U}$  and  $\Phi_2 \in \tilde{\mathcal{L}} \cup \tilde{\mathcal{U}}$ . Assume that  $\Phi_2/\Phi_1$  is nondecreasing. Let  $\alpha, \beta > -1$  and define, for  $0 < t \leq 1$ , the function*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{t^{n+1+\alpha}}\right)}.$$

*Then the following assertions hold.*

(i) *If  $\omega$  is nonincreasing on  $(0, 1]$ , then*

$$\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = H_\omega^\infty(\mathbb{B}^n).$$

(ii) *If  $\omega$  is equivalent to 1, then  $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = H^\infty(\mathbb{B}^n)$ .*

(iii) *If  $\omega$  is nondecreasing on  $(0, 1]$  and  $\lim_{t \rightarrow 0} \omega(t) = 0$ , then*

$$\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = \{0\}.$$

Note that if the function  $\omega$  in the last theorem is nondecreasing and  $\lim_{t \rightarrow 0} \omega(t) \neq 0$ , then  $\omega$  is equivalent to 1.

The proofs of the first two theorems will be given in Section 4 and applications to embeddings between Bergman–Orlicz spaces and the pointwise multipliers of Bergman–Orlicz spaces will be provided in Section 5. In the next section, we recall some results from the literature needed in our proofs.

All through the text, we assume without loss of generality that our growth functions  $\Phi$  are such that  $\Phi(1) = 1$ . Finally, all through the text,  $C$  will be a constant not necessarily the same at each occurrence. We recall that given two positive quantities  $A$  and  $B$ , the notation  $A \lesssim B$  means that  $A \leq CB$  for some positive constant  $C$ . When  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \sim B$ .

### 3. Some preliminaries

We provide in this section some useful results, which are mostly related to growth functions.

For  $a \in \mathbb{B}^n$ ,  $a \neq 0$ , let  $\varphi_a$  denote the automorphism of  $\mathbb{B}^n$  taking 0 to  $a$  and defined by

$$\varphi_a(z) = \frac{a - P_a(z) - (1 - |z|^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle},$$

where  $P_a$  is the projection of  $\mathbb{C}^n$  onto the one-dimensional subspace span of  $a$  and  $Q_a = I - P_a$ , where  $I$  is the identity. It is easy to see that

$$\begin{aligned} \varphi_a(0) &= a, & \varphi_a(a) &= 0, & \varphi_a \circ \varphi_a(z) &= z, \\ 1 - |\varphi_a(z)|^2 &= \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}. \end{aligned}$$

For  $0 < r < 1$  and  $a \in \mathbb{B}^n$ , we write  $r\mathbb{B}^n := \{z \in \mathbb{B}^n : |z| < r\}$  and define the (pseudo-hyperbolic metric) ball  $\Delta(a, r)$  by

$$\Delta(a, r) = \{z \in \mathbb{B}^n : |\varphi_a(z)| < r\}.$$

Clearly,  $\Delta(a, r) = \varphi_a(r\mathbb{B}^n)$ . One easily checks the following (for details, see [20]).

**LEMMA 3.1.** *For any  $a \in \mathbb{B}^n$  and  $0 < r < 1$ , there exist  $\xi \in \mathbb{S}^n$  and  $\delta > 0$  such that  $\Delta(a, r) \subset Q_\delta(\xi)$ . Moreover,  $\delta \sim 1 - |a|^2$ .*

We have the following estimate (see [17, 18]).

**LEMMA 3.2.** *Let  $\Phi \in \mathcal{L} \cup \mathcal{U}$ ,  $-1 < \alpha < \infty$ . There is a constant  $C > 0$  such that for any  $f \in A_\alpha^\Phi(\mathbb{B}^n)$  and any  $a \in \mathbb{B}^n$ ,*

$$|f(a)| \leq C \Phi^{-1}\left(\frac{1}{(1 - |a|^2)^{n+1+\alpha}}\right) \|f\|_{\Phi, \alpha}^{\text{lux}}. \tag{3.1}$$

The next lemma provides a useful function in  $A_\alpha^\Phi(\mathbb{B}^n)$  (see [18]).

**LEMMA 3.3.** *Let  $-1 < \alpha < \infty$ ,  $a \in \mathbb{B}^n$ . Suppose that  $\Phi \in \mathcal{L} \cup \mathcal{U}$ . Then the following function is in  $A_\alpha^\Phi(\mathbb{B}^n)$ :*

$$f_a(z) = \Phi^{-1}\left(\frac{1}{(1-|a|)^{n+1+\alpha}}\right)\left(\frac{1-|a|^2}{1-\langle z, a \rangle}\right)^{2(n+1+\alpha)}.$$

Moreover,  $\|f_a\|_{\Phi, \alpha}^{\text{lux}} \lesssim 1$ .

