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Physics of neutrinos

13.1 Neutrino masses

Neutrinos as elementary particles have remarkable properties. They have only weak and gravitational interactions, which allows them to travel through matter making very few interactions. They carry a global quantum number, known as lepton number, which can be broken without disturbing the conservation of electric charge. The breaking of lepton number resides on the mass matrices which we introduce in this section.

Their unique properties have led to several discoveries, the newest among them being neutrino oscillations, which provide information on their mass differences and mixing parameters. Neutrino oscillations also create new questions concerning their properties, which are now under active investigation.

Masses for quarks and leptons were introduced in Chapters 8 and 9 through Yukawa couplings to the Higgs doublet. The neutrino remained massless because interactions of right-handed neutrinos had not been observed. This is a unique and unfamiliar situation, because all other fermions have right-handed components. In this chapter we shall describe the properties of neutrinos and introduce right-handed neutrinos $\mathrm{N_R}$, which are singlets under $\mathrm{SU(2)_L}$. The representation content for the electron family is

$$\Psi_{\mathrm{L}} = \begin{pmatrix} \nu \\ \mathrm{e} \end{pmatrix}_{\mathrm{L}}, \ \mathrm{N_R} \text{ and } \mathrm{e_R}, \tag{13.1}$$

with $e_{\mathrm{R,L}} = \frac{1}{2}(1 \pm \gamma_5)e$. We should have written the right-handed state as ν_{R}; however, right-handed neutrinos will play later a special role (Majorana), so we decided to denote them $\mathrm{N_R}$. There is an analogous classification of the leptonic states for the muon and tau families.

A mass term of the form

$$m_{\mathrm{D}}\bar{\nu}_{\mathrm{L}} N_{\mathrm{R}} + \text{h.c.} \tag{13.2}$$

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is possible and is generated from the Yukawa coupling

$$\mathcal{L}_{\mathrm{Y}}^{\nu} = \sum f_{ll'} \bar{\Psi}_{\mathrm{L}}^{l} \hat{\Phi} N_{\mathrm{R}}^{l'} + \mathrm{h.c.}, \tag{13.3}$$

where Φ is the standard Higgs doublet and

$$\hat{\Phi} = \mathrm{i}\tau_2 \Phi^* = \begin{pmatrix} \Phi_0 \\ -\Phi^- \end{pmatrix} \overrightarrow{\text{breaking}} \begin{pmatrix} \Phi_0 + v \\ 0 \end{pmatrix}. \tag{13.4}$$

We call this the Dirac mass term. The discussion so far is similar to that of quarks and charged leptons. However, the observed extreme smallness of neutrino masses seems to require a special treatment. It is possible to introduce another term, known as the Majorana mass term, which brings special properties and is a candidate for explaining the small masses.

Before we proceed with new properties of the neutrinos, it is instructive to point out which spinors correspond to the above states. The identification is helpful when we proceed with calculations. The Dirac equation has four solutions: two solutions with $E > 0$ and two with $E < 0$, which describe particle and antiparticle states to be denoted by u and v, respectively. We follow standard textbook notation with

$$\psi_{1,2} = u_i \mathrm{e}^{-\mathrm{i}p\cdot x} \qquad \text{and} \qquad \psi_{3,4} = v_i \mathrm{e}^{\mathrm{i}p\cdot x} \qquad \text{for } i = 1, 2, \tag{13.5}$$

where, for particles moving in the z-direction, the four spinors are

$$u_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ 0 \end{pmatrix}, \qquad u_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-p_z}{E+m} \end{pmatrix},$$
$$v_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \dfrac{p_z}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ \dfrac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}. \tag{13.6}$$

The normalization of the spinors is as follows: $\overline{u_i} u_j = \delta_{ij}$ and $\overline{v_i} v_j = -\delta_{ij}$.

For massless neutrinos the spinors are eigenfunctions of the operator γ_5 and we define the spinor for the neutrino as

$$\nu = \left(\frac{1-\gamma_5}{2}\right) u_2 \tag{13.7}$$

and the spinor for the antineutrino as

$$\bar{\nu} = \left(\frac{1+\gamma_5}{2}\right) v_1. \tag{13.8}$$

In this limit we speak of left-handed neutrinos and right-handed antineutrinos. We also notice that in the limit $p_z \gg m$ there are only two independent spinors.

For massive particles the situation is different. Considering the Hamiltonian of a free Dirac particle,

$$H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2, \tag{13.9}$$

where $\beta\vec{\alpha} = \gamma_i$ and $\beta = \gamma_0$, we notice that γ_5 does not commute with the Hamiltonian since $[\gamma_5, \gamma_0] \neq 0$. Thus, for massive neutrinos, γ_5 is not a good quantum number; i.e. the spinors are not eigenstates of the γ_5 operator. For the massive case we introduce another operator, the helicity

$$h = \frac{1}{2}\frac{\vec{p} \cdot \vec{s}}{|\vec{p}|} = \frac{1}{2}\hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{13.10}$$

It is easy to verify that the spinors u_1 und v_1 are eigenfunctions with helicity $\frac{1}{2}$. Similarly, u_2 und v_2 are eigenfunctions of the helicity operator with eigenvalues $-\frac{1}{2}$. In a given Lorentz frame, helicity in a reaction is conserved. However, the helicity of a massive particle depends on the frame, because, by moving very fast, we can reverse the momentum of a particle, leaving its spin unchanged.

A new mass term of the form

$$\frac{1}{2} M_{\rm m} \bar{N}_{\rm R}^{\rm c} N_{\rm R} + \text{h.c.} \tag{13.11}$$

is allowed to be present, since it is Lorentz- and $SU(2)_L$-invariant. This is known as a Majorana mass term and is unique to neutrinos, which are neutral particles. A Majorana mass term conserves electric charge but changes the lepton number by two units. It may be introduced to the Langrangian as an additional term or as a new interaction term, coupled to a new scalar particle which is an $SU(2)_L$ singlet. A Majorana mass is generated by assigning to the new scalar particle a vacuum expectation value. Consequently, we generate Dirac mass terms through Eq. (13.3) and a Majorana term through Eq. (13.11). On collecting these terms together, one obtains the neutrino mass matrix:

$$\begin{pmatrix} \bar{\nu}_{\rm L} & \bar{N}_{\rm R}^{\rm c} \end{pmatrix} \begin{pmatrix} 0 & m_{\rm D} \\ m_{\rm D}^{\rm T} & M_{\rm m} \end{pmatrix} \begin{pmatrix} \nu_{\rm L} \\ N_{\rm R} \end{pmatrix}. \tag{13.12}$$

Once we have introduced mass terms that do not preserve the original symmetry, we must solve the problem again using the Lagrangian. This means that, after introducing Dirac and/or Majorana mass terms, we should solve the problem using the rules of the new Lagrangian (see Problem 13.3). For our specific case, we can no longer use the eigenfunctions of Eq. (13.6). We must diagonalize the mass matrix and use the new mass eigenstates (physical states) which will have components that couple the various flavors with each other and, in addition, couple particles to antiparticles (when Majorana mass terms are present).

