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CUP PRODUCTS IN SHEAF COHOMOLOGY

BY
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ABSTRACT. Let k be an algebraically closed field, and let ℓ be a prime number not equal to char(k). Let X be a locally fibrant simplicial sheaf on the big étale site for k, and let Y be a k-scheme which is cohomologically proper. Then there is a Künneth-type isomorphism

$$H^*_{et}(X; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*_{et}(Y; \mathbb{Z}/\ell) \cong H^*_{et}(X \times Y; \mathbb{Z}/\ell)$$

which is induced by an external cup-product pairing. Reductive algebraic groups G over k are cohomologically proper, by a result of Friedlander and Parshall. The resulting Hopf algebra structure on $H^*_{et}(G; \mathbb{Z}/\ell)$ may be used together with the Lang isomorphism to give a new proof of the theorem of Friedlander-Mislin which avoids characteristic 0 theory. A vanishing criterion is established for the Friedlander-Quillen conjecture.

Introduction. Let k be an algebraically closed field, and let ℓ be a prime number which is distinct from the characteristic of k. Let X be a simplicial sheaf on the big étale site $(Sch|_k)_{et}$ which is fibrant in the sense that it is locally a Kan complex. The main new result of this paper asserts that the cup product structure for étale cohomology of simplicial sheaves introduced in [5] determines a Künneth-type formula

$$(1) \qquad H^*_{et}(X; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*_{et}(Y; \mathbb{Z}/\ell) \cong H^*_{et}(X \times Y; \mathbb{Z}/\ell),$$

provided that Y is represented by a k-scheme which is cohomologically proper in a suitable sense. Examples of such Y include all complete k-varieties and, by a theorem of Friedlander and Parshall [4], all reductive algebraic groups over k.

The Künneth formula induces a Hopf algebra structure on the étale cohomology $H^*_{et}(G; \mathbb{Z}/\ell)$ of a reductive group G over k which does not depend on a passage to characteristic 0 theory, and hence avoids the classification of reductive group-schemes over arbitrary bases. It also leads to an alternate proof of a recent theorem of Friedlander and Mislin [3], which asserts that the canonical map

$$(2) \qquad \epsilon\colon \Gamma^*BG(k) \to BG$$

of simplicial sheaves induces an isomorphism

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(3) $$H^*_{et}(BG; \mathbb{Z}/\ell) \cong H^*(BG(k); \mathbb{Z}/\ell),$$

provided that k is the algebraic closure of a finite field. This proof appears in the second section of this paper; it is or is not quicker, depending on your point of view, since it relies heavily on the results of [5]. The key point is that the Hopf algebra structure of $H^*_{et}(G; \mathbb{Z}/\ell)$, together with the Lang isomorphism, implies that there are isomorphisms

(4) $$H^j_{et}(G/\Gamma^*G(k); \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell, & j = 0 \\ 0, & j \neq 0, \end{cases}$$

where $G/\Gamma^*G(k)$ is the sheaf-theoretic quotient of G by the action of the constant subgroup-sheaf $\Gamma^*G(k)$ for the group $G(k)$ of k-rational points of G.

It is worth trying to compute $H^*_{et}(G/\Gamma^*G(k); \mathbb{Z}/\ell)$ for all reductive groups G over all algebraically closed fields k. A vanishing result like (4) in that range would imply the Friedlander–Quillen conjecture (also called the generalized isomorphism conjecture [3], [5]), which asserts that the map (2) induces an isomorphism (3) in the same generality. This follows from the argument given for the Friedlander–Mislin result in this paper. One has to learn somehow to live without the Lang isomorphism to be successful with this approach.

This paper is not the end of the cup products story. The Künneth formula (1) is susceptible to obvious generalizations. Furthermore, one would like to know more about the situation over bases which are not algebraically closed fields.