#### 4. Proof of Theorems 2.2 and 2.3

**PROOF OF THEOREM 2.2.** The proof follows the same idea as in the power-function case (see, for example, [24]). We provide details here.

(i)  $\Rightarrow$  (ii): for  $|a| \leq \frac{3}{4}$ , (2.3) is obvious since the measure is necessarily finite. Let  $\frac{3}{4} < |a| < 1$  and choose  $\xi = a/|a|$ . For any nonnegative integer  $k$ , let  $r_k = 2^{k-1}(1-|a|)$ ,  $k = 1, 2, \dots, N$ , where  $N$  is the smallest integer such that  $2^{N-2}(1-|a|) \geq 1$ . Let  $E_1 = Q_{r_1}(\xi)$  and  $E_k = Q_{r_k}(\xi) - Q_{r_{k-1}}(\xi)$ ,  $k \geq 2$ . We have

$$\mu(E_k) \leq \mu(Q_{r_k}(\xi)) \leq \frac{C}{\Phi\left(\frac{1}{2^{(k-1)n}(1-|a|)^n}\right)}.$$

Moreover, if  $k \geq 2$  and  $z \in E_k$ , then

$$\begin{aligned} |1 - \langle a, z \rangle| &= |1 - |a| + |a|(1 - \langle \xi, z \rangle)| \\ &\geq -(1 - |a|) + |a||1 - \langle \xi, z \rangle| \\ &\geq \frac{3}{4}2^{k-1}(1 - |a|) - (1 - |a|) \\ &\geq 2^{k-2}(1 - |a|). \end{aligned}$$

We also have for  $z \in E_1$ ,

$$|1 - \langle z, a \rangle| \geq 1 - |a| > \frac{1}{2}(1 - |a|).$$

Let us put

$$K_a(z) = \frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}},$$

so that for  $z \in E_k$ ,

$$K_a(z) \leq \frac{1}{2^{2(k-2)n}(1 - |a|)^n}.$$



Using the above estimates and putting  $\varepsilon = 1$  if  $\Phi \in \mathcal{U}$ , and  $\varepsilon = p$  if  $\Phi \in \mathcal{L}$  is of lower type  $0 < p < 1$ ,

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi\left(\frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}}\right) d\mu(z) \\ &= \int_{\mathbb{B}^n} \Phi(K_a(z)) d\mu(z) \\ &= \sum_{k=1}^N \int_{E_k} \Phi(K_a(z)) d\mu(z) \\ &\leq C \sum_{k=1}^N \frac{\Phi\left(\frac{1}{2^{2(k-2)n}(1 - |a|)^n}\right)}{\Phi\left(\frac{1}{2^{(k-1)n}(1 - |a|)^n}\right)} \\ &\leq C \sum_{k=1}^N \frac{1}{2^{kn\varepsilon}} < \tilde{C} < \infty. \end{aligned}$$

(ii)  $\Rightarrow$  (i): let  $a \neq 0$ ,  $a \in \mathbb{B}^n$ . Set  $\delta = 1 - |a|^2$  and  $\xi = a/|a|$ . We remark that for  $z \in Q_\delta(\xi)$ ,  $|1 - \langle z, a \rangle| \leq 2(1 - |a|^2)$ . Hence, using (2.1) for  $\Phi \in \mathcal{U}$  and the fact that the function  $\Phi(t)/t$  is nonincreasing for  $\Phi \in \mathcal{L}$ ,

$$\begin{aligned} \mu(Q_\delta(\xi))\Phi\left(\frac{1}{\delta^n}\right) &\lesssim \int_{\mathbb{B}^n} \Phi\left(\frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}}\right) d\mu(z) \\ &\leq C. \end{aligned}$$

The proof is complete. □

**PROOF OF THEOREM 2.3.** We observe that the implication (b)  $\Rightarrow$  (a) follows in the same way as in the proof of the implication (ii)  $\Rightarrow$  (i) in Theorem 2.2. We will then only prove that (a)  $\Rightarrow$  (c)  $\Rightarrow$  (b).