For $M_m \gg m_D$ the mass matrix of the form given in Eq. (13.12) has eigenvalues with specific properties: one eigenvalue is large and the other is suppressed. This method of introducing masses for neutrinos is known as the see-saw mechanism, to which we shall return in the fourth section. In the meantime we shall discuss neutrino oscillations in free space and in matter, which are active fields of research nowadays.

13.2 Neutrino oscillations

The couplings of Dirac neutrinos to charged and neutral currents conserve the lepton number, as has been tested in many experiments. Lepton flavor can be violated in the mass matrix, with the appearance of off-diagonal elements. In this case the eigenfunctions of the Hamiltonian are superpositions of neutrinos with various flavor numbers. We shall call them the mass eigenstates and they have the time development given below in Eq. (13.14). In a physical reaction, however, the neutrinos that are produced have definite flavor number. Their time development requires special attention because we must rewrite the flavor states in terms of mass eigenstates, whose time development is that given in Eq. (13.14). This mis-match between the production of flavor states and the time development of mass states leads to an oscillation of lepton quantum numbers that will be described below.

We demonstrate the mixing phenomenon for two generations of neutrinos, for which the algebra is simpler. We consider ν_e and ν_μ neutrinos with the mass matrix

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \mathcal{H}_{\text{mass}}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}, \quad \text{with} \quad \mathcal{H}_{\text{mass}} = \begin{pmatrix}m_{ee} & m_{e\mu}\\ m_{\mu e} & m_{\mu\mu}\end{pmatrix}. \tag{13.13}$$

The mass matrix mixes electron and muon neutrinos in a manner analogous to atomic physics, where energy levels are mixed through the interactions with external fields, such as magnetic fields. For the sake of simplicity, we assume that the matrix elements are real and in addition that $m_{\mu e} = m_{e\mu}$. A symmetric matrix is diagonalized by an orthogonal matrix to be denoted by O and has the eigenvalues m_1 and m_2. A mass eigenstate with momentum p has the time development

$$|\nu_i(t)\rangle = |\nu_i(0)\rangle \mathrm{e}^{-\mathrm{i}E_i t}, \tag{13.14}$$

with $i = 1$ or 2 and $E_i = \sqrt{p^2 + m_i^2}$. To avoid confusion, we shall use two types of subscripts, with numbers denoting physical eigenstates and letters denoting flavor

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states. Let us parametrize the orthogonal matrix as follows:

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

then the mass eigenstates are given by

$$\begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix}. \tag{13.15}$$

We can easily invert this equation to obtain

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix}, \tag{13.16}$$

which, with the help of Eq. (13.14), can be rewritten as

$$\begin{aligned} |\nu_e(t)\rangle &= e^{-iE_1 t}|\nu_1(0)\rangle\cos\theta + e^{-iE_2 t}|\nu_2(0)\rangle\sin\theta, \\ |\nu_\mu(t)\rangle &= -e^{-iE_1 t}|\nu_1(0)\rangle\sin\theta + e^{-iE_2 t}|\nu_2(0)\rangle\cos\theta. \end{aligned} \tag{13.17}$$

The general structure of these equations is summarized by

$$|\nu_\alpha(t)\rangle = \sum_{k=1,2} U_{\alpha k} e^{-iE_k t}|\nu_k(0)\rangle,$$

with the unitary matrix $U_{\alpha k}$ defining the mixing between flavor and mass states. The mixing matrix was introduced a long time ago and is referred to as the MNS matrix (Maki *et al.*, 1962).

On replacing $\nu_1(0)$ and $\nu_2(0)$ by the flavor eigenstates given in Eq. (13.15), we obtain the final result

$$\begin{aligned} |\nu_e(t)\rangle &= (\cos^2\theta\, e^{-iE_1 t} + \sin^2\theta\, e^{-iE_2 t})|\nu_e(0)\rangle \\ &\quad + \sin\theta\cos\theta(e^{-iE_2 t} - e^{-iE_1 t})|\nu_\mu(0)\rangle, \\ |\nu_\mu(t)\rangle &= \sin\theta\cos\theta(e^{-iE_2 t} - e^{-iE_1 t})|\nu_e(0)\rangle \\ &\quad + (\cos^2\theta\, e^{-iE_1 t} + \sin^2\theta\, e^{-iE_2 t})|\nu_\mu(0)\rangle. \end{aligned} \tag{13.18}$$

This shows explicitly that the flavor content of the wave function changes with time. For example, at $t = 0$ the first equation contains only an electron neutrino. In the course of time a ν_μ component develops. The second equation describes a state which starts as $\nu_\mu(0)$.

We consider a state that starts as $\nu_e(0)$ and compute the probability of finding a $\nu_e(t)$. The masses of the neutrinos are small relative to their momenta and one can

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use the approximation $E_i = \sqrt{p^2 + m_i^2} \approx E[1 + m_i^2/(2E^2)]$ to arrive at

$$P_{\mathrm{ee}}(t) = |\langle \nu_{\mathrm{e}} | \nu_{\mathrm{e}}(t) \rangle|^2 = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E} t\right), \tag{13.19}$$

with $\Delta m^2 = m_2^2 - m_1^2$. Similarly, we compute the probability of finding a ν_μ at time t,

$$P_{\mu\mathrm{e}}(t) = |\langle \nu_\mu | \nu_e(t) \rangle|^2 = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E} t\right). \tag{13.20}$$

The sum of the two probabilities is equal to unity. A mono-energetic neutrino beam thus oscillates with amplitude $\sin^2(2\theta)$ and wave-number $\Delta m^2/(4E)$. For oscillations to occur, we need a non-zero θ and at least one non-zero mass. The amplitude is maximal for $\theta = \pi/4$.