1. **Homological algebra**. Let X be a fibrant simplicial sheaf on $(Sch|_k)_{et}$, and let F be a sheaf of abelian groups. Recall [5] that the group $H^n(X; F) = H^n_{et}(X; F)$ may be defined by

$$H^n(X; F) = [X, K(F, n)],$$

where the square brackets denote morphisms in the associated homotopy category, and the simplicial sheaf $K(F, n)$ is an Eilenberg–MacLane complex. Recall also that the homotopy category $Ho(Sch|_k)_{et}$ is constructed by formally inverting morphisms represented by trivial fibrations (also called hypercovers) in the category $\pi(Sch|_k)_{et}$ whose objects are the fibrant simplicial sheaves, and whose hom sets $\pi(Y, X)$ are obtained from the simplicial sheaf hom sets $hom(Y, X)$ by collapsing by the smallest equivalence relation containing the simplicial homotopy relation. The class of morphisms represented by hypercovers in $\pi(Sch|_k)_{et}$ admits a calculus of fractions, so that there is an isomorphism

(5) $$[X, K(F, n)] \cong \varinjlim_{\substack{T \to X \\ \text{in Triv} \downarrow X}} \pi(T, K(F, n)),$$

where the filtered category Triv $\downarrow X$ is the full subcategory of $\pi(Sch|_k)_{et} \downarrow X$ on those objects which are represented by hypercovers.

The same analysis [1], [5] goes through for simplicial abelian sheaves, chain

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complexes of sheaves, and $\mathbb{Z}$-graded cochain complexes of sheaves on $(Sch|_k)_{et}$. Let $[\ ,\ ]_{ab}$, $[\ ,\ ]_{ch}$ and $[\ ,\ ]_{co}$ denote morphisms in their respective homotopy categories. Brown's adjoint functor lemma [1], together with Theorem 2.5 of [5], implies that there is a natural isomorphism

$$(6) \qquad [X, A] \cong [\mathbb{Z}X, A]_{ab},$$

for fibrant simplicial sheaves X and abelian sheaves A. $X \mapsto \mathbb{Z}X$ is the free abelian sheaf functor. The isomorphism (6) is induced by the standard adjunction isomorphism which relates sets to abelian groups. That same adjunction induces an isomorphism

$$\pi(T, A) \cong \pi_{ab}(\mathbb{Z}T, A),$$

where $\pi_{ab}(\ ,\ )$ denotes simplicial abelian homotopy classes of maps. Let $\gamma: T \to X$ be a hypercover as before. Then $\mathbb{Z}\gamma: \mathbb{Z}T \to \mathbb{Z}X$ is a trivial fibration of simplicial abelian sheaves, and there is a commutative diagram

$$(7) \qquad \begin{array}{ccc} \pi(T, A) & \xrightarrow{[\gamma]^*} & [X, A] \\ \downarrow \cong & & \downarrow \cong \\ \pi_{ab}(\mathbb{Z}T, A) & \xrightarrow[{[\mathbb{Z}\gamma]^*}]{} & [\mathbb{Z}X, A]_{ab}, \end{array}$$

where $[\gamma]^*$ and $[\mathbb{Z}\gamma]^*$ are the canonical maps for the colimit (5) and its analogue for simplicial abelian sheaves.

The usual Dold–Puppe story (see [5]) implies that there is a commutative diagram

$$(8) \qquad \begin{array}{ccc} \pi_{ab}(\mathbb{Z}T, A) & \xrightarrow{[\mathbb{Z}\gamma]^*} & [\mathbb{Z}X, A]_{ab} \\ \downarrow \cong & & \downarrow \cong \\ \pi_{ch}(\mathbb{Z}T, A) & \longrightarrow & [\mathbb{Z}X, A]_{ch}, \end{array}$$

where $\pi_{co}(\ ,\ )$ means chain homotopy classes of maps, and the simplicial abelian sheaves in question are confused notationwise with their associated Moore complexes.

Finally, observe that the inclusion of the chain complex category as the objects concentrated in negative degrees in the $\mathbb{Z}$-graded cochain complex category has a right adjoint which preserves weak equivalences. Putting this together with the analogue of (5), as well as (7) and (8) above yields the following generalization of the Verdier hypercovering theorem:

LEMMA 1.1: *Let $\mathbb{Z}X$ be the Moore complex of a fibrant simplicial sheaf X, and let J be a $\mathbb{Z}$-graded cochain complex. Then there is an isomorphism*

$$[\mathbb{Z}X, J]_{co} \cong \varinjlim_{\substack{T \to X \\ \text{in Triv} \downarrow X}} \pi_{co}(\mathbb{Z}T, J).$$

In particular, the colimit on the right takes cohomology isomorphisms to isomorphisms of abelian groups. $\pi_{co}(\ ,\ )$ is cochain homotopy classes of maps. If J is the cochain complex $F[n]$ which is the abelian sheaf F concentrated in degree $-n$, then Lemma 1.1 specializes to the generalized Verdier hypercovering theorem of [5].