(a)  $\Rightarrow$  (c): we fix  $\frac{1}{2} < r < 1$  and  $z \in \mathbb{B}^n$ . We recall that by Lemma 3.1,  $\Delta(z, r) \subset Q_\delta(\xi)$  for some  $\xi \in \mathbb{S}^n$  and  $\delta > 0$  with  $\delta \simeq 1 - |z|^2$ . Under (a), this implies that

$$\mu(\Delta(z, r)) \leq \mu(Q_\delta(\xi)) \leq \frac{C_1}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)}. \tag{4.1}$$

Next, we recall that if  $\Phi \in \mathcal{L}$  with lower type  $0 < p < 1$ , then the growth  $\Phi_p(t) = \Phi(t^{1/p})$  belongs to  $\mathcal{U}$  (see [18]). We will also use the notation  $\Phi_p$  for  $\Phi \in \mathcal{U}$ , noting that in this case  $p = 1$ . As  $|f|^p$  ( $0 < p \leq 1$ ) is  $\mathcal{M}$ -subharmonic, we obtain using inequality (4.3) of [19] and the convexity of  $\Phi_p$  that

$$\begin{aligned} \Phi(|f(z)|) &= \Phi_p(|f(z)|^p) \\ &\leq C_2 \int_{\Delta(z, 1/2)} \Phi_p(|f(w)|^p)(1 - |w|^2)^{-(n+1+\alpha)} d\nu_\alpha(w) \\ &= C_2 \int_{\Delta(z, 1/2)} \Phi(|f(w)|)(1 - |w|^2)^{-(n+1+\alpha)} d\nu_\alpha(w). \end{aligned}$$

That is,

$$\Phi(|f(z)|) = C_2 \int_{\Delta(z,1/2)} \Phi(|f(w)|)(1 - |w|^2)^{-(n+1+\alpha)} dv_\alpha(w). \tag{4.2}$$

Let  $K_1 = \max\{1, C_1 C_2, (CC_1 C_2)^{1/p}\}$ , where  $C$  is the constant in (2.2),  $C_1$  the constant in (4.1) and  $C_2$  the constant in the last inequality. Put  $K = \max\{C', K_1\}$ , where  $C'$  is the constant in (3.1). Using (4.2) and Fubini's lemma,

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) d\mu(z) \\ &\leq C_2 \int_{\mathbb{B}^n} d\mu(z) \int_{\Delta(z,1/2)} \Phi_2\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right)(1 - |w|^2)^{-(n+1+\alpha)} dv_\alpha(w) \\ &\leq C_2 \int_{\mathbb{B}^n} \left(\int_{\mathbb{B}^n} \chi_{\Delta(z,1/2)}(w) d\mu(z)\right) \Phi_2\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right)(1 - |w|^2)^{-n-1} dv(w). \end{aligned}$$

Using the fact that  $\chi_{\Delta(z,1/2)}(w) \leq \chi_{\Delta(w,r)}(z)$  for each  $z \in \mathbb{B}^n$  and  $w \in \mathbb{B}^n$ , we deduce that

$$L \leq C_2 \int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right)(1 - |w|^2)^{-n-1} \mu(\Delta(w, r)) dv(w).$$

Now, using the fact that the function  $\Phi_2/\Phi_1$  is nondecreasing and (3.1),

$$\begin{aligned} L &\leq C_2 \int_{\mathbb{B}^n} \Phi_1\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) \frac{\Phi_2\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right)}{\Phi_1\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right)} (1 - |w|^2)^{-n-1} \mu(\Delta(w, r)) dv(w) \\ &\leq C_2 \int_{\mathbb{B}^n} \Phi_1\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) \frac{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{(1 - |w|^2)^{n+1+\alpha}}\right)}{\Phi_1 \circ \Phi_1^{-1}\left(\frac{1}{(1 - |w|^2)^{n+1+\alpha}}\right)} \\ &\quad \times (1 - |w|^2)^{-n-1} \mu(\Delta(w, r)) dv(w). \end{aligned}$$

Finally, using (4.1),

$$\begin{aligned} L &\leq C_1 C_2 \int_{\mathbb{B}^n} \Phi_1\left(\frac{|f(w)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) dv_\alpha(w) \\ &\leq \int_{\mathbb{B}^n} \Phi_1\left(\frac{|f(w)|}{\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) dv_\alpha(w) \leq 1. \end{aligned}$$

That is,

$$\int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{K\|f\|_{\Phi_{1,\alpha}}^{\text{lux}}}\right) d\mu(z) \leq 1.$$

(c)  $\Rightarrow$  (b): let  $a \in \mathbb{B}^n$ . Recall with Lemma 3.3 that the function

$$f_a(z) = \Phi_1^{-1}\left(\frac{1}{(1 - |a|)^{n+1+\alpha}}\right) \left(\frac{1 - |a|^2}{1 - \langle z, a \rangle}\right)^{2(n+1+\alpha)}$$

is uniform in  $A_\alpha^{\Phi_1}(\mathbb{B}^n)$ . Thus, the implication follows by testing (c) with  $f_a$  and using the monotonicity of  $\Phi$  or the monotonicity of the function  $\Phi(t)/t$ . The proof is complete.  $\square$

### 5. Embeddings and multipliers between Bergman–Orlicz spaces

We start this section with an embedding result between Bergman–Orlicz spaces.