In vacuum-oscillation experiments there is a redundancy in the values of the mixing angle. The same numerical value of $\sin^2(2\theta)$ appears eight times when θ varies between zero and 2π. This means that in oscillation experiments it suffices to consider angles in the interval between zero and $\pi/4$. In other situations for which the mixing depends on $\cos(2\theta)$ the range for the mixing angle must be extended from 0 to $\pi/2$, as we shall discuss at the end of the next section.

We can express the various quantities in useful units and substitute $t \simeq L$, where the speed of light c is set equal to unity,

$$\frac{\Delta m^2\, L}{4E} = 1.27\, \frac{L}{\mathrm{km}}\, \frac{\Delta m^2}{\mathrm{eV}^2}\, \frac{\mathrm{GeV}}{E}. \tag{13.21}$$

This entity is a dimensionless quantity and we are free to use the units. The units in Eq. (13.21) are now standard and convenient for terrestrial experiments. This formula defines the sensitivity of an experiment, since oscillations occur when $\Delta m^2 L/E$ is of order unity. As a function of the baseline length, the maximum of the oscillation occurs at $L \simeq 2E/\Delta m^2$.

The generalization to three or more families is straightforward and leads to the transition probability

$$|\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \delta_{\alpha\beta} - 2\,\mathrm{Re}\left\{ \sum_{j>i} U_{\alpha i} U_{\alpha j}^* U_{\beta i}^* U_{\beta j} \left[1 - \exp\left(-\mathrm{i} \frac{\Delta m_{ij}^2}{2E} L \right) \right] \right\}. \tag{13.22}$$

For two neutrino families, $U_{\alpha i}$ is a simple 2×2 orthogonal matrix. Upon substitution the probabilities simplify to Eqs. (13.19) and (13.20). For three families a hierarchy of the form $\Delta m_{12}^2 \gg \Delta m_{13}^2 \simeq \Delta m_{23}^2$ and the smallness of one mixing-matrix element (U_{e3}) describes all existing experiments.

Finally, we present a simple description of what happens in oscillations. A $|\nu_\alpha\rangle$ is composed of various waves with different m_i. At a given energy the heavier mass states oscillate faster and the various $|\nu_i\rangle$ components come out of phase, so at a certain distance they do not sum up to a $|\nu_\alpha\rangle$. Since the oscillatory term comes from an interference between the different m_i, a common phase factor of the $|\nu_i\rangle$s plays no role and can be ignored. This will be useful in the next section, when we turn to oscillation in matter.

13.2.1 Oscillation in matter

The neutral-current interaction of neutrinos with matter is extremely weak, but can nonetheless affect oscillations (Wolfenstein, 1978). The reason is that the momentum transfer to the target can have a large wavelength, such that the neutrino interacts coherently with all the particles within its wavelength. At the same time the difference $E_1 - E_2 \simeq \Delta m_{12}^2/(2E)$ may correspond to a wavelength of the same magnitude, so the time development of the mass eigenstates is influenced by the interactions with the medium. In the medium the flavor eigenstates ν_e and ν_μ interact with electrons and protons with different cross sections and modify the development of the mass eigenstates.

The Hamiltonian on the mass basis is given by

$$\mathrm{i}\,\frac{\partial \Psi_i}{\partial t} = \mathcal{H}_0 \Psi_i = E_i \Psi_i \simeq p\left(1 + \frac{m_i^2}{2p^2}\right)\Psi_i \rightarrow \frac{m_i^2}{2E}\Psi_i, \tag{13.23}$$

where after the arrow we omitted the term proportional to the unit matrix. This term does not influence the oscillations because it can be eliminated by the transformation $\nu_i \rightarrow \nu_i \mathrm{e}^{-\mathrm{i}pt}$ common for all neutrinos. For this reason we can simplify the Hamiltonian,

$$\mathcal{H}_0 \Psi = \mathcal{H}_0 \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{13.24}$$

The interactions with matter, however, involve flavor states. For this reason we transform the equations to the flavor basis where we include the interactions with matter. The solution of the new eigenvalue problem describes the propagating eigenfunctions. In the flavor basis

$$\mathcal{H}_{\text{flavor}} \equiv \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger = \frac{1}{4E} \begin{pmatrix} \Sigma - \Delta m^2 \cos(2\theta) & \Delta m^2 \sin(2\theta) \\ \Delta m^2 \sin(2\theta) & \Sigma + \Delta m^2 \cos(2\theta) \end{pmatrix}, \tag{13.25}$$

where $\Sigma = m_1^2 + m_2^2$, $\Delta m^2 = m_1^2 - m_2^2$, and U is the mixing matrix in Eq. (13.16).

In a medium there is an additional interaction Hamiltonian created by the neutral and charged currents. Recall the interaction term from Chapter 8,

$$\mathcal{H}_{\rm int} = \frac{G_{\rm F}}{\sqrt{2}}\,\bar{\psi}_x\gamma_\mu\big(g_{\rm V}^x - g_{\rm A}^x\gamma_5\big)\psi_x\bar{\psi}_{\nu_{\rm e}}\gamma^\mu(1-\gamma_5)\psi_{\nu_{\rm e}}. \tag{13.26}$$

Here x denotes weakly interacting particles in the medium, which in the Sun are electrons, neutrons, and protons. For left-handed neutrinos $(1-\gamma_5)\psi_\nu = 2\psi_\nu$. The term $\langle\bar{\psi}_x\gamma_\mu\gamma_5\psi_x\rangle$ reduces in the non-relativistic limit to the expectation value for the spin operator. In an unpolarized medium, the states ψ_x occupy all possible spin states and this term averages to zero. For the remaining term $\langle\bar{\psi}_x\gamma_\mu\psi_x\rangle$ the space components are proportional to the momentum of the particles, which is small. Consequently only the term $\langle\bar{\psi}_x\gamma_0\psi_x\rangle = n_x$ survives and is equal to the density of the particles n_x. This term produces the potential $g_{\rm V}^x\sqrt{2}G_{\rm F}n_x\psi_{\nu_{\rm e}}^\dagger\psi_{\nu_{\rm e}}$ which is added to the Hamiltonian. Finally, we obtain the Schrödinger equation from the Euler–Lagrange equation for $\psi_{\nu_{\rm e}}^\dagger$.

