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CONFORMAL $\mathfrak{sl}_2$ ENVELOPING ALGEBRAS AS AMBISKEW POLYNOMIAL RINGS

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Abstract We study a three parameter deformation $\mathcal{U}_{abc}$ of $\mathcal{U}(\mathfrak{sl}_2)$ introduced by Le Bruyn in 1995. Working over an arbitrary algebraically closed field of characteristic zero, we determine the centres, the finite-dimensional irreducible representations, and, when the parameter a is not a non-trivial root of unity, the prime ideals of those $\mathcal{U}_{abc}$, with $ac \neq 0$, which are conformal as ambiskew polynomial rings.

Keywords: quantum $\mathfrak{sl}_2$ enveloping algebras; iterated skew polynomial rings; prime ideals

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1. Introduction

In [**19**], a seven-parameter family of deformations of $\mathcal{U}(\mathfrak{sl}_2)$ was introduced, namely the filtered $\mathbb{C}$-algebra with generators x, y and t and defining relations

$$\begin{aligned} tx - \alpha xt &= \beta x, \\ yt - \gamma ty &= \delta y, \\ xy - \epsilon yx &= \zeta t + \eta t^2, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{C}$. As was observed in [**13**], not all of these algebras will have desirable ring-theoretic properties, such as being a domain, having finite global dimension or a PBW basis. In the classical case the associated graded ring of $\mathcal{U}(\mathfrak{sl}_2)$ is a commutative polynomial ring, and so the above properties follow. However, for these algebras the associated graded ring is not necessarily commutative. In [**13**], Le Bruyn gave the definition of a conformal $\mathfrak{sl}_2$ enveloping algebra as being those seven-parameter algebras above whose associated graded ring is an Auslander regular algebra of global dimension three. This allows conformal $\mathfrak{sl}_2$ enveloping algebras to enjoy some of the same good ring-theoretic and homological properties as in the classical case.

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In [**13**] all the conformal $\mathfrak{sl}_2$ enveloping algebras were classified: for $a, b, c \in \mathbb{C}$, let $\mathcal{U}_{abc}$ denote the $\mathbb{C}$-algebra with generators x, y and t and defining relations

$$\begin{aligned} tx - axt &= x, \\ yt - aty &= y, \\ xy - cyx &= t + bt^2. \end{aligned}$$

Then every conformal $\mathfrak{sl}_2$ enveloping algebra is isomorphic to $\mathcal{U}_{abc}$ for some $a, b, c \in \mathbb{C}$ and, conversely, every $\mathcal{U}_{abc}$ is a conformal $\mathfrak{sl}_2$ enveloping algebra. It was also shown in [**13**] that if $ac \neq 0$, then $\mathcal{U}_{abc}$ is Auslander regular of global dimension three and satisfies the Cohen–Macaulay property. For the definition of such terms as Auslander regular and the Cohen–Macaulay property, we refer the reader to [**14**]. Observe that $\mathcal{U}(\mathfrak{sl}_2)$ is a conformal $\mathfrak{sl}_2$ enveloping algebra, and is isomorphic to $\mathcal{U}_{101}$.

The algebras $\mathcal{U}_{abc}$ have previously been considered in [**13**] and [**12**], where the finite-dimensional irreducible representations were studied. In these two papers quite different approaches were taken: [**13**] used non-commutative projective geometry, and [**12**] classified the finite-dimensional irreducible representations for generic values of parameters in certain root-of-unity cases using techniques from non-commutative algebraic geometry developed by Rosenberg. Here we take a more ring-theoretical approach; we use methods developed by Jordan to study $\mathcal{U}_{abc}$, with $ac \neq 0$, over an arbitrary algebraically closed field k of characteristic zero.

When considering conformal $\mathfrak{sl}_2$ enveloping algebras, it is worth noting that two of the most interesting cases (that of $\mathcal{U}_{a0c}$ and $\mathcal{U}_{abc}$ for $a, b, c \in k^*$) lie on the borderline between two types of algebras currently being studied, namely *down–up algebras* and *ambiskew polynomial rings*. The definition of a down–up algebra has a combinatorial origin, relating to the operators of a differential partly ordered set, and was first given by Benkart and Roby in [**4**]. It is known from [**4**, 1.5] that, when a and c are non-zero, $\mathcal{U}_{a0c}$ is isomorphic to a down–up algebra. The finite-dimensional irreducible representations of down–up algebras have been determined in [**4**], [**6**] and [**11**], and the centre of a down–up algebra in [**20**].

An ambiskew polynomial ring (see [**11**] and § 2 below) is a certain skew polynomial ring in two indeterminates, whose commutativity on elements of the base ring is controlled by an automorphism of the base ring. There is a close relation between down–up algebras and ambiskew polynomial rings: it is shown in [**11**, 3.1] that all Noetherian down–up algebras belong to a precise subclass of ambiskew polynomial rings. In § 2 we show that when $a, b, c \in k$, with $ac \neq 0$, then $\mathcal{U}_{abc}$ is an ambiskew polynomial ring. To study these algebras we want to use the techniques developed in [**9**] and [**10**]. The methods of [**9**] and [**10**] apply only to *conformal* ambiskew polynomial rings. In § 3 we explain what this means and make some notational alterations to clear any confusion between the two uses of the adjective 'conformal'. We refer to those $\mathcal{U}_{abc}$, with $ac \neq 0$, which are conformal as ambiskew polynomial rings as *J-conformal*. The main results of this paper are as follows.

(1) Characterization of when certain subclasses of $\mathcal{U}_{abc}$ are J-conformal (Propositions 3.3 and 3.4).

(2) Description of the centre in all cases where $\mathcal{U}_{abc}$ is J-conformal (Theorem 4.6).

(3) Description of the finite-dimensional irreducible representations of J-conformal $\mathcal{U}_{abc}$ (Theorems 5.9, 5.10, 5.11 and 5.12).

(4) The prime spectrum of J-conformal $\mathcal{U}_{abc}$ in all cases except where a is a non-trivial root of unity (Theorems 7.5, 7.6, 7.7 and 7.8).

It should be pointed out that, by [**11**, 2.1], every ambiskew polynomial ring, and therefore every $\mathcal{U}_{abc}$ with $ac \neq 0$, is isomorphic to a Generalized Weyl Algebra in the sense of Bavula [**1**]. We could have employed the techniques of [**1**], [**2**] and [**3**] to determine the finite-dimensional irreducible $\mathcal{U}_{abc}$-modules. However, as the title of this paper suggests, we will be adopting the ambiskew polynomial ring approach throughout.

Crucial to our arguments will be the study of the localization $\mathcal{L}_{abc}$ of J-conformal $\mathcal{U}_{abc}$ at a certain normal element g, which is only defined when $a \neq 1$ (see Notation 2.3 (c) and Definition 3.9 below). In § 5 we show that when we factor by this normal element g we always obtain a familiar algebra, the finite-dimensional irreducible modules and the prime ideals of which are well known. Therefore, successfully obtaining result (3) is equivalent to finding all the finite-dimensional irreducible modules of J-conformal $\mathcal{U}_{1bc}$ and, when $a \neq 1$, the finite-dimensional irreducible modules of $\mathcal{L}_{abc}$; similarly, successfully obtaining result (4) is equivalent to finding all the prime ideals of J-conformal $\mathcal{U}_{1bc}$ and, when a is not a root of unity, the prime ideals of $\mathcal{L}_{abc}$.

Many (if not all) of the particular deformations of $\mathcal{U}(\mathfrak{sl}_2)$ which have been studied over the past 20 years can be realized as particular cases of the algebras $\mathcal{U}_{abc}$ or $\mathcal{L}_{abc}$, or algebras closely related to these. Thus we demonstrate in Examples 3.6, 3.7 and 3.8 and Theorem 3.10 that

(1) the standard quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is isomorphic to $\mathcal{L}_{q^2,((q^2-1)/2),q^{-2}}$ and that $\mathcal{L}_{q^{-4}0q^2}[g^{1/2}]$ exists and is isomorphic to $\mathcal{U}_q(\mathfrak{sl}_2)$;

(2) [**13**] $\mathcal{U}_{q^{-1},((q^{-1}-1)/2),q}$ is isomorphic to the algebra studied in [**8**];

(3) [**13**] $\mathcal{U}_{q,q-1,1}$ is isomorphic to Witten's quantum $\mathfrak{sl}_2$ enveloping algebra [**19**, 5.2]; and

(4) if fourth and twelfth roots of unity are excluded, $\mathcal{U}_{q^2,q^2-1,1}$ is isomorphic to the enveloping algebra of the quantum Lie algebra $\mathfrak{sl}(2)_q$ [**15**].

2. Preliminaries

Throughout, we let k be an algebraically closed field of characteristic zero.

Definition 2.1 (see [13]). Let $a, b, c \in k$. Then the *conformal $\mathfrak{sl}_2$ enveloping algebra*, denoted $\mathcal{U}_{abc}$, is the k-algebra generated by x, y and t with defining relations:

$$tx - axt = x, \tag{2.1}$$

$$yt - aty = y, \tag{2.2}$$

$$xy - cyx = t + bt^2. \tag{2.3}$$

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Definition 2.2 (see [11]). Let T be a commutative k-algebra, let σ be a k-algebra automorphism of T, let $v \in T$ and let $\rho \in k^*$. Extend σ to the skew polynomial ring $T[Y;\sigma]$ by setting $\sigma(Y) = \rho^{-1}Y$. There is a σ^{-1}-derivation δ of $T[Y;\sigma]$ such that $\delta(T) = 0$ and $\delta(Y) = v$, by [**7**, Example 1F]. The *ambiskew polynomial ring* $R(T, \sigma, v, \rho)$ is the skew polynomial ring $T[Y;\sigma][X;\sigma^{-1},\delta]$. Thus $XY - \rho YX = v$ and, for all $h \in T$, $Yh = \sigma(h)Y$ and $X\sigma(h) = hX$.

Notation 2.3. Let $a, b, c \in k$ with $ac \neq 0$ and consider $\mathcal{U}_{abc}$. Throughout we will use the following *fixed* notation.

(a) $A = k[t]$, where t is one of the generators of $\mathcal{U}_{abc}$ exactly as in Definition 2.1.

(b) α is the k-algebra automorphism of A given by $\alpha(t) = at + 1$.

(c) Whenever $a \neq 1$ we let $g = (a-1)t+1$. Observe that $\alpha^i(g) = a^i g$, for all integers i.

(d) For $\gamma \in k^*$ and an integer $j > 1$ we write $[\gamma]_1 = 1$ and $[\gamma]_j = 1 + \gamma + \cdots + \gamma^{j-1}$.