**THEOREM 5.1.** *Let  $\Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U}$ ,  $\alpha, \beta > -1$ . Assume that  $\Phi_2/\Phi_1$  is nondecreasing. Then  $A_\alpha^{\Phi_1}(\mathbb{B}^n)$  embeds continuously into  $A_\beta^{\Phi_2}(\mathbb{B}^n)$  if and only if there is a constant  $C > 0$  such that*

$$\Phi_1^{-1}(t^{n+1+\alpha}) \leq \Phi_2^{-1}(Ct^{n+1+\beta}) \quad \text{for } t \in [1, \infty). \tag{5.1}$$

**PROOF.** That  $A_\alpha^{\Phi_1}(\mathbb{B}^n)$  embeds continuously into  $A_\beta^{\Phi_2}(\mathbb{B}^n)$  is equivalent to saying that there exists a constant  $C > 0$  such that for every  $f \in A_\alpha^{\Phi_1}(\mathbb{B}^n)$ , with  $\|f\|_{\Phi_1, \alpha}^{\text{lux}} \neq 0$ ,

$$\int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{C\|f\|_{\Phi_1, \alpha}^{\text{lux}}}\right) d\nu_\beta(z) \leq 1,$$

which is equivalent by Theorem 2.3 to saying that  $\nu_\beta$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure. Hence, we only have to prove that  $\nu_\beta$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure if and only if (5.1) holds.

Let us first assume that (5.1) holds. Then we easily obtain that for any  $\xi \in \mathbb{S}^n$  and any  $0 < \delta < 1$ ,

$$\nu_\beta(Q_\delta(\xi)) \sim \delta^{n+1+\beta} = \frac{C}{\Phi_2 \circ \Phi_2^{-1}\left(\frac{C}{\delta^{n+1+\beta}}\right)} \leq \frac{C}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right)}.$$

That is,  $\nu_\beta$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure.

Now assume that  $\nu_\beta$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure. Then, by Theorem 2.3, there exists a constant  $K > 0$  such that

$$\int_{\mathbb{B}^n} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{(1-|a|^2)^{n+1+\alpha}}\right) \frac{(1-|a|^2)^{2(n+1+\alpha)}}{|1-\langle a, z \rangle|^{2(n+1+\alpha)}}\right) d\nu_\beta(z) \leq K \tag{5.2}$$

for all  $a \in \mathbb{B}^n$ .

For  $a \in \mathbb{B}^n$  given, let  $\delta = 1 - |a|^2$  and  $\xi = a/|a|$  ( $a \neq 0$ ). Then, using the type of  $\Phi_2$  or the monotonicity of  $\Phi_2(t)/t$ , we obtain from (5.2) that

$$\delta^{n+1+\beta} \Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right) \simeq \nu_\beta(Q_\delta(\xi)) \Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right) \leq K,$$

that is,

$$\Phi_1^{-1}\left(\frac{1}{\delta^{n+1+\alpha}}\right) \leq \Phi_2^{-1}\left(\frac{C}{\delta^{n+1+\beta}}\right)$$

for some constant  $C > 0$ . Thus, (5.1) holds. The proof is complete.  $\square$

We next consider multipliers between two Bergman–Orlicz spaces. We start with the following general result.

**THEOREM 5.2.** *Let  $\Phi_1, \Phi_2 \in \mathcal{L} \cup \mathcal{U}$ . Assume that  $\Phi_2/\Phi_1$  is nondecreasing. Let  $\alpha, \beta > -1$  and define, for  $0 < t \leq 1$ , the function*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{t^{n+1+\alpha}}\right)}.$$

Then the following assertions hold.

- (i) *If  $\omega$  is equivalent to 1, then  $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = H^\infty(\mathbb{B}^n)$ .*
- (ii) *If  $\omega$  is nondecreasing on  $(0, 1]$  and  $\lim_{t \rightarrow 0} \omega(t) = 0$ , then*

$$\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = \{0\}.$$

**PROOF.** Let us start by proving assertion (i). We first prove the inclusion  $H^\infty(\mathbb{B}^n) \subset \mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$ . Let us assume that  $g \in H^\infty(\mathbb{B}^n)$ . Then, for any  $f \in A_\alpha^{\Phi_1}(\mathbb{B}^n)$ ,

$$\begin{aligned} \int_{\mathbb{B}^n} \Phi_2\left(\frac{|g(z)||f(z)|}{C\|g\|_\infty\|f\|_{\Phi_1, \alpha}^{\text{lux}}}\right) d\nu_\beta(z) &\leq \int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{C\|f\|_{\Phi_1, \alpha}^{\text{lux}}}\right) d\nu_\beta(z) \\ &\leq 1, \end{aligned}$$

where the last inequality follows from Theorem 5.1, with  $C$  an appropriate constant.