We already know the couplings of neutrinos to electrons from Section 8.3, where they are summarized in Table 8.1. The couplings to nucleons are obtained in a similar way:

$$\begin{aligned} g_{\rm V}^{\rm e} &= \tfrac{1}{2} + 2\sin^2\theta_{\rm W} && \text{for } \nu_{\rm e},\\ g_{\rm V}^{\rm e} &= -\tfrac{1}{2} + 2\sin^2\theta_{\rm W} && \text{for } \nu_\mu,\\ g_{\rm V}^{\rm p} &= \tfrac{1}{2} - 2\sin^2\theta_{\rm W} && \text{for } \nu_\alpha,\\ g_{\rm V}^{\rm n} &= -\tfrac{1}{2} && \text{for } \nu_\alpha, \end{aligned} \tag{13.27}$$

with α being either the electron neutrino or the muon neutrino. The difference of 1 in the $g_{\rm V}^{\rm e}$ contribution between $\nu_{\rm e}$ and ν_μ comes from the Fierz-transformed charged-current term in the Hamiltonian. In an electrically neutral medium $n_{\rm e} = n_{\rm p}$, therefore

$$V = \sqrt{2}G_{\rm F}\times\begin{cases} -\frac{1}{2}n_{\rm n} + n_{\rm e} & \text{for } \nu_{\rm e},\\ -\frac{1}{2}n_{\rm n} & \text{for } \nu_\mu. \end{cases} \tag{13.28}$$

The new contribution from the scattering with matter must be added to the ee and $\mu\mu$ elements of $\mathcal{H}_{\rm flavor}$, yielding a propagation equation

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\nu_{\rm e}\\ \nu_\mu\end{pmatrix} = \frac{1}{2E}\begin{pmatrix} m_{\rm ee}^2 + 2\sqrt{2}EG_{\rm F}n_{\rm e} & m_{\rm e\mu}^2\\ m_{\rm e\mu}^2 & m_{\mu\mu}^2\end{pmatrix}\begin{pmatrix}\nu_{\rm e}\\ \nu_\mu\end{pmatrix}, \tag{13.29}$$

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with matrix elements

$$m^2_{ee} = -\tfrac{1}{2}\,\Delta m^2 \cos(2\theta),$$
$$m^2_{\mu\mu} = \tfrac{1}{2}\,\Delta m^2 \cos(2\theta), \qquad (13.30)$$
$$m^2_{e\mu} = \tfrac{1}{2}\,\Delta m^2 \sin(2\theta).$$

One sees that effects of matter become more important at higher energy. Note that we ignored terms proportional to the unit matrix. Apart from the aforementioned fact that they can be transformed away, there is another justification: after diagonalization the mixing angle and the difference between the eigenvalues are independent of equal diagonal terms (see Problem 13.1). Exactly such terms appear in the formula (13.30) for the oscillation probability and have been omitted. Diagonalization of the matrix gives the mass difference

$$m^2_{2M} - m^2_{1M} = \Delta m^2_M = \Delta m^2 \sqrt{[A - \cos(2\theta)]^2 + \sin^2(2\theta)}, \qquad (13.31)$$

where

$$A \equiv \frac{2\sqrt{2} E G_F n_e}{\Delta m^2},$$

and the mixing angle (Wolfenstein, 1978; Mikheyev and Smirnov, 1985)

$$\tan(2\theta_M) = \frac{\sin(2\theta)}{\cos(2\theta) - A} \Rightarrow \sin^2(2\theta_M) = \frac{\sin^2(2\theta)}{[\cos(2\theta) - A]^2 + \sin^2(2\theta)}. \qquad (13.32)$$

In the limit $n_e \to 0$, we recover the formulas for oscillation in vacuum. The careful reader will notice that there is a resonance effect when $A = \cos(2\theta)$ (Mikheyev and Smirnov, 1985). For this value of A the mixing in the medium is maximal, even though the mixing in the vacuum can be very small.

The angle θ_M expresses the matter eigenstates in terms of the flavor states

$$\begin{pmatrix} \nu_{1M} \\ \nu_{2M} \end{pmatrix} = \begin{pmatrix} \cos\theta_M & -\sin\theta_M \\ \sin\theta_M & \cos\theta_M \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \qquad (13.33)$$

We discuss two cases realized in the Sun. We know that electron-type neutrinos are created in the interior of the Sun, where n_e is very large. In that region

$$\tan(2\theta_M) \simeq -\frac{2\theta}{A} \quad \text{and} \quad \theta_M \simeq \frac{\pi}{2}, \qquad (13.34)$$

since $\tan(2\theta_M)$ approaches zero from negative values. Equation (13.33) now gives $\nu_{1M} \simeq -\nu_\mu$ and $\nu_{2M} \simeq \nu_e$, i.e. in the interior of the Sun the state with the heavier effective mass is the electron neutrino. As the beam transverses the Sun, n_e decreases and $\tan(2\theta_M) \simeq \tan(2\theta)$, therefore $\theta_M \simeq \theta$ and $\nu_{2M} \simeq \nu_\mu$. The flavor content of the beam changes. This phenomenon is called the MSW effect. It will be interesting to measure in terrestrial experiments the lepton-number content of the neutrinos arriving from the Sun.

Next we discuss terrestrial experiments with a ν_μ beam going through the Earth. This situation can be realized in long-baseline accelerator experiments or via atmospheric neutrinos. We assume that the ν_μ oscillates into another flavor state, as reported from the Superkamiokande experiment. Two cases are of interest.

Case 1

The ν_μ mixes with the ν_τ. The interaction with matter proceeds with the exchange of a Z^0 boson and is identical for μ and τ neutrinos. It will thus add the same term to the diagonal elements of $\mathcal{H}_{\text{flavor}}$, which does not affect the oscillation. This means that no effects of matter appear.