Let ℓ be the prime number chosen in the Introduction. As usual, $\mathbb{Z}/\ell$ will mean both the abelian group and the associated constant abelian sheaf $\Gamma^*\mathbb{Z}/\ell$ on $(Sch|_k)_{et}$. Let N be an ℓ-torsion abelian sheaf (meaning sheaf on $\mathbb{Z}/\ell$-modules), and let $N \to J^*$ be an injective resolution in the ℓ-torsion category. The bicomplex

$$hom(\mathbb{Z}/\ell(X_n), J^p) \cong hom(\mathbb{Z}X_n, J^p)$$

determines a spectral sequence, with

$$(9) \qquad E_2^{p,q} \cong \mathrm{Ext}^q(H_p(X; \mathbb{Z}/\ell); N) \Rightarrow \pi_{co}(\mathbb{Z}X, J[p+q]).$$

Some explanations are in order; $X \mapsto \mathbb{Z}/\ell X)$ is the free ℓ-torsion sheaf functor, and $H_p(X; \mathbb{Z}/\ell)$ is the p^{th} homology sheaf of the associated chain complex. Also, $J[p+q]^n = J^{n+(p+q)}$ defines the cochain complex $J[p+q]$. Observe that the Ext group is computed in the ℓ-torsion category.

It follows that the functor

$$X \mapsto \pi_{co}(\mathbb{Z}X, J[n])$$

takes weak equivalences to group isomorphisms. In particular, using Lemma 1.1, one sees

LEMMA 1.2: *Let X be a fibrant simplicial sheaf and let $N \to J^*$ be a resolution of the ℓ-torsion sheaf N by ℓ-torsion injectives. Then the canonical map*

$$(10) \qquad \pi_{co}(\mathbb{Z}X, J[n]) \to [\mathbb{Z}X, J[n]]_{co}$$

is an isomorphism.

Actually, the map (10) is an isomorphism when $J[n]$ is replaced by any cochain complex of injectives which is bounded below, ℓ-torsion or not. One modifies the spectral sequence (9) to see this. Similarly, if $X = K(U, O)$ is the constant (hence fibrant) simplicial sheaf represented by a scheme $U \in (Sch|_k)_{et}$, then the integral version of Lemma 1.2 (see [5]) implies the existence of a canonical isomorphism

$$[\mathbb{Z}K(U, O), F[n]] \cong H^n_{et}(U; F),$$

where $H^n_{et}(U; F)$ is the usual étale cohomology group with coefficients in the restriction of F to the étale site ét$|_U$ for U.

Let Y be another fibrant simplicial sheaf on $(Sch|_k)_{et}$, and choose hypercovers π: $S \to Y$ and γ: $T \to X$. Then the $\mathbb{Z}/\ell$-structure of the ℓ-torsion sheaf N induces a cross-product pairing

$$(11) \qquad hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(T_n, N) \xrightarrow{\times} hom(S_m \times T_n, N)$$

in the obvious way. The Eilenberg–Zilber map induces a weak equivalence of cochain complexes

$$(12) \qquad Tot_{m,n} hom(S_m \times T_n, N) \xrightarrow{f^*} hom(S \times T, N).$$

Now consider the induced composition

$$(13)\quad \begin{array}{l} H^p hom(S, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^q hom(T, N) \to H^{p+q} Tot_{m,n} hom(S_m \times T_n, N) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow f^* \\ \qquad\qquad\qquad\qquad\qquad\qquad H^{p+q} hom(S \times T; N) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow [\pi \times \gamma]^* \\ \qquad\qquad\qquad\qquad\qquad\qquad H^{p+q}(Y \times X; N) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad [\mathbb{Z}(Y \times X), N[p+q]]_{co}. \end{array}$$