Now assume that  $g \in \mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$ . First, by Lemma 3.2, there exists a constant  $C > 0$  such that for any  $f \in A_\alpha^{\Phi_1}(\mathbb{B}^n)$  and any  $z \in \mathbb{B}^n$ ,

$$|g(z)||f(z)| = |f(z)g(z)| \leq C\Phi_2^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\beta}}\right)\|fg\|_{\Phi_2, \alpha}^{\text{lux}}.$$

Hence,

$$|g(z)||f(z)| \leq C\Phi_2^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\beta}}\right)\|f\|_{\Phi_1, \alpha}^{\text{lux}}. \tag{5.3}$$

Taking in the above inequality

$$f(z) = f_a(z) = \Phi_1^{-1}\left(\frac{1}{(1 - |a|)^{n+1+\alpha}}\right)\left(\frac{1 - |a|^2}{1 - \langle z, a \rangle}\right)^{2(n+1+\alpha)}$$

with  $a \in \mathbb{B}^n$  fixed, we obtain for the same constant in (5.3) that for any  $z \in \mathbb{B}^n$ ,

$$\begin{aligned} |g(z)|\Phi_1^{-1}\left(\frac{1}{(1 - |a|)^{n+1+\alpha}}\right)\left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|}\right)^{2(n+1+\alpha)} \\ \leq C\Phi_2^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\beta}}\right). \end{aligned}$$

Taking in particular  $z = a$  in the last inequality, we obtain that for any  $a \in \mathbb{B}^n$ ,

$$|g(a)| \lesssim C.$$

Hence,  $g \in H^\infty(\mathbb{B}^n)$ . The proof of assertion (i) is complete.

Proof of (ii): let us assume that  $g \in \mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$ ; then, as above, there is a constant  $C > 0$  such that for any  $f \in A_\alpha^{\Phi_1}(\mathbb{B}^n)$  and any  $z \in \mathbb{B}^n$ , (5.3) holds. Testing (5.3) with  $f(z) = f_a(z) = \Phi_1^{-1}(1/(1 - |a|)^{n+1+\alpha})((1 - |a|^2)/(1 - \langle z, a \rangle))^{2(n+1+\alpha)}$  with  $a \in \mathbb{B}^n$  fixed and then taking  $z = a$ , we obtain that for any  $a \in \mathbb{B}^n$ ,

$$|g(a)| \lesssim \frac{\Phi_2^{-1}\left(\frac{1}{(1 - |a|^2)^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{(1 - |a|)^{n+1+\alpha}}\right)} = C\omega(1 - |a|^2).$$

From the hypotheses on the function  $\omega$ , we have that the right-hand side of the last inequality goes to 0 as  $|a| \rightarrow 1$ . Hence, by the maximum principle,  $g(a) = 0$  for all  $a \in \mathbb{B}^n$ . The proof is complete.  $\square$

We finish with the following restriction to target growth functions in  $\tilde{\mathcal{L}} \cup \tilde{\mathcal{U}}$ .

**THEOREM 5.3.** *Let  $\Phi_1 \in \mathcal{L} \cup \mathcal{U}$  and  $\Phi_2 \in \tilde{\mathcal{U}}$ . Assume that  $\Phi_2/\Phi_1$  is nondecreasing. Let  $\alpha, \beta > -1$  and define, for  $0 < t \leq 1$ , the function*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{t^{n+1+\alpha}}\right)}.$$

Then, if  $\omega$  is nonincreasing on  $(0, 1]$ ,

$$\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = H_\omega^\infty(\mathbb{B}^n).$$

**PROOF.** That any function in  $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$  is an element of  $H_\omega^\infty(\mathbb{B}^n)$  can be proved following the same idea in the proof of the necessity in assertion (ii) of the previous theorem.