Case 2

The ν_μ mixes with a sterile neutrino ν_s. The interaction with matter in the transition $\nu_\mu \to \nu_\mu$ proceeds through the exchange of a Z^0 boson, but there is no such process for the ν_s. We must add an interaction with matter only in the $\mu\mu$ element and the propagation Hamiltonian has the form

$$\mathcal{H}_{\text{flavor}} = \frac{1}{2E}\begin{pmatrix} m^2_{\mu\mu} - \sqrt{2}EG_{\text{F}}n_{\text{n}} & m^2_{\mu\text{s}} \\ m^2_{\mu\text{s}} & m^2_{\text{ss}} \end{pmatrix}. \tag{13.35}$$

The notation is analogous to that in Eq. (13.30) with mass elements $m^2_{\mu\mu}$, $m^2_{\mu\text{s}}$, and m^2_{ss} similar to those in Eq. (13.30). The angle θ now governs the oscillation between ν_μ and ν_s. We note that the sign of the matter term is reversed and the neutron density n_{n} replaces the density of electrons.

Finally, Eqs. (13.31) and (13.32) depend on $\cos(2\theta)$ and the range $0 \le \theta \le \pi/4$ does not cover negative values of $\cos(2\theta)$. The range must now be extended to $0 \le \theta \le \pi/2$.

13.3 Experimental results

There has been a long search for neutrino oscillations. The experiments were carried out with neutrino beams from nuclear reactors, the Sun, and accelerators, and recently with atmospheric neutrinos. In recent years several experiments have provided evidence for neutrino oscillation. We will not discuss the experiments in detail but only summarize the main results. For an overview of the current status, see Mohapatra *et al.* (2005).

- *Atmospheric neutrinos* Compelling evidence for oscillation comes from experiments with atmospheric neutrinos. They are produced in the atmosphere by meson decays, which in turn are produced by the interaction of cosmic rays with nuclei of the atmosphere. Both electron and muon neutrinos are produced in pion and muon decays and the ratio at production is roughly two muon neutrinos to one electron neutrino. Neutrino interactions are detected in a huge underground target/detector – the Superkamiokande. The experiment established that there is a decrease in the number of muon-type neutrinos and no decrease in number of electron neutrinos. This implies that the ν_μ neutrinos oscillate into another type, such as τ or sterile (see below) neutrinos. A zenith-angle dependence of the disappearance is observed, whereby more of the neutrinos coming from underneath the detector disappear. They are produced in the atmosphere and travel through the Earth in order to reach the detector. These observations require a mass difference of $\Delta m^2 \simeq 2 \times 10^{-3}$ eV^2 and the mixing angle to be maximal.
- *Solar neutrinos* Nuclear reactions in the Sun produce a tremendous amount of heat, which is radiated and at the same time produces electron neutrinos. The cycles which produce them have been studied and the neutrino spectra have been calculated. For three decades experiments measuring the flux of neutrinos from the Sun have observed deficits, indicating that ν_e neutrinos, during their journey from Sun to Earth, oscillate into other types of neutrinos. Previous experiments trying to verify this result with reactor neutrinos failed to find a deficit, indicating that the distance of 1 km from the reactors is too small. The explanation of the solar deficit requires that it is either oscillation in vacuum with a large mixing angle and $\Delta m^2 \simeq 10^{-10}$ eV^2 or oscillations inside the Sun where effects of matter according to the MSW effect are important. The latter possibility gives two solutions with either large mixing (LAMSW) or small mixing (SAMSW) and with $\Delta m^2 \simeq 8 \times 10^{-5}$ eV^2.
- *Reactor experiments* Reactor experiments search for a decrease of the antineutrino flux far away from the reactor. The CHOOZ collaboration detected $\bar{\nu}_e$ at a distance of 1 km away but saw no deficit. These data restrict the mixing angle θ_{13}, as we will discuss below. The experiment KamLAND measured the flux of $\bar{\nu}_e$ neutrinos from distant nuclear reactors. The experiment uses a target containing 1000 tons of liquid scintillator viewed by more than 1800 light-detecting photomultiplier tubes. It detects electron antineutrinos emitted by $\sim$70 nuclear reactors in Japan and South Korea arriving from an average distance of 180 km. The ratio of the observed antineutrino interactions to the expected number without disappearance of $\bar{\nu}_e$ is 0.611 ± 0.085 (statistical uncertainty) $\pm\, 0.041$ (systematic uncertainty) for $\bar{\nu}_e$ energies $>$3.4 MeV. In the two-flavor analysis only the "large-mixing-angle" solution is allowed. The best fit of their data gives $\Delta m^2_{12} = 6.9 \times 10^{-5}$ eV^2 and $\sin^2(2\theta) = 1.0$, which selects the large-mixing-angle solution.
- *The Sudbury experiment* The Sudbury Neutrino Observatory (SNO) detects ^{8}B solar neutrinos through the reactions

$$\nu_e + d \rightarrow p + p + e^-,$$
$$\nu_x + d \rightarrow p + n + \nu_x,$$
$$\nu_x + e^- \rightarrow \nu_x + e^-,$$

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where the subscript x denotes any type of flavor. Only electron neutrinos produce charged-current interactions, while the neutral-current (NC) interactions and elastic scattering are sensitive to all types of flavors. The NC reaction measures the total flux of all active neutrino flavors produced in the Sun with energy above the threshold of 2.2 MeV. This provides a measurement of the solar neutrino flux as

$$5.21 \pm 0.27\,(\text{statistical uncertainty}) \pm 0.38\,(\text{systematic uncertainty}) \times 10^6\,\text{cm}^{-2}\,\text{s}^{-1},$$

which is in agreement with the standard solar models. A global analysis of these and other solar- and reactor-neutrino results yields $\Delta m^2 = 7.1^{+1.2}_{-0.6} \times 10^{-5}$ eV^2 and $\theta = 32.5^{+2.4}_{-2.3}$ degrees.

- *Accelerator neutrinos* The LSND experiment has found an electron-antineutrino excess in a muon-antineutrino beam. The solution would be small mixing and a mass-squared difference of 0.1–1 eV^2. However, the very similar KARMEN experiment has found no such effect, but it cannot rule out LSND's complete parameter space. The LSND results are referred to as the LSND anomaly because the large mass difference cannot be reconciled with the three-family model. The MiniBoone experiment at Fermilab is currently running with the aim of checking the LSND result.
- *Long-base-line experiments* There are several experiments that will use two detectors, one close to the accelerator, where the neutrino beam is produced, and a second detector at a distance of 200–400 km. The nearby detector will be used for calibration reasons to determine the beam and properties of the neutrino reactions. The faraway detector will be looking at the changes that take place because muon neutrinos oscillate to neutrinos of another flavor. A small-scale experiment, K2K, operated in Japan and two others are under construction, MINOS in the USA and OPERA in Europe.