A simplicial homotopy of maps in either S or T induces a chain homotopy in all relevant chain complexes. Thus, the functor

$$\varinjlim_{\substack{T \to X \\ \text{in Triv} \downarrow X}} \ \varinjlim_{\substack{T \to X \\ \text{in Triv} \downarrow X}} (\quad)$$

may be applied, inducing an external cup product pairing

$$H^p(Y; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^q(X; N) \xrightarrow{\cup} H^{p+q}(Y \times X; N).$$

A scheme $U \in (Sch|_k)_{et}$ is said to be *cohomologically proper* if each $H^p(U; \mathbb{Z}/\ell)$ is finite and if the presheaf map

$$V \mapsto H^*(U; \mathbb{Z}/\ell) \xrightarrow{pr^*} H^*(V \times U; \mathbb{Z}/\ell)$$

induces an isomorphism of sheaves on $(Sch|_k)_{et}$. Computing stalkwise, one sees that examples of cohomologically proper k-schemes include all complete k-varieties, by the proper base change theorem, and all reductive algebraic groups over k, by the Friedlander-Parshall theorem [4]. The main result of this paper is

THEOREM 1.3: *Suppose that U in $(Sch|_k)_{et}$ is cohomologically proper, and that Y is a fibrant simplicial sheaf. Then the external cup product pairing*

$$H^*(Y; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*(U; \mathbb{Z}/\ell) \xrightarrow{\cup} H^*(Y \times U; \mathbb{Z}/\ell)$$

is an isomorphism.

PROOF: Choose hypercovers $\pi: S \to Y$ and $y: T \to U$, and choose an ℓ-torsion injective resolution $i: \mathbb{Z}/\ell \to J^*$. There is a commutative diagram of tricomplexes

$$\begin{array}{ccc} hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(U, J^p) & \xrightarrow{\times} & hom(S_m \times U, J^p) \\ \downarrow 1 \otimes \gamma^* & & \downarrow (1 \times \gamma)^* \\ hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(T_n, J^p) & \xrightarrow{\times} & hom(S_m \times T_n, J^p) \\ \uparrow 1 \otimes i_* & & \uparrow i_* \\ hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(T_n, \mathbb{Z}/\ell[0])^p) & \xrightarrow{\times} & hom(S_m \times T_n, (\mathbb{Z}/\ell[0])^p), \end{array}$$

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The maps induced on the total complex level by γ and the Eilenberg–Zilber map are weak equivalences. It follows (see (13)) that the map induced by the bicomplex map

$$hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(U, J^p) \xrightarrow{\times} hom(S_m \times U, J^p)$$

is isomorphic to the external cup rpoduct pairing, after applying the functor

$$\varinjlim_{\substack{T \to Y \\ \text{in Triv} \downarrow Y}} H^* Tot_{m,p}(\quad).$$

Recall that the sheaf $\mathbf{hom}(U, J^p)$ is defined by

$$\mathbf{hom}(U, J^p)(V) = hom(U|_V, J^p|_V)$$

over $(Sch|_V)_{et}$, and that the evaluation map

$$ev: \mathbf{hom}(U, J^p) \times U \to J^p$$

induces an adjunction isomorphism

$$ev_*: hom(Z, \mathbf{hom}(U, J^p)) \xrightarrow{\cong} hom(Z \times U, J^p).$$

Observe also that there is a natural isomorphism

$$can: hom(\mathbb{Z}/\ell, M) \xrightarrow{\cong} M(k)$$

for all ℓ-torsion sheaves M. Then there is a commutative diagram

$$\begin{array}{ccc}
hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(\mathbb{Z}/\ell, \mathbf{hom}(U, J^p)) & \xrightarrow{\subset} & hom(S_m, \mathbf{hom}(U, J^p)) \\
\cong \downarrow 1 \otimes can & & \cong \downarrow ev_* \\
hom(S_m, \mathbb{Z}/\ell) \times_{\mathbb{Z}/\ell} \mathbf{hom}(U, J^p)(k) & & hom(S_m \times U, J^p) \\
\| & \nearrow \times & \\
hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(U, J^p), & &
\end{array}$$

where the map $\subset$ is defined by composition. It therefore suffices to show that $\subset$ induces the desired isomorphism.