Let us prove that any  $g \in H_\omega^\infty(\mathbb{B}^n)$  is an element of  $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$ . Let  $K = \max\{1, C_1 C_2 C_3\}$ , where  $C_1$  is the constant in (2.1);  $C_2$  and  $C_3$  are respectively from conditions (2.6) and (2.7) in the definition of the class  $\tilde{\mathcal{U}}$ . For  $C > 0$ , a constant whose existence has to be proved, using the condition on  $g$  and (2.6),

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi_2\left(\frac{|g(z)||f(z)|}{KC\|g\|_{H_\omega^\infty}\|f\|_{\Phi_{1,\alpha}^{\text{lux}}}}\right) dv_\beta(z) \\ &\leq \int_{\mathbb{B}^n} \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)} \frac{|f(z)|}{KC\|f\|_{\Phi_{1,\alpha}^{\text{lux}}}}\right) dv_\beta(z) \\ &\leq C_2 \int_{\mathbb{B}^n} \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)}\right) \Phi_2\left(\frac{|f(z)|}{KC\|f\|_{\Phi_{1,\alpha}^{\text{lux}}}}\right) dv_\beta(z). \end{aligned}$$

Now, using (2.7) and (2.1), we deduce that

$$\begin{aligned} L &\leq C_2 C_3 \int_{\mathbb{B}^n} \frac{1}{(1 - |z|^2)^{n+1+\beta} \left( \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \right)^q} \Phi_2 \left( \frac{|f(z)|}{KC \|f\|_{\Phi_{1,\alpha}}^{\text{lux}}} \right) dv_\beta(z) \\ &\leq C_1 C_2 C_3 \int_{\mathbb{B}^n} \frac{1}{(1 - |z|^2)^{n+1+\beta} \Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)} \Phi_2 \left( \frac{|f(z)|}{KC \|f\|_{\Phi_{1,\alpha}}^{\text{lux}}} \right) dv_\beta(z) \\ &\leq \int_{\mathbb{B}^n} \frac{1}{(1 - |z|^2)^{n+1+\beta} \Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)} \Phi_2 \left( \frac{|f(z)|}{C \|f\|_{\Phi_{1,\alpha}}^{\text{lux}}} \right) dv_\beta(z). \end{aligned}$$

To conclude, we only have to prove the existence of a constant  $C > 0$  such that

$$\int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{C \|f\|_{\Phi_{1,\alpha}}^{\text{lux}}} \right) d\mu(z) \leq 1,$$

where

$$d\mu(z) = \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)} (1 - |z|^2)^{-(n+1)} dv(z).$$

For this, we know from Theorem 2.3 that it is enough to prove that  $\mu$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure.

Following the proof of the implication (a)  $\Rightarrow$  (c) in Theorem 2.3, we see that to prove that  $\mu$  is a  $(\Phi_2 \circ \Phi_1^{-1}, \alpha)$ -Carleson measure it is enough to prove that for any  $a \in \mathbb{B}^n$  and  $0 < r < 1$ ,

$$\mu(\Delta(a, r)) \leq C \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right)}.$$

Using that for any  $z \in \Delta(a, r)$ ,  $1 - |z|^2 \sim 1 - |a|^2$  and that

$$\nu(\Delta(a, r)) \sim (1 - |a|^2)^{n+1},$$

we easily obtain

$$\begin{aligned} \mu(\Delta(a, r)) &= \int_{\Delta(a,r)} \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)} (1 - |z|^2)^{-(n+1)} dv(z) \\ &\sim \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right)} (1 - |a|^2)^{-(n+1)} \nu(\Delta(a, r)) \\ &\sim \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |a|^2)^{n+1+\alpha}} \right)}. \end{aligned}$$

The proof is complete. □

We also have the following result.

**THEOREM 5.4.** *Let  $\Phi_1 \in \mathcal{L}$  and  $\Phi_2 \in \tilde{\mathcal{L}}$ . Assume that  $\Phi_2/\Phi_1$  is nondecreasing. Let  $\alpha, \beta > -1$  and define, for  $0 < t \leq 1$ , the function*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{t^{n+1+\alpha}}\right)}.$$

Then, if  $\omega$  is nonincreasing on  $(0, 1]$ ,

$$\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n)) = H_\omega^\infty(\mathbb{B}^n).$$

**PROOF.** Again, that any function in  $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{B}^n), A_\beta^{\Phi_2}(\mathbb{B}^n))$  is an element of  $H_\omega^\infty(\mathbb{B}^n)$  can be proved following the same idea in the proof of Theorem 5.2.

Let us prove that any  $g \in H_\omega^\infty(\mathbb{B}^n)$  is a multiplier from  $A_\alpha^{\Phi_1}(\mathbb{B}^n)$  to  $A_\beta^{\Phi_2}(\mathbb{B}^n)$ .