For three flavors of neutrinos there are only two independent Δm^2. If the result from the LSND experiment is correct, then there must be a fourth flavor, which has to be "sterile," i.e. it does not couple to the gauge bosons and therefore does not contribute to the Z^0 width. The results of the other experiments are summarized as follows:

$$(\Delta m^2\ (\text{eV}^2),\ \sin^2\theta) \simeq \begin{cases} (2 \times 10^{-3}, \simeq 0.5), & \text{atmospheric,} \\ (7 \times 10^{-5}, \simeq 0.3)\ \text{LAMSW}, & \text{solar, KamLAND.} \end{cases} \quad (13.36)$$

We can now estimate the magnitude of the leptonic MNS matrix. Let us ignore the LSND result and assume that there are two mass-squared differences governing the atmospheric and solar oscillation. They obey $\Delta m^2_\odot \ll \Delta m^2_\text{A}$. If we identify $\Delta m^2_{21} = \Delta m^2_\odot \ll \Delta m^2_\text{A} = \Delta m^2_{31} \simeq \Delta m^2_{32}$, then we have, for a short-baseline reactor experiment such as CHOOZ (see Problem 2),

$$P^\text{CHOOZ}_\text{ee} = 1 - 4|U_\text{e3}|^2(1 - |U_\text{e3}|^2)\sin^2\Delta_{31}, \quad (13.37)$$

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where $\Delta_{ij} = \Delta m^2_{ij}\, L/(4E)$ and L is the distance for the CHOOZ experiment. The absence of disappearance of neutrinos in the experiment means that $|U_{e3}|$ is either small or close to unity. The probability for solar neutrinos is (Bilenky, 2003)

$$P^{\odot}_{ee} = (1 - |U_{e3}|^2)^2 \left(1 - 4\frac{|U_{e1}|^2 |U_{e2}|^2}{(1 - |U_{e3}|^2)^2}\sin^2\Delta_{21}\right) + |U_{e3}|^4. \tag{13.38}$$

Experimentally, $P^{\odot}_{ee}$ is significantly less than unity and also energy-dependent. On combining the results of the CHOOZ experiment with the solar deficit, we conclude that $|U_{e3}|^2 \ll 1$. Finally, atmospheric neutrinos oscillate with

$$P^{A}_{\mu\tau} = 4|U_{\mu 3}|^2 |U_{\tau 3}|^2 \sin^2\Delta_{31}. \tag{13.39}$$

Note that oscillations of reactor and atmospheric neutrinos are triggered by the same Δm^2. If for the latter we use $\Delta m^2 = 3 \times 10^{-3}\ \text{eV}^2$, CHOOZ gives $|U_{e3}|^2 \lesssim 0.05$. As a first approximation we can assume $|U_{e3}| \simeq 0$, maximal atmospheric mixing, and the LAMSW solution for solar mixing. With these results, we can approximate the MNS matrix by

$$U_{\alpha i} \simeq \begin{pmatrix} c & s & 0 \\ -s/2 & c/2 & 1/\sqrt{2} \\ s/2 & -c/2 & 1/\sqrt{2} \end{pmatrix} \text{LAMSW}, \tag{13.40}$$

where $c = \cos\theta$, $s = \sin\theta$, and $s^2 \simeq 0.3$. Note that there can be a phase in the matrix, resulting in CP violation as in the quark sector. However, for $U_{e3} = 0$ the theory is effectively a two-flavor theory. Only for a non-vanishing U_{e3} element could one establish CP violation in long-baseline experiments.

If neutrinos are Majorana particles (see below), then there are two additional phases in the mixing matrix. They can be shown to have no influence on oscillation physics and reveal their presence only in neutrinoless double beta decay, which we will discuss below.

13.4 Majorana neutrinos

The results of the last section give strong evidence in favor of massive neutrinos. The oscillations indicate that in the leptonic sector flavor number is not conserved, i.e. muon neutrinos can become tau neutrinos, etc. This is a change from one family to the next. So far there has been no discussion of a particle changing into its antiparticle. The mixing among the families is produced when we introduce a mass term of the form

$$\mathcal{L}^{D} = \sum_{i,j} m^{ij}_{D}\,\overline{\psi}_i \psi_j + \text{ h.c.}, \tag{13.41}$$

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with i and j running over the families. For the electroweak theory, special attention is required because the states are classified according to helicities – with left-handed particles in doublets and right-handed particles in singlets of weak SU(2). The mass term is now

$$\mathcal{L}^{\mathrm{D}} = \sum_{i,j} m_{\mathrm{D}}^{ij} \bar{\psi}_{\mathrm{L}i} \psi_{\mathrm{R}j} + \text{ h.c.} \tag{13.42}$$

It is known as the Dirac mass term and is produced by the Higgs mechanism, as described at the beginning of this chapter.

There is also the possibility of introducing new mass terms,

$$\frac{1}{2} M_{\mathrm{R}} \bar{N}_{\mathrm{R}}^{\mathrm{c}} N_{\mathrm{R}} \quad \text{and} \quad \frac{1}{2} M_{\mathrm{L}} \bar{\nu}_{\mathrm{L}}^{\mathrm{c}} \nu_{\mathrm{L}} \tag{13.43}$$

and their Hermitian conjugates, which are called Majorana mass terms. Obviously, Majorana mass terms can include mixing among the generations by introducing indices i and j, as in Eq. (13.42). They also mix neutrinos with antineutrinos. The new terms are Lorentz-invariant, and they carry lepton number two, thus introducing violation of the lepton number. Again, special attention must be paid to the peculiar property of the electroweak theory which classifies the states according to helicities. This introduces special requirements on the terms which are allowed. For instance, the Lorentz structure gives the identities $\bar{N}_{\mathrm{R}} N_{\mathrm{R}} = \bar{\nu}_{\mathrm{L}} \nu_{\mathrm{L}} = 0$ and $\bar{N}_{\mathrm{R}} \nu_{\mathrm{L}}^{\mathrm{c}} = \bar{\nu}_{\mathrm{L}} N_{\mathrm{R}}^{\mathrm{c}} = 0$. A Majorana state is defined as one with equal components of particles and antiparticles. It follows from the above identities that

$$\left(\bar{N}_{\mathrm{R}}^{\mathrm{c}} + \bar{N}_{\mathrm{R}}\right)\left(N_{\mathrm{R}}^{\mathrm{c}} + N_{\mathrm{R}}\right) = \bar{N}_{\mathrm{R}}^{\mathrm{c}} N_{\mathrm{R}} + \bar{N}_{\mathrm{R}} N_{\mathrm{R}}^{\mathrm{c}},$$

which indicates that the mass terms introduced do indeed correspond to Majorana particles.