Let $\mathbf{hom}(U, J^*) \to I^{**}$ be an Eilenberg–Cartan resolution in the ℓ-torsion category, with resolutions $j: H^p \to K^{p,*}$ of the cohomology sheaves $H^p = H^p\mathbf{hom}(U, J^*)$. Then it is enough to show that the tricomplex map

$$hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(\mathbb{Z}/\ell, I^{p,q}) \xrightarrow{\subset} hom(S_m, I^{p,q})$$

induces an isomorphism after applying the functor

$$\varinjlim_{\substack{S \to Y \\ \text{in Triv} \downarrow Y}} H^* Tot_{m,p,q}(\quad).$$

But the map induced by computing cohomology in the p-direction is

$$hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(\mathbb{Z}/\ell, K^{p,q}) \xrightarrow{\subset} hom(S_m, K^{p,q})$$

since the resolution I^{**} splits by construction. There is also a commutative diagram of bicomplexes

$$\begin{array}{ccc} hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(\mathbb{Z}/\ell, (H^p[0])^q) & \xrightarrow{\subset} & hom(S_m, (H^p[0])^q) \\ \downarrow 1 \otimes j_* & & \downarrow j_* \\ hom(S_m, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} hom(\mathbb{Z}/\ell, K^{p,q}) & \xrightarrow{\subset} & hom(S_m, K^{p,q}). \end{array} \tag{14}$$

The top composition map of (14) is an isomorphism, since H^p is assumed to be a finite direct sum of copies of $\mathbb{Z}/\ell$. Thus, since j is a weak equivalence, the bottom composition map of (14) is an isomorphism after applying

$$\varinjlim_{\substack{S \to Y \\ \text{in Triv} \downarrow Y}} H^* Tot_{m,q}(\quad).$$

The result follows. QED

COROLLARY 1.4: *Suppose that G is a reductive algebraic group over k, and that X is a fibrant simplicial sheaf on $(Sch|_k)_{et}$. Then the map*

$$H^*(X; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*(G; \otimes_{\mathbb{Z}/\ell}) \to H^*(X \times G; \mathbb{Z}/\ell)$$

defined by

$$u \otimes v \mapsto \mathrm{pr}_X{}^*(u) \cup \mathrm{pr}_G{}^*(v)$$

is an isomorphism.

Observe that the preservation of weak equivalence by the global sections functor for sheaves on $(Sch|_k)_{et}$ is required in the proof of Theorem 1.3. This follows from the assumption that k is an algebraically closed field.

2. **Algebraic groups**

Let k be the algebraic closure of the finite field $\mathbb{F}_p$, and choose a prime number $\ell \neq p$. The main result of [3], when translated into the context of simplicial sheaves on $(Sch|_k)_{et}$, is the following:

THEOREM 2.1: *Let G be a reductive affine algebraic group over $k = \overline{\mathbb{F}}_p$, and choose ℓ as above. The the canonical map*

$$\epsilon: \Gamma^* BG(k) \to BG$$

of simplicial sheaves induces an isomorphism

$$\epsilon^*: H^*_{et}(BG; \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BG(k); \mathbb{Z}/\ell).$$

The purpose of this section is to outline a proof of this theorem from the point of view of the homotopy theory of simplicial sheaves, using the results of the previous section.

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The first step is to observe that $\Gamma^*G(k)$ is a subgroup-sheaf of G, via ϵ: $\Gamma^*G(k) \to G$. Thus, there is a commutative diagram of simplicial sheaves

$$\begin{array}{ccccc} E\Gamma^*G(k) & \longrightarrow & EG & \searrow & \\ \downarrow & & \downarrow & & BG \\ B\Gamma^*G(k) & \underset{\tilde{\epsilon}_*}{\longrightarrow} & EG/\Gamma^*G(k), & \nearrow_{\pi} & \end{array}$$

where $\pi\tilde{\epsilon}_*$ is isomorphic to ϵ. $\tilde{\epsilon}_*$ induces an isomorphism of homology sheaves, so the theorem is proved by showing that π induces an isomorphism in cohomology.