Let  $K = \max\{1, C'(C_1 C_3)^{1/p}\}$ , where  $C'$  is the constant in (2.1),  $C_1$  is the constant in (2.6) and  $C_3$  is the constant in (2.8). For  $C > 0$ , a constant whose existence has to be proved, using the condition on  $g$  and (2.6),

$$\begin{aligned} L &:= \int_{\mathbb{B}^n} \Phi_2\left(\frac{|g(z)||f(z)|}{KC\|g\|_{H_\omega^\infty}\|f\|_{\Phi_1,\alpha}^{\text{lux}}}\right) dv_\beta(z) \\ &\leq \int_{\mathbb{B}^n} \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)} KC\|f\|_{\Phi_1,\alpha}^{\text{lux}}\right) dv_\beta(z) \\ &\leq C_1 \int_{\mathbb{B}^n} \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)}\right) \Phi_2\left(\frac{|f(z)|}{KC\|f\|_{\Phi_1,\alpha}^{\text{lux}}}\right) dv_\beta(z). \end{aligned}$$

Next, using (2.8) and the fact that as  $\Phi_2 \in \mathcal{L}_p$ ,  $\Phi_2^{-1} \in \mathcal{U}^{1/p}$ ,

$$\begin{aligned} L &\leq C_1 C_3 \int_{\mathbb{B}^n} \frac{\left(\Phi_2^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\beta}}\right)\right)^p}{\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)\right)} \Phi_2\left(\frac{|f(z)|}{KC\|f\|_{\Phi_1,\alpha}^{\text{lux}}}\right) dv_\beta(z) \\ &\leq C_1 C_3 (C')^p \int_{\mathbb{B}^n} \frac{1}{(1-|z|^2)^{n+1+\beta} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)\right)} \\ &\quad \times \Phi_2\left(\frac{|f(z)|}{KC\|f\|_{\Phi_1,\alpha}^{\text{lux}}}\right) dv_\beta(z) \\ &\leq \int_{\mathbb{B}^n} \Phi_2\left(\frac{|f(z)|}{C\|f\|_{\Phi_1,\alpha}^{\text{lux}}}\right) d\mu(z), \end{aligned}$$

where

$$d\mu(z) = \frac{1}{\Phi_2 \circ \Phi_1^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right)} (1 - |z|^2)^{-(n+1)} d\nu(z).$$

Again, to conclude, we only have to prove the existence of a constant  $C > 0$  such that

$$\int_{\mathbb{B}^n} \Phi_2 \left( \frac{|f(z)|}{C \|f\|_{\Phi_{1,\alpha}}^{\text{lux}}} \right) d\mu(z) \leq 1.$$

This follows as at the end of the proof of Theorem 5.3. The proof is complete.  $\square$

Theorem 2.4 clearly follows from Theorems 5.2–5.4.

Let  $f \in \mathcal{H}(\mathbb{B}^n)$ . The radial derivative  $Rf$  of  $f$  is given by

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

An analytic function  $f$  in  $\mathbb{B}^n$  belongs to  $\Lambda_\omega$  if  $Rf \in \mathcal{H}_\omega^\infty(\mathbb{B}^n)$ , that is,

$$\sup_{z \in \mathbb{B}^n} \frac{(1 - |z|)|Rf(z)|}{\omega(1 - |z|)} < \infty.$$

The space  $\Lambda_\omega$  is a Banach space under

$$\|f\|_{\Lambda_\omega} := |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{(1 - |z|)|Rf(z)|}{\omega(1 - |z|)}.$$

Note that if  $\omega(t) = t^{1-\lambda}$ ,  $0 < \lambda < \infty$ ,  $\Lambda_\omega$  is just the  $\lambda$ -Bloch space usually denoted  $\mathcal{B}^\lambda$ ;  $\mathcal{B} = \mathcal{B}^1$  being the Bloch space.

It is not hard to see that for  $\lambda > -1$ , one has  $\mathcal{H}_\omega^\infty(\mathbb{B}^n) = \Lambda_\omega$ , when  $\omega(t) = t^{-\lambda}$ . Indeed, in this case, the proof of the continuous embedding  $\Lambda_\omega \hookrightarrow \mathcal{H}_\omega^\infty(\mathbb{B}^n)$  follows as in [17, proof of Lemma 2.10], while the proof of the continuous embedding  $\mathcal{H}_\omega^\infty(\mathbb{B}^n) \hookrightarrow \Lambda_\omega$  follows the first lines of [17, proof of Lemma 2.11].

It follows from the last observation and Theorem 2.4 that we have the following result.