Proposition 2.4. *Suppose that $a, b, c \in k$ with $ac \neq 0$. Then $\mathcal{U}_{abc} \cong R(A, \alpha, t+bt^2, c)$. It follows that $\mathcal{U}_{abc}$ is a right and left Noetherian ring, and that, whenever $a \neq 1$, g is a normal element of $\mathcal{U}_{abc}$.*

Proof. By the definition of α we have that $\alpha^{-1}(t) = a^{-1}t - a^{-1}$. Thus we have by (2.1) that $xt = a^{-1}tx - a^{-1}x = (a^{-1}t - a^{-1})x = \alpha^{-1}(t)x$ and by (2.2) that $yt = y + aty = (at+1)y = \alpha(t)y$. Since $\{t^i y^j x^k : i, j, k \geqslant 0\}$ is a PBW type basis of $\mathcal{U}_{abc}$ (as can be seen by a straightforward application of the *Diamond Lemma* [**5**, 1.2]), and as $\{X^i Y^j : i, j \geqslant 0\}$ is linearly independent over T, the isomorphism follows by (2.3) and Definition 2.2. That $\mathcal{U}_{abc}$ is right and left Noetherian follows from [**7**, 1.12]. Now suppose that $a \neq 1$. Observe that $tg = gt$, $yg = \alpha(g)y = agy$ and $gx = x\alpha(g) = ag$. As $\mathcal{U}_{abc}$ is generated as a k-algebra by t, y and x, g is a normal element of $\mathcal{U}_{abc}$. $\square$

The representation theory of the algebras $\mathcal{U}_{abc}$ is influenced by the maximal ideals of A and the action of α on these maximal ideals. The proof of the next lemma is routine.

Lemma 2.5. *Let $a \in k^*$.*

(i) *Suppose that $a = 1$. Then every maximal ideal of A has infinite orbit under α.*

(ii) *Suppose $a \neq 1$. The only maximal ideal of $k[t]$ invariant under α is $gk[t]$.*

(iii) *Suppose $a \neq 1$. Let m be a positive integer and choose any $\mu \in k$. Then $\alpha^m(t-\mu) = a^m t + [a]_m - \mu$ and $\alpha^{-m}(t-\mu) = a^{-m}(t - [a]_m - a^m\mu)$, and therefore we have $\alpha^m((t-\mu)k[t]) = (t + a^{-m}[a]_m - a^{-m}\mu)k[t]$ and $\alpha^{-m}((t-\mu)k[t]) = (t - [a]_m - a^m\mu)k[t]$.*

(iv) *Suppose $a \neq 1$. If a is not a root of unity, then the only maximal ideal of $k[t]$ with finite orbit under α is the α-invariant maximal ideal $gk[t]$.*

(v) *If a is a non-trivial root of unity, of multiplicative order $n > 1$ in k^*, then every maximal ideal of $k[t]$ not equal to the α-invariant maximal ideal $gk[t]$ has orbit of order n under α.*

Corollary 2.6. *Let $a \in k^*$.*

(i) *Suppose that a is a non-root of unity. Then the only non-zero, proper α-invariant ideals of $k[t]$ are $g^i k[t]$, $\forall i \geqslant 1$. In fact, if I is a non-zero, proper ideal with finite orbit under α, then I is necessarily α-invariant, and so $I = g^i k[t]$ for some $i \geqslant 1$.*

(ii) *Suppose that $a = 1$. Then every non-zero, proper ideal of A has infinite orbit under α.*

Proof. We know by Lemma 2.5 (iv) that the only maximal ideal of finite orbit under α is $gk[t]$. Let I be a non-zero, proper ideal of $k[t]$. Then $I = fk[t]$ for some $f \in k[t]$ of degree $m \geqslant 1$, say. Now, I is contained in a finite number of maximal ideals of $k[t]$, corresponding to the linear factors of f. Therefore, if I has finite orbit under α, then so too must each of the maximal ideals containing I. Hence f is a non-zero scalar multiple of g^m and $I = g^m k[t]$. This proves part (i). Part (ii) is proved in a similar way, noting Lemma 2.5 (i). □

3. *J*-conformality

In [**11**], an ambiskew polynomial ring $R(T, \sigma, v, \rho)$ (as in Definition 2.2) is said to be *conformal* if $XY - \rho YX = v = w - \rho\sigma(w)$ for some $w \in T$. We let $Z = XY - w$. As Z is a generalization of the definition of the Casimir element of $\mathcal{U}(\mathfrak{sl}_2)$ to an arbitrary conformal ambiskew polynomial ring, Z is referred to as the Casimir element of $R(T, \sigma, w - \rho\sigma(w), \rho)$. Recall that the adjective 'conformal' has already been used in [**13**] in the title of the algebras $\mathcal{U}_{abc}$. To avoid confusion we make the following definition.

Definition 3.1. Let $a, b, c \in k$ with $ac \neq 0$.

(i) We will say that the conformal enveloping algebra $\mathcal{U}_{abc}$ is J-*conformal (with respect to $w \in A$, of degree $m > 0$)* if it is conformal as an ambiskew polynomial ring, with $w - c\alpha(w) = t + bt^2$, where w is of degree $m > 0$ in A.

(ii) Let $\mathcal{U}_{abc}$ be J-conformal (with respect to $w \in A$, of degree $m > 0$). Then, as in [**11**], $z := xy - w = c(yx - \alpha(w))$ is the Casimir element of $\mathcal{U}_{abc}$; we extend α to the polynomial k-algebra in two indeterminates $A[z]$ by setting $\alpha(z) = c^{-1}z$. Observe that $yz = c^{-1}zy = \alpha(z)y$ and $xz = czx = \alpha^{-1}(z)x$.

Notation 3.2. Let $a, b, c \in k$, $ac \neq 0$, and suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. Then we set $r(w)$ to be the number of distinct roots in k of the equation $w = 0$ (so $1 \leqslant r(w) \leqslant m$); we denote these roots by $\rho_1, \ldots, \rho_{r(w)}$.

Proposition 3.3. *Let $a, c \in k^*$.*

(i) *$\mathcal{U}_{a0c}$ is J-conformal with respect to $w \in A$, of degree 1, if and only if $c \notin \{1, a^{-1}\}$.*

(ii) *$\mathcal{U}_{a0c}$ is J-conformal with respect to $w \in A$, of degree 2, if and only if $a = 1 = c$, that is $\mathcal{U}_{abc} = \mathcal{U}_{101} \cong \mathcal{U}(\mathfrak{sl}_2)$.*

(iii) *$\mathcal{U}_{a01}$ and $\mathcal{U}_{a0a^{-1}}$ are not J-conformal for all non-roots of unity $a \in k^*$.*

(iv) *$\mathcal{U}_{-101}$ and $\mathcal{U}_{-10-1}$ are not J-conformal.*

Proof. We do not include the proofs here. Parts (i), (ii) and (iii) can be proved through straightforward calculation; (iv) is proved using elementary linear algebra. □

Proposition 3.4. *Let $a, b, c \in k^*$.*

(i) *(a) $\mathcal{U}_{1bc}$ is J-conformal of degree 2 for all $c \neq 1$. (b) $\mathcal{U}_{1b1}$ is not J-conformal of degree 2, but is J-conformal of degree 3.*

(ii) *Suppose that $a \neq 1$. There exists an element $w \in k[t]$ of degree 2 such that $w - c\alpha(w) = t + bt^2$ if and only if one of the following holds:*

(a) $(a, b, c) \in \mathcal{F} = \{(a, b, c) \in k^* \times k^* \times k^* : a \neq 1,\ c \notin \{1, a^{-1}, a^{-2}\}\}$;

(b) $(a, b, c) \in \{(d, \frac{1}{2}(d-1), d^{-1}) : d \in k \setminus \{0, 1\}\}$;

(c) $(a, b, c) \in \{(d, d-1, 1) : d \in k \setminus \{0, 1\}\}$.

(iii) *For $a \neq 1$, suppose that a and c are not both roots of unity, or that $c \neq a^{-n}$ for all integers $n > 2$. Then $\mathcal{U}_{abc}$ is conformal as an ambiskew polynomial ring if and only if there exists an element $w \in k[t]$ of degree 2 such that $w - c\alpha(w) = t + bt^2$.*

Proof. Parts (i) and (ii) can be proved by careful, though straightforward, calculation. We now prove (iii). The reverse direction is certainly true, by the definition of conformality of an ambiskew polynomial ring. For the forward direction suppose that there exists $w = \sum_{i=0}^{n} \mu_i t^i$, for some $n \geqslant 0$, $\mu_i \in k$ with $\mu_n \neq 0$, satisfying $w - c\alpha(w) = t + bt^2$. Clearly $n \geqslant 2$. If $n = 2$ we are done. Suppose then that $n > 2$. Then $\mu_n - ca^n\mu_n = 0$, and, therefore, since μ_n is non-zero, $1 = ca^n$. It is clear that if $c \neq a^{-n}$ for all $n > 2$, or if only one of a and c is a root of unity, then no such w of degree strictly greater than two can exist. Thus in these cases such an element w, if it exists, must be of degree 2. Now suppose that a and c are both non-roots of unity. We show that $1 = ca^n$ implies that $(a, b, c) \in \mathcal{F}$. By our hypothesis on a and c, neither is equal to 1. Suppose that $c \in \{a^{-1}, a^{-2}\}$. Then $1 \in \{a^{n-1}, a^{n-2}\}$, where $n - 1 > n - 2 > 0$, contradicting our hypothesis on a. Hence $(a, b, c) \in \mathcal{F}$, and so we are done. This proves the proposition. □

Lemma 3.5. *Let $a, b, c \in k$, $ac \neq 0$, with $a \neq 1$, and suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. (i) When $c = 1$, if g divides w, then $b = a - 1$; when $c \neq 1$, g divides w if and only if $b = a - 1$. (ii) Suppose that $m > 1$. Then g^i does not divide w for each $i = 2, \ldots, m$.*

Proof. (i) Let $\eta \in k$ be such that $-\eta$ is equal to the evaluation of w at $t = 1/(1-a)$ (recall that $w \in A = k[t]$). Then g always divides $w + \eta$ and, since $\alpha(g) = ag$, g always divides $w + \eta - c\alpha(w + \eta) = t + bt^2 + (1 - c)\eta$. Hence

$$\frac{1}{1-a} + \frac{b}{(1-a)^2} = (c-1)\eta.$$

Suppose that $c = 1$. Then we must have that $b = a - 1$. Now suppose that $c \neq 1$, and so

$$\eta = \frac{1-a+b}{(c-1)(a-1)^2}.$$

Thus g divides w, i.e. $\eta = 0$, if and only if $b = a - 1$. (ii) A straightforward calculation shows that g^2 divides w only if $b = \frac{1}{2}(a - 1)$. Since g must also divide w in this case we have, by (i), that $b = a - 1$. Since $a \neq 1$ this gives a contradiction. Hence we have the result. $\square$

Example 3.6. As was noted in [**13**, 2.2], when (a, b, c) is as in Proposition 3.4 (ii) (b), $\mathcal{U}_{abc}$ is isomorphic to the deformation $\mathcal{U}_q$ of $\mathcal{U}(\mathfrak{sl}_2)$ defined and studied in [**8**]. In fact, for $q \in k^*$ we have that $\mathcal{U}_q$ is isomorphic to $\mathcal{U}_{q^{-1},((q^{-1}-1)/2),q}$.