The map π, on the simplex level, is the projection map

$$G/\Gamma^*G(k) \times (G \times \ldots \times G) \xrightarrow{\text{pr}} (G \times \ldots \times G),$$

and so, by Corollary 1.4 and a spectral sequence argument, it is enough to show that

$$H^j(G/\Gamma^*G(k); \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell, & j = 0, \\ 0, & j > 0. \end{cases}$$

Observe that

$$G/\Gamma^*G(k) = \varinjlim G/\Gamma^*G(\mathbb{F}_{p^n}),$$

where $G(\mathbb{F}_{p^n})$ is the finite group of $\mathbb{F}_{p^n}$-valued points of G, and the filtered colimit is indexed over the natural numbers, ordered by divisibility. Proposition 2.13 of [5] implies that there is a short exact sequence

$$(15) \qquad 0 \to \varprojlim{}^1 H^{j-1}(G/\Gamma^*G(\mathbb{F}_{p^n}); \mathbb{Z}/\ell) \to H^j(G/\Gamma^*G(k) : \mathbb{Z}/\ell)$$
$$\to \varprojlim H^j(G/\Gamma^*(\mathbb{F}_{p^n}); \mathbb{Z}/\ell) \to 0.$$

In effect, the directed system above may be replaced by a cofinal subsystem

$$G/\Gamma^*(\mathbb{F}_p) \to G/\Gamma^*(\mathbb{F}_{p^{n_1}}) \to G/\Gamma^*(\mathbb{F}_{p^{n_2}}) \to \ldots .$$

Then each of the resulting maps (of constant simplicial sheaves) may be replaced up to weak equivalence by a monomorphism, by using a pointwise telescope on the presheaf level and then sheafifying.

Now let $\varphi: G \to G$ be the Frobenius homomorphism. The sequence of sheaf homomorphisms

$$\Gamma^*G(\mathbb{F}_{p^n}) \to G \xrightarrow{1/\varphi^n} G$$

is short exact in the sense that $1/\varphi^n$ determines a canonical isomorphism of sheaves

$$G/\Gamma^*G(\mathbb{F}_{p^n}) \cong G$$

on $(Sch|_k)_{et}$. This is the Lang isomorphism [2, 12.1], [7] in this context. It follows that the $\varprojlim{}^1$ term of the sequence (15) vanishes, since the groups $H^*(G; \mathbb{Z}/\ell)$ are finite. It also follows easily that

$$H^0(G/\Gamma^*G(k);\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell.$$

Let $\psi = \varphi^n$ for a fixed n. Then there is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{1/\psi^d} & G = G/\Gamma^*G(\mathbb{F}_{p^{nd}}) \\ \| & & \uparrow \gamma \\ G & \longrightarrow & G = G/\Gamma^*G(\mathbb{F}_{p^n}), \end{array} \tag{16}$$

where γ is the composite

$$G \xrightarrow{\Delta} G \times \ldots \times G \xrightarrow{1 \times \psi \times \psi^2 \times \ldots \times \psi^{d-1}} G \times \ldots \times G \xrightarrow[m]{} G.$$

Δ is the diagonal map and m is the multiplication map; they give $H^*(G;\mathbb{Z}/\ell)$ a Hopf algebra structure, via Corollary 1.4. It follows that

$$\gamma^*(x) = x + \psi^*(x) + \ldots + (\psi^*)^{d-1}(x) + \text{decomposables}$$

in positive degrees.

Recall finally that φ^* is an automorphism of the finite ℓ-torsion group $H^j(G;\mathbb{Z}/\ell)$, $j > 0$, [6] so that $(\psi^*)^N = 1$ for some N. Thus, for each n and $j > 0$, there is a d such that, for γ in (16), $\gamma^*(H^j(G;\mathbb{Z}/\ell))$ is decomposeable. It follows by induction on j that the system of groups

$$\{H^j(G/\Gamma^*G(\mathbb{F}_{p^n});\mathbb{Z}/\ell)\}$$

is pro-trivial. This implies that

$$\varprojlim H^j(G/\Gamma^*G(\mathbb{F}_{p^n});\mathbb{Z}/\ell) = 0$$

for each $j > 0$. QED

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