**COROLLARY 5.5.** *Let  $0 < p \leq q < \infty$ ,  $\alpha, \beta > -1$ . Define  $\lambda = (n + 1 + \beta)/q - (n + 1 + \alpha)/p$ . Then the following assertions hold.*

- (i) *If  $\lambda > 0$ , then  $\mathcal{M}(A_\alpha^p(\mathbb{B}^n), A_\beta^q(\mathbb{B}^n)) = \mathcal{B}^{\lambda+1}$ .*
- (ii) *If  $\lambda = 0$ , then  $\mathcal{M}(A_\alpha^p(\mathbb{B}^n), A_\beta^q(\mathbb{B}^n)) = H^\infty(\mathbb{B}^n)$ .*
- (iii) *If  $\lambda < 0$ , then  $\mathcal{M}(A_\alpha^p(\mathbb{B}^n), A_\beta^q(\mathbb{B}^n)) = \{0\}$ .*

## References

- [1] K. R. M. Attele, ‘Analytic multipliers of Bergman spaces’, *Michigan Math. J.* **31** (1984), 307–319.
- [2] S. Axler, ‘Multiplication operators on Bergman spaces’, *J. reine angew. Math.* **336** (1982), 26–44.
- [3] S. Axler, ‘Zero-multipliers of Bergman spaces’, *Canad. Math. Bull.* **28** (1985), 237–242.



- [4] L. Carleson, 'An interpolation problem for bounded analytic functions', *Amer. J. Math.* **80** (1958), 921–930.
- [5] L. Carleson, 'Interpolation by bounded analytic functions and corona problem', *Ann. of Math. (2)* **76** (1962), 547–559.
- [6] S. Charpentier, 'Composition operators on weighted Bergman–Orlicz spaces on the ball', *Complex Anal. Oper. Theory* **7**(1) (2013), 43–68.
- [7] S. Charpentier and B. F. Sehba, 'Carleson measure theorems for large Hardy–Orlicz and Bergman–Orlicz spaces', *J. Funct. Spaces Appl.* **2012** (2012), 792763, 21 pages.
- [8] J. A. Cima and W. Wogen, 'A Carleson measure theorem for the Bergman space on the unit ball of  $\mathbb{C}^n$ ', *J. Operator Theory* **7**(1) (1982), 157–165.
- [9] P. L. Duren, 'Extension of a theorem of Carleson', *Bull. Amer. Math. Soc. (N.S.)* **75** (1969), 143–146.
- [10] W. Hastings, 'A Carleson measure theorem for Bergman spaces', *Proc. Amer. Math. Soc.* **52** (1975), 237–241.
- [11] L. Hörmander, ' $L^p$ -estimates for (pluri)subharmonic functions', *Math. Scand.* **20** (1967), 65–78.
- [12] D. Luecking, 'A technique for characterizing Carleson measures on Bergman spaces', *Proc. Amer. Math. Soc.* **87** (1983), 656–660.
- [13] D. Luecking, 'Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives', *Amer. J. Math.* **107**(1) (1985), 85–111.
- [14] D. Luecking, 'Multipliers of Bergman spaces into Lebesgue spaces', *Proc. Edingb. Math. Soc. (2)* **29** (1986), 125–131.
- [15] D. Luecking, 'Embedding theorems for spaces of analytic functions via Khinchine's inequality', *Michigan Math. J.* **40**(2) (1993), 333–358.
- [16] S. C. Power, 'Hörmander's Carleson theorem for the ball', *Glasg. Math. J.* **26**(1) (1985), 13–17.
- [17] B. F. Sehba and S. Stevic, 'On some product-type operators from Hardy–Orlicz and Bergman–Orlicz spaces to weighted-type spaces', *Appl. Math. Comput.* **233** (2014), 565–581.
- [18] B. F. Sehba and E. Tchoundja, 'Hankel operators on holomorphic Hardy–Orlicz spaces', *Integral Equations Operator Theory* **73**(3) (2012), 331–349.
- [19] M. Stoll, *Invariant Potential Theory in the Unit Ball of  $\mathbb{C}^n$*  (Cambridge University Press, Cambridge, 1994).
- [20] S. Ueki, 'Weighted composition operators between weighted Bergman spaces in the unit ball of  $\mathbb{C}^n$ ', *Nihonkai Math. J.* **16**(1) (2005), 31–48.
- [21] I. V. Videnskii, 'On an analogue of Carleson measures', *Dokl. Akad. Nauk SSSR* **298** (1988), 1042–1047; *Soviet Math. Dokl.* **37** (1988), 186–190.
- [22] D. Vukotić, 'Pointwise multiplication operators between Bergman spaces on simply connected domains', *Indiana Univ. Math. J.* **48** (1999), 793–803.
- [23] R. Zhao, 'Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces', *Ann. Acad. Sci. Fenn. Math.* **29** (2004), 139–150.
- [24] R. Zhao and K. Zhu, 'Theory of Bergman spaces in the unit ball of  $\mathbb{C}^n$ ', *Mém. Soc. Math. Fr. (N.S.)* **115** (2008), 103 pages.

**BENOÎT F. SEHBA**, Department of Mathematics, University of Ghana,  
Legon, P.O. Box LG 62, Legon Accra, Ghana  
e-mail: [bfsehba@ug.edu.gh](mailto:bfsehba@ug.edu.gh)