Example 3.7. As was noted in [**13**, 2.2], when (a, b, c) is as in Proposition 3.4 (ii) (c), $\mathcal{U}_{abc}$ is isomorphic to the quantum $\mathfrak{sl}_2$ enveloping algebra of Witten [**19**, 5.2].

Example 3.8. When (a, b, c) is as in Proposition 3.4 (ii) (c) with $a = q^2$, where $q \in k^*$ is not a fourth, nor a primitive twelfth, root of unity, then $\mathcal{U}_{abc}$ is isomorphic to the enveloping algebra of the quantum Lie algebra $\mathfrak{sl}(2)_q$ as defined in [**15**].

Definition 3.9. Let $a, b, c \in k$ with $ac \neq 0$ and $a \neq 1$. Recall Proposition 2.4. Since g is a normal element of $\mathcal{U}_{abc}$, $\{g^i : i \geqslant 0\}$ is a right Ore set in the Noetherian ring $\mathcal{U}_{abc}$, and so is automatically a right denominator set by [**16**, 1.13 (iii)]. We can therefore localize to $\mathcal{L}_{abc} := \mathcal{U}_{abc}[g^i : i \geqslant 0]^{-1}$. Note that $\mathcal{L}_{abc}$ is isomorphic to $R(S, \alpha, t + bt^2, c)$, where $S := A[g^i : i \geqslant 0]^{-1}$ and α is extended by setting $\alpha(g^{-1}) = \alpha(g)^{-1}$. Whenever $a, b, c \in k$ with $ac \neq 0$ and $a \neq 1$, this notation for S and $\mathcal{L}_{abc}$ will be fixed throughout.

Theorem 3.10. *For $q \in k^*$ with $q^2 \neq 1$, let $\mathcal{U}_q(\mathfrak{sl}_2)$ denote the quantized enveloping algebra of $\mathfrak{sl}_2$, namely the k-algebra generated by K, K^{-1}, E and F with defining relations $KK^{-1} = 1 = K^{-1}K$, $KE = q^2EK$, $KF = q^{-2}FK$ and $EF - FE = (q - q^{-1})^{-1}(K - K^{-1})$.*

(i) *Let $q \in k^*$, with $q^2 \neq 1$. Recall Example 3.6. Then $\mathcal{L}_{q^2,((q^2-1)/2),q^{-2}}$ is isomorphic to $\mathcal{U}_q(\mathfrak{sl}_2)$ as a k-algebra.*

(ii) *Let $q \in k^*$, with $q^4 \neq 1$. Then we have that $\mathcal{U}_{q^{-4}0q^2}[g^{1/2}]$ exists, and its localization with respect to $\{g^{i/2} : i \geqslant 0\}$ is isomorphic as a k-algebra to $\mathcal{U}_q(\mathfrak{sl}_2)$ (cf. [**18**, 2.6]).*

Proof. (i) Note that $g = (q^2 - 1)t + 1$. Then $\mathcal{L} = \mathcal{L}_{q^2,((q^2-1)/2),q^{-2}}$ is generated by x, y, g and g^{-1}. Let $e = g^{-1}x$ and $f = 2qy$. Then it is clear that e, f, g and g^{-1} generate $\mathcal{L}$. It is easily seen that $gg^{-1} = 1 = g^{-1}g$, $ge = q^2eg$, $gf = q^{-2}fg$, and that $ef - fe = (q - q^{-1})^{-1}(g - g^{-1})$. Thus the map $\psi : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{L}_{q^2,((q^2-1)/2),q^{-2}}$ given by $\psi(K) = g, \psi(K^{-1}) = g^{-1}, \psi(E) = e$ and $\psi(F) = f$ is a k-algebra homomorphism, which is clearly surjective. Since $\{K^{i_1}E^{i_2}F^{i_3} : i_i, i_2, i_3 \in \mathbb{Z},\ i_2, i_3 \geqslant 0\}$ is a PBW-type basis for $\mathcal{U}_q(\mathfrak{sl}_2)$ and $\{g^{i_1}e^{i_2}f^{i_3} : i_i, i_2, i_3 \in \mathbb{Z},\ i_2, i_3 \geqslant 0\}$ is a PBW-type basis for $\mathcal{L}_{q^2,((q^2-1)/2),q^{-2}}$, we have that ψ is injective. Thus ψ is a k-algebra isomorphism.

(ii) Let $\mathcal{U} = \mathcal{U}_{q^{-4}0q^2}$. Define $\phi : \mathcal{U} \to \mathcal{U}_q(\mathfrak{sl}_2)$ by $\phi(x) = K^{-1}E$, $\phi(g) = K^{-2}$ and $\phi(y) = \lambda F$, where $\lambda = q(1+q^{-2})^{-1} \in k$. By straightforward calculation we have that ϕ respects the relations between the generators x, y and g of $\mathcal{U}$, namely that $\phi(g)\phi(x) = q^{-4}\phi(x)\phi(g)$, $\phi(y)\phi(g) = q^{-4}\phi(g)\phi(y)$ and $\phi(x)\phi(y) - q^2\phi(y)\phi(x) = (q^{-4}-1)^{-1}(\phi(g)-1)$, where we note that $t = (q^{-4}-1)^{-1}(g-1)$. By a similar argument involving PBW-type bases as in (i), we have that $\operatorname{Ker}\phi = 0$. Thus $\mathcal{U}$ is isomorphic to $B_0 := \operatorname{Im}\phi$, a k-subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$. Note that $K^{-1} = \phi(g)^{1/2} \in \mathcal{U}_q(\mathfrak{sl}_2)$. Thus the k-algebra $\mathcal{U}[g^{1/2}]$ exists and is isomorphic, by extending $\phi(g^{1/2}) = K^{-1}$, to $B := \langle B_0, K^{-1}\rangle$, which is just the subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ generated by $K^{-1}E$, F and K^{-1}. Hence $\mathcal{U}[g^{-1/2}]$, the localization of $\mathcal{U}[g^{1/2}]$ at $\{g^{i/2} : i \geqslant 0\}$, is isomorphic to the localization of B with respect to $\{K^{-i} : i \geqslant 0\}$, which is, of course, equal to $\mathcal{U}_q(\mathfrak{sl}_2)$. □

Note 3.11. Let $q \in k^*$ with $q \neq 1$. Then Example 3.6 and Theorem 3.10 (i) together show that Jing and Zhang's algebra $\mathcal{U}_q$ of [**8**] is isomorphic to the subalgebra $k\langle K, KE, F\rangle$ of $\mathcal{U}_{q^{-1/2}}(\mathfrak{sl}_2)$.

4. Centre of J-conformal $\mathcal{U}_{abc}$ for $a, b, c \in k$, $ac \neq 0$

Definition 4.1 (see 1.7 in [9]). Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be a conformal ambiskew polynomial ring where T is a commutative domain which is finitely generated as a k-algebra, and $0 \neq w \in T$. Suppose that there exists a non-zero element $h \in T$, such that $\sigma(h) = \rho^n h$ for some $n \geqslant 1$. Let $n \geqslant 1$ be minimal for the existence of such h. Then any non-zero element $h \in T$ satisfying $\sigma(h) = \rho^n h$ will be called a *principal eigenvector* and n will be its degree.

Theorem 4.2 (see 2.1(ii) in [9]). *Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be a conformal ambiskew polynomial ring where T is a commutative domain which is finitely generated as a k-algebra, and $0 \neq w \in T$; let Z denote the corresponding Casimir element. Suppose that T is σ-simple, i.e. T has no non-zero, proper σ-invariant ideals. (i) If T has no principal eigenvectors then $Z(R) = k$. (ii) If $T \neq k$ and has a principal eigenvector h of degree n, then $Z(R) = k[hZ^n]$.*

Lemma 4.3. *Let $a, b, c \in k$, $ac \neq 0$ with a not a root of unity and suppose that $\mathcal{U}_{abc}$ is J-conformal. Recall that $\mathcal{L}_{abc} := \mathcal{U}_{abc}[g^i : i \geqslant 0]^{-1} = R(S, \alpha, t + bt^2, c)$.*

(i) *Suppose that c is not a root of unity.*

(a) *There exist principal eigenvectors of S if and only if there exists integers l, n with $n \geqslant 1$ and $l \neq 0$ such that $c^n = a^l$.*

(b) *Suppose there exist principal eigenvectors of S. Let $N \geqslant 1$ be minimal such that c^N is a non-zero integer power of a. Then there is a unique non-zero $l \in \mathbb{Z}$ such that $c^N = a^l$, and $\{\lambda g^l : \lambda \in k^*\}$ is a complete set of principal eigenvectors of S. Each has degree N.*

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(ii) *Suppose that c is a root of unity, of multiplicative order $l \geqslant 1$ in k^*. Then principal eigenvectors of S exist; k^* is a complete list of principal eigenvectors, and each has degree l.*

Proof. (i) (a) ($\Rightarrow$) Let $v \in S$ be a principal eigenvector of degree N, for some integer $N \geqslant 1$. Then $0 \neq v = fg^{-i}$, for some $0 \neq f \in A$, $i \geqslant 0$, and $\alpha(v) = c^N v$. Hence $c^N fg^{-i} = \alpha(f)\alpha(g^{-i}) = a^{-i}\alpha(f)g^{-i}$, since $\alpha(g) = ag$. Therefore

$$c^N a^i f = \alpha(f). \tag{4.1}$$

Thus the ideal of A generated by f is α-invariant. Therefore, by Corollary 2.6, $f = \lambda g^j$ for some $0 \neq \lambda \in k$, $j \geqslant 0$. By (4.1), $c^N a^i \lambda g^j = \lambda a^j g^j$ and so $c^N = a^{j-i}$. Since c is not a root of unity, the integer $j - i$ is non-zero, as claimed. Notice that $v = \lambda g^l$, where $0 \neq l \in \mathbb{Z}$ and $c^N = a^l$. ($\Leftarrow$). Suppose that there exist integers N, M with $N \geqslant 1$ and $M \neq 0$ such that $c^N = a^M$. Choose $N \geqslant 1$ minimal for the existence of such an M. Now $g^M \in S \setminus \{0\}$ and $\alpha(g^M) = a^M g^M = c^N g^M$. If $N = 1$, then g^M is certainly a principal eigenvector of degree N. Let $N > 1$. Suppose that there exists $0 \neq v \in S$ such that $\alpha(v) = c^n v$ for some $n \in \mathbb{Z}$, $1 \leqslant n < N$. By the proof of the ($\Rightarrow$) direction, c^n must be a non-zero integer power of a which contradicts the minimality of N. Hence no such v exists, and so g^M is a principal eigenvector, as required. (b) Suppose that principal eigenvectors exist. We know that $c^N = a^l$ for some $0 \neq l \in \mathbb{Z}$. Since a is not a root of unity, l is necessarily unique. By the proof of the ($\Leftarrow$) direction of (i) (a), $\{\lambda g^l : 0 \neq \lambda \in k\}$ are all principal eigenvectors of S, with degree N. By the proof of the ($\Rightarrow$) direction of (i) (a) we know that if v is a principal eigenvector of S (necessarily of degree N since we already have explicit eigenvectors of degree N), then $v = \lambda g^{l'}$, where $0 \neq \lambda \in k$, $0 \neq l' \in \mathbb{Z}$ and $c^N = a^{l'}$. Hence $l' = l$, $v = \lambda g^l$ and so we are done.