Similarly, specific terms are permitted by the SU(2) symmetry. We can introduce the term $M_{\mathrm{R}} \bar{N}_{\mathrm{R}}^{\mathrm{c}} N_{\mathrm{R}}$, since it is Lorentz-invariant and SU(2) singlet. We cannot introduce the term $M_{\mathrm{L}} \bar{\nu}_{\mathrm{L}}^{\mathrm{c}} \nu_{\mathrm{L}}$, because it is the direct product of a doublet with a doublet, which decomposes into an SU(2) singlet plus a triplet. Their product with the Higgs doublet cannot produce a singlet. Similarly, their product with a singlet produces a singlet and a triplet. SU(2) symmetry dictates that $M_{\mathrm{L}} = 0$, unless we introduce triplet representations of Higgses. The possible mass terms are a Dirac mass term as in Eq. (13.42) and, in addition, a Majorana term appearing as the first term in Eq. (13.43). We sum up these terms in a matrix notation:

$$\mathcal{L}_{\text{mass}} = \left(\bar{\nu}_{\mathrm{L}} \bar{N}_{\mathrm{R}}^{\mathrm{c}}\right) \begin{pmatrix} 0 & m_{\mathrm{D}} \\ m_{\mathrm{D}} & M_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} \nu_{\mathrm{L}}^{\mathrm{c}} \\ N_{\mathrm{R}} \end{pmatrix} + \text{h.c.} \tag{13.44}$$

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We can diagonalize this mass matrix with an orthogonal transformation. The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(M_R \pm \sqrt{M_R^2 + 4m_D^2}\right), \tag{13.45}$$

which for $m_D/M_R \ll 1$ become

$$\lambda_1 = -\frac{m_D^2}{M_R} \quad \text{and} \quad \lambda_2 = M_R + \frac{m_D^2}{M_R}. \tag{13.46}$$

The case we studied is called the seesaw mechanism (Gell-Mann *et al.*, 1979; Yanagida, 1979; Minkowski, 1977) because when one mass is large the other is small. Now, since the neutrino masses are very small relative to lepton and quark masses, for instance $m_\nu \approx 10^{-2}$ eV, the seesaw mechanism supplies an explanation provided that the Majorana mass, M_R, is large.

The corresponding wave functions, omitting a normalization constant, are

$$\begin{aligned} \psi_1 &= \nu_L + \frac{m}{M_R} N_R^c, \\ \psi_2 &= -\frac{m}{M} \nu_L^c + N_R, \end{aligned} \tag{13.47}$$

indicating that ψ_1 consists primarily of the normal neutrino with a small admixture of N_R^c. The other state ψ_2 is the heavy state with a large component N_R and a small admixture of ν_L^c.

Identifying ν_L or ψ_1 with the standard-model neutrinos explains their lightness by introducing a heavy scale. It follows a hierarchical mass scheme for the neutrinos. Neutrino masses $m_\nu \simeq 10^{-2}$ eV require $m_R \simeq 10^{16}$ GeV for a Dirac scale of 1 GeV. This is a typical scale of grand unified theories, which is one of the reasons why seesaw models are very popular.

13.5 Neutrinoless double beta decay

Much effort has been devoted to discovering the nature of the neutrino. The experimental results on oscillations are independent of the Majorana character of the neutrinos. They observe oscillation of flavors. Experiments also search for evidence of Majorana neutrinos. A reaction that is conceptually simple concerns the conversion

$$\nu_e + N \to e^+ + \text{hadrons},$$

which has, unfortunately, a very small cross section. Theoretical estimates indicate that the rate for a neutrino with an energy of 1 GeV is 10^{-18}–10^{-22} times smaller

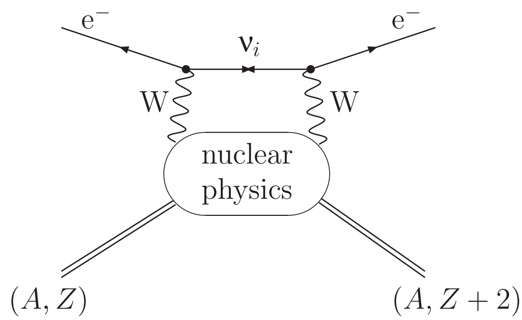

Figure 13.1. Neutrinoless double beta decay.

than the normal charged-current cross section. Experiments with neutrino beams are far away from these limits.

A favorable reaction is

$$2\mathrm{n} \to 2\mathrm{p} + (\mathrm{e}^-\mathrm{e}^-),$$

which is known as neutrinoless double beta decay and is indicated as $0\nu\beta\beta$. The process is shown in Fig. 13.1 where a nucleus with Z protons and $A - Z$ neutrons emits two W^- bosons and decays into a nucleus with $Z + 2$ protons and $A - Z - 2$ neutrons. The two W^- bosons interact with each other and convert into two electrons. The lepton propagator is a Majorana neutrino containing both particles and antiparticles. The amplitude for the process contains the leptonic tensor

$$L_{\alpha\beta}(0\nu\beta\beta) \propto U_{\mathrm{e}i}^2 \bar{e}\gamma_\alpha\gamma_- \frac{\not{p} + m_i}{p^2 - m_i^2}\gamma_\beta\gamma_+ e = U_{\mathrm{e}i}^2 m_i \frac{1}{p^2}\bar{e}\gamma_\alpha\gamma_-\gamma_\beta e, \quad (13.48)$$

where p is the momentum of the propagating neutrino, m_i its mass, and e are the electron fields. The element U_{ei} gives the coupling of the electron to the ith neutrino mass state. There is also a corresponding hadronic tensor $W^{\alpha\beta}$, whose structure is rather complicated. Its structure is unfortunately left out in most theoretical articles and discussions in spite of the fact that it introduces considerable uncertainty in the predictions. The decay rate of the process is obtained by first summing the amplitudes over all intermediate states and then squaring the total amplitude to arrive at