(ii) We know by [**9**, 1.7 (ii)] that 1 is a principal eigenvector of degree l. Thus every element of k^* is a principal eigenvector of degree l. Now suppose that v is a principal eigenvector of S, necessarily of degree l. Then v is non-zero and $\alpha(v) = c^l v = v$. Thus v is a non-zero element of S fixed by α. Now $v = fg^r$ for some integer r and non-zero polynomial $f \in A$, where g does not divide f in A. As v is fixed by α we have that $\alpha(f) = a^{-r} f$. Therefore fA is a non-zero α-invariant ideal of A. If $fA \neq A$ then, by Corollary 2.6, g divides f in A. Thus $fA = A$, and so $f \in k^*$. Therefore $v = \alpha(v) = a^r fg^r = a^r v$. As a is not a root of unity, $r = 0$. Hence $v = f \in k^*$, and the result follows. □

Notation 4.4. Let $a, b, c \in k$ with $ac \neq 0$ and suppose that $\mathcal{U}_{abc}$ is J-conformal, with respect to $w \in A$. By [**10**, 1.7] we can form the localization of $\mathcal{U}_{abc}$ with respect to y, denoted $\mathcal{U}_y$, which is equal to $A[z][y, y^{-1}; \alpha]$. Similarly, we can form $\mathcal{U}_x$ which is equal to $A[z][x, x^{-1}; \alpha^{-1}]$. When $a \neq 1$, we also have, using analogous notation, that $\mathcal{L}_y = S[z][y, y^{-1}; \alpha]$ and $\mathcal{L}_x = S[z][x, x^{-1}; \alpha^{-1}]$.

Lemma 4.5. *Let $a, b, c \in k$ with $ac \neq 0$, and suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$. Recall the notation of 4.4. Suppose that c is a non-root of unity. Then*

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every non-zero element of $Z(\mathcal{U}_y)$ *is of the form* $\sum_{i=0}^{n} a_i z^i$, *where* $0 \leqslant n \in \mathbb{Z}$, $a_i \in A$, $a_n \neq 0$ *and* $\alpha(a_i) = c^i a_i$ *for each* $i = 0, \ldots, n$. *When* $a \neq 1$, *every non-zero element of* $Z(\mathcal{L}_y)$ *is of the form* $\sum_{i=0}^{n} s_i z^i$, *where* $0 \leqslant n \in \mathbb{Z}$, $s_i \in S$, $s_n \neq 0$ *and* $\alpha(s_i) = c^i s_i$ *for each* $i = 0, \ldots, n$.

Proof. We only prove the $\mathcal{U}_y$ case, the other being similar. Let $0 \neq f \in Z(\mathcal{U}_y)$. Then f is expressible as the finite sum $\sum_{i \geqslant 0,\ j \in \mathbb{Z}} a_{ij} z^i y^j$, where each $a_{ij} \in A$. Since $yz = c^{-1} zy$,

$$\sum a_{ij} z^{i+1} y^j = zf = fz = \sum a_{ij} c^{-j} z^{i+1} y^j.$$

Hence $a_{ij} = c^{-j} a_{ij}$ for all i, j; so $a_{ij} \neq 0$ implies $c^{-j} = 1$. Since c is not a root of unity, $j = 0$. Thus, without loss of generality, $f = \sum_{i=0}^{n} a_i z^i$, where $0 \leqslant n \in \mathbb{Z}$, $a_i \in A$, $a_n \neq 0$. Since $yf = fy$ we must have that $\alpha(a_i) = c^i a_i$ for each $i = 0, \ldots, n$. □

Theorem 4.6. *Let* $a, b, c \in k$, $ac \neq 0$, *and let* $\mathcal{U}_{abc}$ *be J-conformal, with respect to* $w \in A$ *of degree* $m > 0$.

(i) (a) *Suppose that* c *is not a root of unity. Then* $Z(U_{1bc}) = k$.

(b) *Suppose that* c *is a primitive* l*th root of unity, where* $0 < l \in \mathbb{Z}$. *Then* $Z(U_{1bc}) = k[z^l]$.

(ii) *Suppose that* a *is not a root of unity.*

(a) *Suppose that* c *is a root of unity, with multiplicative order* $l \geqslant 1$ *in* k^*. *Then* $Z(\mathcal{L}_{abc}) = k[z^l]$ and $Z(\mathcal{U}_{abc}) = k[z^l]$.

(b) *Suppose that* c *is not a root of unity and that there exists* $N \geqslant 1$ *minimal with respect to the property that* c^N *is a non-zero integer power of* a. *Let* $0 \neq l \in \mathbb{Z}$ *be such that* $c^N = a^l$. *(i)* $Z(\mathcal{L}_{abc}) = k[g^l z^N]$. *(ii) If* $l > 0$, *then* $Z(\mathcal{U}_{abc}) = k[g^l z^N]$. *(iii) If* $l < 0$, *then* $Z(\mathcal{U}_{abc}) = k$.

(c) *Suppose that* c *is not a root of unity and that there does not exist an integer* $N \geqslant 1$ *such that* c^N *is a non-zero integer power of* a. *Then* $Z(\mathcal{L}_{abc}) = k$ *and* $Z(\mathcal{U}_{abc}) = k$.

(iii) *Suppose that* a *is a primitive* N*th root of unity, for some* $1 < N \in \mathbb{Z}$, *and that* c *is not a root of unity. Then* $Z(\mathcal{L}_{abc}) = k[g^{\pm N}]$ and $Z(\mathcal{U}_{abc}) = k[g^N]$.

(iv) *Let* a *be a primitive* n*th and* c *a primitive* l*th root of unity, for integers* $n > 1$ *and* $l > 0$. *Set* $\mathcal{B} = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 0 \leqslant i < n,\ 0 \leqslant j < l,\ a^i = c^j\}$, *and note that* $|\mathcal{B}| < \infty$ *and* $(0, 0) \in \mathcal{B}$. *Let* $s = \operatorname{lcm}(n, l)$.

(a) *Let*

$$Z_0 = \sum_{(i,j) \in \mathcal{B}} g^i z^j k[g^n, z^l] \subseteq \mathcal{U}_{abc}.$$

Then Z_0 *is a central* k*-subalgebra of* $\mathcal{U}_{abc}$, *the sum is direct over* $k[g^n, z^l]$ *and, as a* $\mathbb{Z}$*-graded* k*-algebra,*

$$Z(\mathcal{U}_{abc}) = \sum_{r>0}^{\oplus} Z_0 x^{rs} \bigoplus Z_0 \bigoplus \sum_{r>0}^{\oplus} Z_0 y^{rs}.$$

(b) *Let*

$$Z_1 = \sum_{(i,j)\in\mathcal{B}} g^i z^j k[g^{\pm n}, z^l] \subseteq \mathcal{L}_{abc}.$$

Then Z_1 is a central k-subalgebra of $\mathcal{L}_{abc}$, the sum is direct over $k[g^{\pm n}, z^l]$ and, as a $\mathbb{Z}$-graded k-algebra,

$$Z(\mathcal{L}_{abc}) = \sum_{r>0}^{\oplus} Z_1 x^{rs} \bigoplus Z_1 \bigoplus \sum_{r>0}^{\oplus} Z_1 y^{rs}.$$

Proof. (i) (a) By Corollary 2.6, A has no non-zero, proper α-invariant ideals. Suppose that there exists a principal eigenvector $h \in A$. Then $h \neq 0$ and $\alpha(h) = c^n h$ for some $0 < n \in \mathbb{Z}$. Consider the non-zero ideal $I = hA$ of A. Then $\alpha(I) = c^n hA = I$, and so we must have that $I = A$. Therefore $h \in k$ and so $c^n = 1$, a contradiction. It follows that principal eigenvectors do not exist, and the result follows by Theorem 4.2. (b) By [**9**, 1.7(ii)], 1 is a principal eigenvector of degree l. Apply Theorem 4.2.

(ii) The results on $Z(\mathcal{L}_{abc})$ are clear from Theorem 4.2 and Lemma 4.3. Observe that $Z(\mathcal{U}_{abc}) = Z(\mathcal{L}_{abc}) \cap \mathcal{U}_{abc}$. Therefore in cases (a), (b) (ii) and (c) we have that $Z(\mathcal{U}_{abc}) = Z(\mathcal{L}_{abc})$. It remains to prove (b) (iii). Let $l = -p$, for some $0 < p \in \mathbb{Z}$. Set

$$B = k[g^{-p} z^N] \cap \mathcal{U}_{abc} = Z(\mathcal{U}_{abc}) = Z(\mathcal{U}_y) \cap \mathcal{U}_{abc}.$$

Let $0 \neq f \in B$. Then, by Lemma 4.5, $f = \sum_{i=0}^n a_i z^i$, where $0 \leqslant n \in \mathbb{Z}$, $a_i \in A$, $a_n \leqslant 0$. However, we also have that $f = \sum_{r=0}^q \gamma_r g^{-pr} z^{Nr}$ for some $0 \leqslant q \in \mathbb{Z}$, $\gamma_r \in k$ with $\gamma_q \neq 0$. Hence $g^{pq} f = \sum_{r=0}^q \gamma_r g^{p(q-r)} z^{Nr} \in A[z]$, and $g^{pq} f = \sum_{i=0}^n a_i g^{pq} z^i \in A[z]$. Comparing leading coefficients, $Nq = n$ and $\gamma_q = a_n g^{pq}$. If $q > 0$ this gives a contradiction since $g \notin k$. Thus $q = 0$; therefore $n = 0$ and $f \in k$. Hence $B \subseteq k \subseteq B$, i.e. $B = k$. This proves (b) (iii).

(iii) Let $0 \neq f \in Z(\mathcal{L}_y)$. Then, by Lemma 4.5, $f = \sum_{i=0}^n s_i z^i$, where $0 \leqslant n \in \mathbb{Z}$, $s_i \in S$, $s_n \neq 0$ and $\alpha(s_i) = c^i s_i$ for each $i = 0, \dots, n$. Observe that s_n is expressible as a finite sum $\sum_{i\in\mathbb{Z}} \gamma_i g^i$, for some $\gamma_i \in k$. Therefore $\alpha(s_n) = \sum_{i\in\mathbb{Z}} \gamma_i a^i g^i$, and so, since $\alpha(s_n) = c^n s_n$, $\gamma_i a^i = \gamma_i c^n$ for each i. Since $s_n \neq 0$, we can choose a non-zero γ_i. Then $1 = a^{Ni} = c^{Nn}$ and, as c is not a root of unity, $Nn = 0$. Therefore $n = 0$, since $N \neq 0$. Thus $f = s_0$, where $\alpha(s_0) = s_0$. Thus $Z(\mathcal{L}_y) \subseteq S^\alpha \subseteq \mathcal{L}_{abc}$, and so $Z(\mathcal{L}_y) = Z(\mathcal{L}_{abc})$. Since $\mathcal{L}_{abc}$ is generated as a k-algebra by S, y and x, we have $S^\alpha \subseteq Z(\mathcal{L}_{abc})$. Hence $Z(\mathcal{L}_{abc}) = S^\alpha$. It is clear that $k[g^{\pm N}] \subseteq S^\alpha$. Let $s \in S^\alpha$. Then s is expressible as a finite sum $\sum_{i\in\mathbb{Z}} \mu_i g^i$ for some $\mu_i \in k$, and $\mu_i a^i g^i = \mu_i g^i$ for each i. Therefore, if μ_i is non-zero for some i, we have $a^i = 1$, i.e. i is a multiple of N. Thus s, and so S^α, is contained in $k[g^{\pm N}]$. Hence $Z(\mathcal{L}_{abc}) = S^\alpha = k[g^{\pm N}]$. Now let $B = Z(\mathcal{L}_{abc}) \cap \mathcal{U}_{abc} = Z(\mathcal{U}_{abc})$. Choose $0 \neq h \in B$. Then $h = h_- + h_+$ for some $h_- \in k[g^{-N}]$ and $h_+ \in k[g^N]$. Since $h, h_+ \in \mathcal{U}_{abc}$ so too must h_-. Hence h_- is expressible as a finite sum $\sum_{r,s,t\geqslant 0} \lambda_{rst} g^r y^s x^t$ for some $\lambda_{rst} \in k$. However, we also have $h_- = \sum_{i=0}^M \lambda_i g^{-Ni}$ for some $0 \leqslant M \in \mathbb{Z}$, $\lambda_i \in k$. Thus

$$g^{NM} h_- = \sum_{i=0}^M \lambda_i g^{N(M-i)} \in k[g^N] \quad \text{and} \quad g^{NM} h_- = \sum \lambda_{rst} g^{NM+r} y^s x^t.$$

By the k-linear independence of $\{g^{i_1}y^{i_2}x^{i_3} : i_1, i_2, i_3 \geqslant 0\}$, $\lambda_{rst} = 0$ whenever s or t is non-zero. Hence $h_- \in k[g] = A$, and so $h \in A$. Then $h \in S^\alpha$ implies that $h \in A^\alpha = k[g^N]$. Thus $k[g^N] \subseteq B \subseteq k[g^N]$, i.e. $Z(\mathcal{U}_{abc}) = B = k[g^N]$.

(iv) We prove (a) only; (b) is proved in a similar manner. Set $\mathcal{U}_0 = A[z]$, $\mathcal{U}_r = A[z]y^r (r > 0)$ and $\mathcal{U}_r = A[z]x^{-r} (r < 0)$. Then

$$\sum_{r\in\mathbb{Z}}^{\oplus} \mathcal{U}_r$$

is a $\mathbb{Z}$-grading for $\mathcal{U}_{abc}$, by [**9**, 1.9]. Since $\mathcal{U}_{abc}$ is generated by homogeneous elements,

$$Z(\mathcal{U}_{abc}) = \sum_{r\in\mathbb{Z}}^{\oplus} Z(\mathcal{U}_{abc}) \cap \mathcal{U}_r.$$

Let $r > 0$ and suppose that $Z(\mathcal{U}_{abc}) \cap \mathcal{U}_r$ is non-zero. Then there exists $0 \neq v = fy^r \in \mathcal{U}_r$, for some $f \in A[z]$, such that v is central in $\mathcal{U}_{abc}$. Since $vz = zv$ we have $zfy^r = c^{-r}zfy^r$, which implies that $c^{-r} = 1$, i.e. r is a multiple of l. Also $gv = vg$, which gives $gfy^r = a^r gfy^r$. Thus $a^r = 1$, and so r is a multiple of n. We therefore have that r is a multiple of s. It is clear that y^s commutes with $A[z]$. Since $\mathcal{U}_y = A[z][y, y^{-1}; \alpha]$, $y^s \in Z(\mathcal{U}_y) \cap \mathcal{U}_{abc} = Z(\mathcal{U}_{abc})$. Therefore $y^r \in Z(\mathcal{U}_{abc})$ and, since $\mathcal{U}_{abc}$ is a domain, $v = fy^r \in Z(\mathcal{U}_{abc})$ if and only if $f \in Z(\mathcal{U}_{abc})$. So $\mathcal{U}_r \cap Z(\mathcal{U}_{abc}) \subseteq (A[z] \cap Z(\mathcal{U}_{abc}))y^r$. As $(A[z] \cap Z(\mathcal{U}_{abc}))y^r \subseteq \mathcal{U}_r \cap Z(\mathcal{U}_{abc})$, $\mathcal{U}_r \cap Z(\mathcal{U}_{abc}) = (A[z] \cap Z(\mathcal{U}_{abc}))y^r$. Similarly, when $r < 0$, $\mathcal{U}_r \cap Z(\mathcal{U}_{abc}) \neq 0$ implies that r is a multiple of s. Since x^s commutes with $A[z]$ and $\mathcal{U}_x = A[z][x, x^{-1}; \alpha^{-1}]$, $x^s \in Z(\mathcal{U}_x) \cap \mathcal{U}_{abc} = Z(\mathcal{U}_{abc})$ and so we also have $\mathcal{U}_r \cap Z(\mathcal{U}_{abc}) = (A[z] \cap Z(\mathcal{U}_{abc}))x^r$. Thus $Z(\mathcal{U}_{abc})$ is equal to

$$\sum_{r>0}^{\oplus} (A[z] \cap Z(\mathcal{U}_{abc}))x^{rs} \oplus (A[z] \cap Z(\mathcal{U}_{abc})) \oplus \sum_{r>0}^{\oplus} (A[z] \cap Z(\mathcal{U}_{abc}))y^{rs}.$$

Notice that $k[g^n, z^l] \subseteq A[z] \cap Z(\mathcal{U}_{abc})$. It is clear that

$$A[z] = \sum_{i=0}^{n-1}{}^{\oplus} \sum_{j=0}^{l-1}{}^{\oplus} g^i z^j k[g^n, z^l]$$

as a $k[g^n, z^l]$-module. Choose i, j with $0 \leqslant i < n$, $0 \leqslant j < l$. Then $g^i z^j \in Z(\mathcal{U}_{abc})$ implies that $xg^i z^j = g^i z^j x$, which is equivalent to $a^{-i}c^j g^i z^j x = g^i z^j x$, i.e. $(i, j) \in \mathcal{B}$. Similarly, $yg^i z^j = g^i z^j y$ if and only if $a^i = c^j$, i.e. $(i, j) \in \mathcal{B}$. Therefore $g^i z^j \in Z(\mathcal{U}_{abc})$ if and only if $(i, j) \in \mathcal{B}$. Thus

$$Z_0 = \sum_{(i,j)\in\mathcal{B}}^{\oplus} g^i z^j k[g^n, z^l] \subseteq A[z] \cap Z(\mathcal{U}_{abc}).$$

Let $f \in A[z] \cap Z(\mathcal{U}_{abc})$. Since $\{g^i z^j : 0 \leqslant i < n,\ 0 \leqslant j < l\}$ form a k-basis for $A[z]$ over $k[g^n, z^l]$, and since $\{g^{r_1}z^{r_2}x^{r_3} : r_1, r_2, r_3 \geqslant 0\}$ are linearly independent over k, it is clear, on considering the relation $fx = xf$, that Z_0 is the whole of $A[z] \cap Z(\mathcal{U}_{abc})$. The result now follows. $\square$

5. Finite-dimensional irreducible representations of J-conformal $\mathcal{U}_{abc}$ for $a, b, c \in k$, $ac \neq 0$

Let $a, b, c \in k$, $ac \neq 0$ and suppose that $a \neq 1$. As was noted in Definition 3.9, g is a normal element of $\mathcal{U}_{abc}$; we consider the k-algebra $\overline{\mathcal{U}} := \mathcal{U}_{abc}/g\mathcal{U}_{abc}$. For $c \in k^*$, $c \neq 1$, recall the quantum plane $\Lambda_c = k\langle X, Y : XY = cYX\rangle$ and the quantum Weyl algebra $A_1^c = k\langle X, Y : XY - cYX = 1\rangle$.

Lemma 5.1. *Let $a, b, c \in k$ with $ac \neq 0$ and $a \neq 1$.*

(i) *Suppose that $b = a - 1$. When $c = 1$, $\overline{\mathcal{U}}$ is isomorphic to a commutative polynomial k-algebra in two indeterminates; when $c \neq 1$, $\overline{\mathcal{U}}$ is isomorphic to the quantum plane Λ_c.*

(ii) *Suppose that $b \neq a - 1$. When $c = 1$, $\overline{\mathcal{U}}$ is isomorphic to the first Weyl algebra; when $c \neq 1$, $\overline{\mathcal{U}}$ is isomorphic to the quantum Weyl algebra A_1^c.*

Proof. Let $X = x + g\mathcal{U}_{abc}$ and $Y = y + g\mathcal{U}_{abc}$ be elements of $\overline{\mathcal{U}}$. Then $\overline{\mathcal{U}}$ is generated by X and Y and, noting that t is equal to $(1-a)^{-1}$ modulo g, is subject to the relation $XY - cYX = (1-a)^{-1}(1 + b(1-a)^{-1})$. When $b = a - 1$ we have that $XY - cYX = 0$. Otherwise $XY - cYX$ is equal to a non-zero scalar, which we can assume without loss of generality to be 1. Hence the result. □

Remark 5.2. Let $a, b, c \in k$ with $ac \neq 0$. Suppose that $\mathcal{U}_{abc}$ is J-conformal. The aim of this section is to determine all the finite-dimensional irreducible $\mathcal{U}_{abc}$-modules. Consider the case $a \neq 1$. The finite-dimensional irreducible $\mathcal{U}_{abc}$-modules that are annihilated by g are precisely the finite-dimensional irreducible $\overline{\mathcal{U}}$-modules, and those that are not annihilated by g are precisely the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules. By Lemma 5.1, $\overline{\mathcal{U}}$ is a well-known algebra. When $c = 1$ the finite-dimensional irreducible representation theory of $\overline{\mathcal{U}}$ has long been known; when $c \neq 1$ the finite-dimensional irreducible representation theory of $\overline{\mathcal{U}}$ has been determined in [**2**], [**3**] and [**17**]. Therefore, to achieve our aim for $\mathcal{U}_{abc}$ when $a \neq 1$, it is the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules that we must calculate. The $a = 1$ case will be considered separately. We will be using the methods of [**10**], and for the reader's convenience we will state the results that we need here.