$$\Gamma(0\nu\beta\beta) = A\left|\sum_i U_{\mathrm{e}i}^2 m_i\right|^2, \quad (13.49)$$

where A is a non-trivial factor representing nuclear matrix elements. The quantity

$$m_{\mathrm{ee}} = \sum_i U_{\mathrm{e}i}^2 m_i, \quad (13.50)$$

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with the sum running over all neutrinos, is called the effective electron-neutrino Majorana mass, or in short the effective neutrino mass. The neutrino masses in the propagator are physical masses and are very small relative to q^2 involved in nuclear decays. The masses m_i and the mixing elements U_{ei} also appear in the oscillations of neutrinos. Unfortunately, the oscillation experiments measure Δm_{ij}^2, i.e. mass differences, rather than absolute values for the masses. They also measure the squares of the U_{ei}, which in general are complex functions. Thus, extracting mass differences and mixing angles from oscillation experiments, there is still a large range of values possible for m_{ee} (Pascoli *et al.*, 2005; Choubey and Rodejohann, 2005). Processes depending on the effective Majorana mass have branching ratios and cross sections that can be very small. The current limit for m_{ee} comes from neutrinoless double beta decay of ^{76}Ge and is approximately 0.2 eV. There is only one experiment (Klapdor-Kleingrothaus *et al.*, 2004) in which it has been claimed that the Majorana mass has been measured. The best value given by the group is $\langle m_{\text{ee}} \rangle = 0.39$ eV, but the data analysis of this experiment has been criticized (Aalseth *etal.*, 2004). There are plans to build improved experiments. This is a difficult but very interesting and exciting field, regarding which the reader can consult other articles (Aalseth *et al.*, 2004).

Problems for Chapter 13

1. Show that the mixing angle θ in the matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which diagonalizes the symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

is given by

$$\tan(2\theta) = \frac{2b}{d-a}$$

and the eigenvalues

$$E_{1,2} = \frac{1}{2}\left[(a+d) \pm \frac{2b}{\sin(2\theta)}\right].$$

What happens to the mixing angle and the difference of the eigenvalues of M when one adds a term proportional to the unit matrix to M?

2. For three generations there are only two independent mass differences. For this reason the observation of two oscillations, $\mathrm{e} \to \mu$ and $\mu \to \tau$, determines the mass-squared differences

$$\Delta m_{\odot}^2 = \Delta m_{12}^2,$$
$$\Delta m_{\mathrm{A}}^2 = \Delta m_{23}^2 = \Delta m_{31}^2.$$

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For the CHOOZ experiment the distance is approximately a kilometer, so $\Delta m^2_{12}\, L/E$ is very small and does not contribute to the oscillation.

(i) Use Eq. (13.22) and approximations suggested by the description of this problem to derive Eq. (13.39).

(ii) Use again Eq. (13.22), the same approximations, and the unitarity of the mixing matrix to derive Eq. (13.37).

3. The general Lagrangian with Dirac and Majorana mass terms is

$$-L = \bar{\psi}\,\partial\!\!\!/\psi + M_{\mathrm{D}}\big[\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \text{h.c.}\big] + \frac{M_{\mathrm{L}}}{2}\big[(\bar{\psi}_{\mathrm{L}})^{\mathrm{c}}\psi_{\mathrm{L}} + \text{h.c.}\big] + \frac{M_{\mathrm{R}}}{2}\big[(\bar{\psi}_{\mathrm{R}})^{\mathrm{c}}\psi_{\mathrm{R}} + \text{h.c.}\big],$$

with M_{L} and M_{R} being Majorana mass terms for the left- and right-handed neutrinos.

(i) Show that it can be written in the form

$$-L = \bar{V}\,\partial\!\!\!/ V + \bar{V}[M]V,$$

where V is a column matrix,

$$V = \frac{1}{2}\begin{pmatrix}\psi_{\mathrm{L}} + (\psi_{\mathrm{L}})^{\mathrm{c}}\\ \psi_{\mathrm{R}} + (\psi_{\mathrm{R}})^{\mathrm{c}}\end{pmatrix},$$

and $[M]$ is a symmetric matrix,

$$[M] = \begin{pmatrix}M_{\mathrm{L}} & M_{\mathrm{D}}\\ M_{\mathrm{D}} & M_{\mathrm{R}}\end{pmatrix}.$$

$[M]$ is the neutrino mass matrix. For $M_{\mathrm{L}} = 0$ it reduces to the seesaw case.

(ii) Let ψ_1 and ψ_2 be the eigenvector fields of the mass matrix with eigenvalues λ_1 and λ_2, respectively. Then show that L can be rewritten as

$$-L = \bar{\psi}_1\,\partial\!\!\!/\psi_1 + \bar{\psi}_2\,\partial\!\!\!/\psi_2 + \lambda_1\bar{\psi}_1\psi_1 + \lambda_2\bar{\psi}_2\psi_2.$$

The physics content of the Lagrangian is now clear: it is the free Lagrangian for two particles ψ_1 and ψ_2.

4. Consider the Dirac Lagrangian and add the term

$$\bar{\psi}(x)\gamma_5\psi(x)\phi.$$

The Dirac equation now becomes

$$[\mathrm{i}\,\partial\!\!\!/ - \gamma_5\phi - m]\psi(x) = 0.$$

When $\phi(x)$ is a function of x_μ, the addition term represents the interaction of the fermion with a scalar field. We may also consider ϕ to be a constant, i.e. independent of space and time. Consider the latter case and search for plane-wave solutions

$$\psi(x) = u(p)\mathrm{e}^{-\mathrm{i}p\cdot x}.$$

(a) Which condition must p_μ satisfy in order for solutions to exist?

(b) Once p_μ satisfies the conditions, which are the linearly independent spinors?

(c) Find a chiral transformation $T = \mathrm{e}^{\mathrm{i}\gamma_5\theta}$ so that the Dirac equation is brought into the form $(\mathrm{i}\,\partial\!\!\!/ - \tilde{m})\psi(x) = 0$.

Comment This exercise shows that, whenever ϕ is a constant, the term $\phi\bar{\psi}(x)\gamma_5\psi(x)$ is a mass term. It also follows that the sign of the fermion mass can be changed by a chiral transformation.

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