Theorem 5.3 (see 2.6, 3.3 and 3.4 in [10]). *Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be an arbitrary conformal ambiskew polynomial ring as in Definition 2.2, where T is a finitely generated commutative k-algebra and $w \in T$. Let $Z = XY - w$, and set $\sigma(Z) = \rho^{-1}Z$. For $r = Y$ or X, we say that a right R-module is r-torsion (respectively r-torsion free) if it is torsion (respectively torsion free) with respect to $\{r^i : i \geqslant 1\}$. A module which is both Y-torsion and X-torsion will be called XY-torsion. Then every finite-dimensional irreducible R-module is isomorphic to one of the following, for a suitable maximal ideal M of T; conversely, for M a maximal ideal of T, all of the following modules, when they exist, are irreducible.*

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(i) *A d-dimensional XY-torsion module*

$$L(M) = \frac{R}{MR + XR + Y^d R},$$

where $d > 0$ is minimal with $w - \rho^d \sigma^d(w) \in M$.

(ii) *An n-dimensional Y-torsion-free module*

$$C(M,\xi) = \frac{R}{MR + ZR + (Y^n - \xi)R},$$

or an n-dimensional X-torsion-free module

$$C^+(M,\xi) = \frac{R}{MR + ZR + (X^n - \xi)R},$$

where M has finite orbit of order n under σ and $0 \neq \xi \in k$.

(iii) *An s-dimensional Y-torsion-free module*

$$B(M,\xi,\eta) = \frac{R}{MR + (Z - \eta)R + (Y^s - \xi)R},$$

or an s-dimensional X-torsion-free module

$$B^+(M,\xi,\eta) = \frac{R}{MR + (Z - \eta)R + (X^s - \xi)R},$$

where M has finite orbit under σ, ρ is a root of unity, s is the least common multiple of the orders of M and ρ, and $\xi, \eta \in k^*$.

No pair of modules of different types is isomorphic.

Notation 5.4. Let $a, b, c \in k$ with $ac \neq 0$. Suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$).

(i) When $a = 1$, let $\mathcal{D}_A = \{M \in \mathrm{Maxspec}(A) : \exists\ d > 0$ minimal such that $w - c^d \alpha^d(w) \in M\}$. When $a \neq 1$, let $\mathcal{D}_S = \{M \in \mathrm{Maxspec}(S) : \exists\ d > 0$ minimal such that $w - c^d \alpha^d(w) \in M\}$.

(ii) Suppose that $a \neq 1$. Whenever M is a maximal ideal of S and $\eta \in k$, set $N_{M,\eta} = MS[z] + (z - \eta)S[z]$, a maximal ideal of $S[z]$.

(iii) Suppose that $a \neq 1$. For M a maximal ideal of S or of $S[z]$, set $\Omega(M) = \{\alpha^i(M) : i \in \mathbb{Z}\}$.

Proposition 5.5. *Let $a, b, c \in k$, $ac \neq 0$, $a \neq 1$, and suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). The set $\{L(M) : M \in \mathcal{D}_S\}$ form a complete and repetition-free list of the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form listed in Theorem 5.3 (i). When $b = 0$ there can exist no more than one, and when $b \neq 0$ no more than two, distinct n-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form $L(M)$ for each positive integer n.*

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Proof. As a polynomial in A, we have that $w - c\alpha(w)$ has degree one when $b = 0$, and degree 2 when $b \neq 0$. It can be shown by a straightforward induction argument that, for each positive integer n, the degree of $w - c^n\alpha^n(w)$ is always less than or equal to the degree of $w - c\alpha(w)$ (when $n > 1$, we write $w - c^n\alpha^n(w)$ as

$$w-c^{n-1}\alpha^{n-1}(w)+c^{n-1}\alpha^{n-1}(w)-c^n\alpha^n(w) = w-c^{n-1}\alpha^{n-1}(w)+c^{n-1}\alpha^{n-1}(w-c\alpha(w)),$$

and then apply the inductive step). The stated result is then immediate from Theorem 5.3, and the fact that $w - c^n\alpha^n(w)$ can lie in no more than one maximal ideal of S when $b = 0$, and no more than two maximal ideals of S when $b \neq 0$. □

Proposition 5.6. *Let $a, b, c \in k$, $ac \neq 0$, and let a be a non-trivial root of unity of multiplicative order $n > 1$ in k^*. Let $\mathcal{U}_{abc}$ be J-conformal (with respect to $w \in A$, of degree $m > 0$). Recall the notation introduced in Notations 3.2 and 5.4. Set $\mathcal{P} = \{\Omega(M) : M \in \operatorname{Maxspec}(S)\}$ and*

$$\mathcal{Q} = \left\{\Omega((t-\rho_i)S) : 1 \leqslant i \leqslant r(w),\ \rho_i \neq \frac{1}{1-a}\right\}.$$

(i) (a) *There is a bijective correspondence between distinct pairs $(\Gamma, \xi) \in \mathcal{P} \times k^*$ and n-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form $C(M, \xi)$. Denote this module $W_{\Gamma,\xi}$.*

(b) *There is a bijective correspondence between distinct pairs $(\Gamma, \xi) \in \mathcal{P} \times k^*$ and n-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form $C^+(M, \xi)$. Denote this module $W^+_{\Gamma,\xi}$.*

(ii) *Let $\Gamma \in \mathcal{P}$. Consider the isomorphism classes of $\mathcal{L}_{abc}$-modules $\mathcal{S}_1 = \{W_{\Gamma,\xi} : \xi \in k^*\}$ and $\mathcal{S}_2 = \{W^+_{\Gamma,\xi} : \xi \in k^*\}$.*

(a) *Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint if and only if $\Gamma \in \mathcal{Q}$, and coincide otherwise. Always, $1 \leqslant |\mathcal{Q}| \leqslant r(w)$ $(\leqslant m)$. However, when $m > 1$ with $b = a - 1$ and $c \neq 1$ we have that $|\mathcal{Q}| < r(w)$.*

(b) *Let $\Gamma' \in \mathcal{P}$. If $\Gamma \neq \Gamma'$, then $\mathcal{S}_1 \cap \{W^+_{\Gamma',\xi} : \xi \in k^*\} = \emptyset$.*

Proof. (i) Let $(\Gamma, \xi) \in \mathcal{P} \times k^*$. Then $\Gamma = \Omega(M)$ for some maximal ideal M of S. We have by Lemma 2.5 (v) that every maximal ideal of A not equal to gA has order n under α. Therefore, every maximal ideal of S has order n under α. Thus we can form the n-dimensional irreducible $\mathcal{L}_{abc}$-module $C(M, \xi)$. It is clear from [**10**, 3.5] that $C(M, \xi) \cong C(M', \xi')$ if and only if $\alpha^i(M) = M'$ for some $i \in \mathbb{Z}$ and $\xi = \xi'$. Therefore, the correspondence in (a) is bijective. Part (b) is similar.

(ii) Let $M \in \operatorname{Maxspec}(S)$ and $\xi \in k^*$. Then, by [**10**, 3.4], $C(M, \xi)$ is x-torsion free, i.e. $C(M, \xi) \cong V \in \{C^+(M', \xi') : \xi' \in k^*\}$ if and only if $\alpha^i(w) \notin M$ for all $i = 0, 1, \ldots, n-1$; in fact we can take, up to isomorphism, $V = C^+(M, \xi')$, for some $\xi' \in k^*$. By symmetry, $C^+(M, \xi)$ is y-torsion free if and only if $\alpha^i(w) \notin M$ for all $i = 0, 1, \ldots, n-1$. Hence, for $\Gamma = \Omega(M)$, the isomorphism classes $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint if and only if $\alpha^i(w) \in M$ for some $0 \leqslant i \leqslant n-1$, which is equivalent to $\prod_{j=0}^{n-1} \alpha^j(w) \in M$, which is

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equivalent to $\Gamma \in \mathcal{Q}$; $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide otherwise. Suppose $m = 1$. Then $b = 0$ and, since $a \neq 1$, g does not divide w by Lemma 3.5. Hence $\mathcal{Q} \neq \emptyset$, and so $|\mathcal{Q}| = 1$. Now suppose that $m > 1$. By Lemma 3.5, g^j does not divide w for all $j = 2, \dots, m$. Hence $1 \leqslant |\mathcal{Q}| \leqslant r(w)$. When $b = a - 1$ and $c \neq 1$, Lemma 3.5 states that g divides w, and so $|Q| < r(w)$ in this case. This proves (a). Let $\xi \in k^*$. By the above remarks, if $W_{\Gamma,\xi}$ is x-torsion free, then $W_{\Gamma,\xi} \cong W^+_{\Gamma,\xi'}$, for some $\xi' \in k^*$. Part (b) is now clear from the correspondence given in (i). □

Proposition 5.7. *Let $a, b, c \in k$ with a and c both roots of unity, of multiplicative orders $n > 1$ and $l \geqslant 1$ in k^*, respectively. Suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Let $s = \operatorname{lcm}(n, l)$. In the notation of 5.4, let $\mathcal{W} = \{\Omega(N_{M,\eta}) : M \in \operatorname{Maxspec}(S),\ 0 \neq \eta \in k\}$. Note that $|\Omega(N)| = s$ for all maximal ideals N of $S[z]$. For each $\eta \in k^*$, we set $\mathcal{M}(\eta) = \{M \in \operatorname{Maxspec}(S) : \alpha^j(w) + c^{-j}\eta \in M$ for some $j = 0, 1, \dots, s-1\}$, and note that $|\mathcal{M}(\eta)| \leqslant ms$.*

- (i) (a) *There is a bijective correspondence between distinct pairs $(\Gamma, \xi) \in \mathcal{W} \times k^*$ and isomorphism classes of s-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form $B(M, \xi, \eta)$. Denote the corresponding module $V_{\Gamma,\xi}$.*
  - (b) *There is a bijective correspondence between distinct pairs $(\Gamma, \xi) \in \mathcal{W} \times k^*$ and isomorphism classes of s-dimensional irreducible $\mathcal{L}_{abc}$-modules of the form $B^+(M, \xi, \eta)$. Denote the corresponding module $V^+_{\Gamma,\xi}$.*
- (ii) *Let $\Gamma \in \mathcal{W}$. Then $\Gamma = \Omega(N_{M,\eta})$ for some $M \in \operatorname{Maxspec}(S)$ and $\eta \in k^*$. Set $\mathcal{V}_1 = \{V_{\Gamma,\xi} : \xi \in k^*\}$ and $\mathcal{V}_2 = \{V^+_{\Gamma,\xi'} : \xi' \in k^*\}$.*
  - (a) *Then $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ if and only if $\Gamma \in \{\Omega(N_{M',\eta}) : M' \in \mathcal{M}(\eta)\}$, a set of no more than ms elements. Otherwise $\mathcal{V}_1 = \mathcal{V}_2$.*
  - (b) *Let $\Gamma' \in \mathcal{W}$. If $\Gamma \neq \Gamma'$, then $\mathcal{V}_1 \cap \{V^+_{\Gamma',\xi} : \xi \in k^*\} = \emptyset$.*

Proof. (i) Let $(\Gamma, \xi) \in \mathcal{W} \times k^*$. Then $W = \Omega(N_{M,\eta})$, for some $M \in \operatorname{Maxspec}(S)$ and $\eta \in k^*$. By Lemma 2.5 (v), M has finite order n under α. Hence we can form the s-dimensional irreducible $\mathcal{L}_{abc}$-module $B(M, \xi, \eta)$. Now, for some $M' \in \operatorname{Maxspec}(S)$ and $\xi', \eta' \in k^*$, we have by [**10**, 3.5] that $B(M, \xi, \eta) \cong B(M', \xi', \eta')$ if and only if $N_{M',\eta'} \in \Omega(N_{M,\eta})$ and $\xi = \xi'$. Therefore the correspondence in (a) is bijective. Part (b) is similar.

(ii) Let $M \in \operatorname{Maxspec}(S)$ and $\eta, \xi \in k^*$. Then it is implicit in [**10**, 3.1, 3.3] that $B(M, \xi, \eta)$ is x-torsion free, i.e. $B(M, \xi, \eta) \cong V \in \{B^+(M', \xi', \eta') : M' \in \operatorname{Maxspec}(S)$ and $\xi', \eta' \in k^*\}$ if and only if $\eta + c^i\alpha^i(w) \notin M$ for all $i = 0, 1, \dots, s-1$; in fact we can take, up to isomorphism, $V = B^+(M, \xi', \eta)$ for some $\xi' \in k^*$. By symmetry, $B^+(M, \xi, \eta)$ is y-torsion free if and only if $\eta + c^i\alpha^i(w) \notin M$, for all $i = 0, 1, \dots, s-1$. Hence the isomorphism classes $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint if and only if $\eta + c^j\alpha^j(w) \in M$ for some $0 \leqslant j \leqslant s-1$, i.e. $M \in \mathcal{M}(\eta)$, and coincide otherwise. Since $1 \leqslant |\mathcal{M}(\eta)| \leqslant ms$, (a) is now proved. Let $\xi \in k^*$. By the above remarks, if $V_{\Gamma,\xi}$ is x-torsion free, then $V_{\Gamma,\xi} \cong V^+_{\Gamma,\xi'}$, for some $\xi' \in k^*$. Part (b) is now clear by the correspondence given in (i). □

Remark 5.8. Let $a, b, c \in k$ with $ac \neq 0$ and suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Consider the remarks in 5.2. We can therefore split the task of determining all the finite-dimensional irreducible representations for $\mathcal{U}_{abc}$ into the following four subproblems, namely the determination of

(A) the finite-dimensional irreducible $\mathcal{U}_{1bc}$-modules;

(B) the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules, when $a \neq 1$ is not a root of unity;

(C) the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules when $a \neq 1$ is a root of unity, but c is not a root of unity; and

(D) the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules when $a \neq 1$ and both a and c are roots of unity.

The next four theorems give the answers to subproblems (A), (B), (C) and (D), respectively.

Theorem 5.9. *Let $b, c \in k$, with $c \neq 0$. Suppose that $\mathcal{U}_{1bc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Then $\{L(M) : M \in \mathcal{D}_A\}$ is a complete and repetition-free list of the finite-dimensional irreducible $\mathcal{U}_{1bc}$-modules. When $b = 0$ there can exist no more than one, and when $b \neq 0$ no more than two, distinct n-dimensional irreducible $\mathcal{U}_{1bc}$-modules for each positive integer n.*

Proof. By Lemma 2.5 (i), every maximal ideal of A has infinite orbit under α. By Theorem 5.3, every finite-dimensional irreducible $\mathcal{U}_{1bc}$-module is of the form stated. The result follows by arguing as in the proof of Proposition 5.5. □

Theorem 5.10. *Let $a, b, c \in k$, $ac \neq 0$, $a \neq 1$, and suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Suppose that a is not a root of unity. Then $\{L(M) : M \in \mathcal{D}_S\}$, as in Proposition 5.5, form a complete and repetition-free list of the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules.*

Proof. By Lemma 2.5 (iv) we have that every maximal ideal of A that is not equal to gA has infinite orbit under α. Therefore every maximal ideal of S has infinite orbit under α, and so, by Theorem 5.3, every finite-dimensional irreducible $\mathcal{L}_{abc}$-module is of the form $L(M)$ for some $M \in \mathcal{D}_S$. □

Theorem 5.11. *Let $a, b, c \in k$, $ac \neq 0$, $a \neq 1$, and suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Suppose that a is a root of unity of multiplicative order $n > 1$, and that c is not a root of unity. Then the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules are, without repetition,*

(i) *$\{L(M) : M \in \mathcal{D}_S\}$ as in Proposition 5.5 and*

(ii) *the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules listed in Proposition 5.6.*

Proof. Apply Theorem 5.3 and Propositions 5.5 and 5.6. □

Theorem 5.12. *Let $a, b, c \in k$, $ac \neq 0$, $a \neq 1$, and suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$). Suppose that a is a primitive nth root of unity, for some $n > 1$, and that c is a primitive lth root of unity, for some $l \geqslant 1$. Let $s = \operatorname{lcm}(l, n)$. Then the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules are, without repetition,*

(i) *$\{L(M) : M \in \mathcal{D}_S\}$ as in Proposition 5.5,*

(ii) *the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules listed in Proposition 5.6, and*

(iii) *the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules listed in Proposition 5.7.*

Proof. Apply Theorem 5.3 and Propositions 5.5, 5.6 and 5.7. □

Remark 5.13. Compare Theorems 5.11 and 5.12 with [**12**, Theorem 4.2.1], where it is stated that for generic $a, b, c \in k$ there exists at most two n-dimensional irreducible $\mathcal{U}_{abc}$-modules for each positive integer n, and with [**13**, Proposition 4.1.5], where it is stated that for generic a, b, c and $n > 1$ there will be precisely two n-dimensional irreducible $\mathcal{U}_{abc}$-modules.

6. Height one prime ideals of $\mathcal{U}_{abc}$ for $a, b, c \in k$, $ac \neq 0$ and a not a non-trivial root of unity

Remark 6.1. Let $a, b, c \in k$, $ac \neq 0$ with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal (with respect to $w \in A$, of degree $m > 0$).

(i) Consider the case $a \neq 1$. The set of prime ideals of $\mathcal{U}_{abc}$ that contain g is in 1–1 correspondence with the prime ideals of $\overline{\mathcal{U}}$. The set of prime ideals of $\mathcal{U}_{abc}$ that do not contain g is in 1–1 correspondence with the prime ideals of $\mathcal{L}_{abc}$. By Lemma 5.1, $\overline{\mathcal{U}}$ is always isomorphic to a familiar algebra, whose prime ideal structure is well known. Therefore, in order to determine $\operatorname{Spec}(\mathcal{U}_{abc})$ when a is not a root of unity, it will be enough to determine $\operatorname{Spec}(\mathcal{L}_{abc})$.

(ii) We split the study of the prime ideals of $\mathcal{U}_{abc}$ when a is not a non-trivial root of unity into the following subcases.

(a) c is a primitive lth root of unity, for some $l \geqslant 1$. When $a = 1$, set $R = \mathcal{U}_{1bc}$. When $a \neq 1$, set $R = \mathcal{L}_{abc}$. We study $\operatorname{Spec}(R)$.

(b) c is not a root of unity, and for all $i, j \in \mathbb{Z}$ with $i > 0$ we have that $c^i \neq a^j$. When $a = 1$, set $R = \mathcal{U}_{1bc}$. When $a \neq 1$, set $R = \mathcal{L}_{abc}$. We study $\operatorname{Spec}(R)$.

(c) c is not a root of unity, but there exists $N \geqslant 1$ minimal such that $c^N = a^l$ for some (necessarily unique) non-zero integer l. Note that $a \neq 1$ in this case. We study $\operatorname{Spec}(\mathcal{L}_{abc})$ for the two cases $l > 0$ and $l < 0$.

Definition 6.2 (see 2.10 in [9]). Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be a conformal ambiskew polynomial ring (as in Definition 2.2), where T is a commutative domain

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which is a finitely generated k-algebra, and w is a non-zero element of T. For each prime ideal P of T we can form the right R-module $V(P)$ which, as a T-module, can be written

$$V(P) = \sum_{i \geqslant 0}^{\oplus} T/\sigma^{-i}(P)$$

and which, for all $i \geqslant 0$ and $h \in T$, has R-module action

$$(h + \sigma^{-i}(P))X = \sigma(h)(\sigma(w) - \rho^{-i}\sigma^{-(i-1)}(w)) + \sigma^{-(i-1)}(P)$$

and

$$(h + \sigma^{-i}(P))Y = \sigma^{-1}(h) + \sigma^{-(i+1)}(P).$$

Observe that $V(P)$ is isomorphic to $R/PR + XR$ as a right R-module via the map that takes, for each $i \geqslant 0$, $1 + \sigma^{-i}(P)$ to the right coset $Y^i + PR + XR$. Suppose that there exists $j > 0$ minimal such that $w - \rho^j\sigma^j(w) \in P$. Then we have that

$$\sum_{i \geqslant j}^{\oplus} T/\sigma^{-i}(P)$$

is an R-submodule of $V(P)$. We denote the corresponding factor module by $L(P)$ and set $Q(P) := \mathrm{Ann}_R(L(P))$.

Remark 6.3. Let $a, b, c \in k$, $ac \neq 0$ with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$.

(i) Suppose that $a \neq 1$. Let $M \in \mathcal{D}_S$. Then the $\mathcal{L}_{abc}$-module $L(M)$ as defined in Remark 6.1 is, by the remarks in [**11**, 2.1], isomorphic to the finite-dimensional irreducible $\mathcal{L}_{abc}$-module $L(M)$ as defined in Theorem 5.3 (i), and so there is no clash of notation. Analogous statements are true when $a = 1$ regarding the equivalence, up to isomorphism, of the two definitions of the $\mathcal{U}_{1bc}$-module $L(M)$ for some $M \in \mathcal{D}_A$.

(ii) Recall from Theorem 5.9 that $\{L(M) : M \in \mathcal{D}_A\}$ is a complete and repetition-free list of the finite-dimensional irreducible $\mathcal{U}_{1bc}$-modules, and from Theorem 5.10 (when $a \neq 1$) that $\{L(M) : M \in \mathcal{D}_S\}$ is a complete and repetition-free list of the finite-dimensional irreducible $\mathcal{L}_{abc}$-modules. Therefore, in view of (i), $\{Q(M) : M \in \mathcal{D}_A\}$ are precisely all the prime ideals of $\mathcal{U}_{1bc}$ with finite codimension; when $a \neq 1$, $\{Q(M) : M \in \mathcal{D}_S\}$ are precisely all the prime ideals of $\mathcal{L}_{abc}$ with finite codimension.

(iii) Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be a conformal ambiskew polynomial ring where T is a commutative domain which is a finitely generated k-algebra, and $0 \neq w \in T$. Let P be a prime ideal of T. Then, by [**9**, 2.12], the prime ideal $Q(P)$ of R, if it exists, is non-principal. In particular, by the remarks in (ii), all the prime ideals of $\mathcal{U}_{1bc}$ with finite codimension are non-principal ideals; when $a \neq 1$ all the prime ideals of $\mathcal{L}_{abc}$ with finite codimension are non-principal ideals.

Theorem 6.4 (see 2.17 in [9]). *Let $R = R(T, \sigma, w - \rho\sigma(w), \rho)$ be a conformal ambiskew polynomial ring (as in Definition 2.2) where T is a commutative domain which* *is a finitely generated k-algebra, and $0 \neq w \in T$; let Z denote the corresponding Casimir element. Suppose that $T \neq k$ and that T is σ-simple. Then the height one prime ideals of R are as follows:*

(i) *ZR;*

(ii) *$(vZ^n - \varepsilon)R$ when there is a principle eigenvector v with degree n, $\varepsilon \in k^*$ and $v(-w)^n \neq \varepsilon$;*

(iii) *Y^nR and X^nR when w is a unit and w^{-n} is a principal eigenvector with degree n; and*

(iv) *a non-principal ideal $Q(P)$ for each height one prime P of T such that $w - \rho^d\sigma^d(w) \in P$ for some $d > 0$, $w \notin P$ and, if v is a principal eigenvector with degree n, $v(-u)^n - \varepsilon \notin P$ for all $\varepsilon \in k^*$.*

Notation 6.5. Let $a, b, c \in k$, $ac \neq 0$, and suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. Recall the notation of 3.2. Define $\mathcal{H}_A := \mathcal{D}_A \cap \{(t - \rho_i)A : 1 \leqslant i \leqslant r(w)\}$. When $a \neq 1$, define $\mathcal{H}_S := \mathcal{D}_S \cap \{(t - \rho_i)S : 1 \leqslant i \leqslant r(w)\}$.

Theorem 6.6. *Let $a, b, c \in k$, $ac \neq 0$, and suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$.*

(i) *Suppose that c is a primitive lth root of unity, where $l \geqslant 1$. When $a = 1$, set $R = \mathcal{U}_{1bc}$ (and $T = A$); when $a \neq 1$, set $R = \mathcal{L}_{abc}$ (and $T = S$). Note that $Z(R) = k[z^l]$. Then $\{zR\} \cup \{(z^l - \gamma)R : 0 \neq \gamma \in k\}$ is a complete, repetition-free list of the height one prime ideals of R.*

(ii) *Suppose that c is not a root of unity and that, for all $i, j \in \mathbb{Z}$ with $i > 0$ and $j \neq 0$, we have that $c^i \neq a^j$. When $a = 1$, set $R = \mathcal{U}_{1bc}$ (and $T = A$); when $a \neq 1$, set $R = \mathcal{L}_{abc}$ (and $T = S$). Note that $Z(R) = k$. Then the height one prime ideals of R are, without repetition, $\{zR\} \cup \{\mathrm{Ann}_R(L(M)) : M \in \mathcal{D}_T \setminus \mathcal{H}_T\}$.*

(iii) *Suppose that c is not a root of unity, and that there exists an integer $N \geqslant 1$ minimal such that $c^N = a^l$ for some (necessarily unique) non-zero $l \in \mathbb{Z}$. Note that $Z(\mathcal{L}_{abc}) = k[g^l z^N]$. Then the set $\{z\mathcal{L}_{abc}\} \cup \{(g^l z^N - \gamma)\mathcal{L}_{abc} : 0 \neq \gamma \in k\}$ is a complete and repetition-free list of the height one prime ideals of $\mathcal{L}_{abc}$.*

Proof. (i) In the case $a = 1$, 1 is a principal eigenvector of degree l, by [**9**, 1.7(ii)]. The α-simplicity of A then implies that k^* is a complete list of principal eigenvectors of A, each of degree l. In the case $a \neq 1$, we have by Lemma 4.3 (ii) that principal eigenvectors of S exist and are precisely the non-zero elements of k, each of degree l. By Theorem 6.4 (i), zR is a height one prime ideal of R. Choose any $\lambda \in k^*$. Then λ is a principal eigenvector of degree l and, since $\lambda(-w)^l \notin k^*$, $(\lambda z^l - \mu)R$ is a height one prime ideal of R, by Theorem 6.4 (ii), for all $\mu \in k^*$. Setting $\lambda = 1$, we have that $\{(z^l - \gamma)R : 0 \neq \gamma \in k\}$ is a complete, repetition-free list of the height one prime ideals of R of the type corresponding to Theorem 6.4 (ii). It is clear that w is not a unit in A. Suppose that $a \neq 1$. Note that the integer powers of g are, up to scalar, precisely the units of S. Suppose $m = 1$. Since it is impossible for g to divide w in this case, w cannot be a unit in S. Suppose $m > 1$. Then it follows from Lemma 3.5 that w cannot be a unit in S. Thus w is not a unit in T. Hence there do not exist any height one primes of R of the type listed in Theorem 6.4 (iii). Since T is an α-simple domain, of Krull dimension one, and possesses principal eigenvectors, all height one prime ideals of R are principal, by [**9**, 2.18]. Thus by Theorem 6.4 (iv) there are no more height one prime ideals of R.

(ii) Suppose that $a = 1$. Then it was shown in the proof of Theorem 4.6 (i) (a) that there do not exist principal eigenvectors of A. Suppose that $a \neq 1$. Then, by Lemma 4.3 (i) (a), principal eigenvectors of S do not exist. Thus T does not have principal eigenvectors, and so the height one prime ideals of R are zR and those non-principal height one primes listed in Theorem 6.4 (iv). Now, the height one prime ideals of T are precisely the maximal ideals of T. Thus, noting 6.3 (ii) and (iii), we have by Theorem 6.4 that $\{\mathrm{Ann}_R(L(M)) : M \in \mathcal{D}_T \text{ and } w \notin M\}$ is a complete and repetition-free list of the non-principal height one prime ideals of R. The result follows by the definition of $\mathcal{H}_T$.

(iii) By Lemma 4.3 (i) (b) principal eigenvectors of S exist; they are precisely the elements of the form $\{\lambda g^l : 0 \neq \lambda \in k\}$ and are of degree N. There exist height one prime ideals of $\mathcal{L}_{abc}$ of the type listed in Theorem 6.4 (iii) only if w is a unit in S. By the proof of part (i) we know that w is not a unit of S. Hence there do not exist any height one primes of $\mathcal{L}_{abc}$ of the type listed in Theorem 6.4 (iii). Since S is an α-simple domain, of Krull dimension one, and possesses principal eigenvectors, there are no non-principal height one prime ideals of $\mathcal{L}_{abc}$, by [**9**, 2.18]. Therefore, all the height one prime ideals of $\mathcal{L}_{abc}$ are of the types listed in Theorem 6.4 (i) and (ii). □

7. Prime spectrum of J-conformal $\mathcal{U}_{abc}$ with $a, b, c \in k$, $ac \neq 0$ and a not a non-trivial root of unity

Definition 7.1. Let R be a ring. Let P be a height one prime ideal of R. We set $\Gamma_P = \{Q \in \mathrm{Spec}(R) : P \subsetneq Q\}$.

Theorem 7.2. *Let $a, b, c \in k$, $ac \neq 0$, with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. When $a = 1$ set $R = \mathcal{U}_{1bc}$ (and $T = A$); when $a \neq 1$, set $R = \mathcal{L}_{abc}$ (and $T = S$).*

(i) *Let P be a height one prime ideal of R. Then either Γ_P is empty, or $\Gamma_P \subseteq \{\mathrm{Ann}_R(L(M)) : M \in \mathcal{D}_T\}$.*

(ii) *Let P and Q be distinct height one prime ideals of R. Then $\Gamma_P \cap \Gamma_Q = \emptyset$.*

Proof. We have that $R = R(T, \alpha, w - c\alpha(w), c)$. (i) Let P be a height one prime ideal of R and suppose that there exists a prime ideal Q of R such that $P \subsetneq Q$. Then Q/P is a non-zero, proper ideal of R/P. Since T has principal eigenvectors, the localizations of R/P with respect to the right denominator sets $\{x^i + P : i \geqslant 0\}$ and $\{y^i + P : i \geqslant 0\}$, respectively, are both simple rings, by [**9**, 2.3]. Therefore there exists an integer $n \geqslant 1$ such that $x^n + P, y^n + P \in Q/P$, and so $x^n, y^n \in Q$. By [**9**, 2.12] there exists $I \in \mathrm{Spec}(T)$ satisfying the conditions for the existence of $Q(I)$, as in Definition 6.2, and such that

$Q(I) \subseteq Q$. Suppose that $I = 0$. Then $w = c^d \alpha^d(w)$ for some $d > 0$, and so wA is an ideal of $A(\subseteq T)$ of finite orbit under σ. However, when $a = 1$ this is impossible, by Corollary 2.6. When $a \neq 1$ we must have, by Corollary 2.6, that $w = \lambda g^m$, for some $\lambda \in k^*$. However, this is clearly impossible when $m = 1$, and for $m > 1$, by Lemma 3.5. Thus in both cases we have that I is non-zero, and since $K\dim(T) = 1$, I is a maximal ideal of T; so $I \in \mathcal{D}_T$. Therefore $Q(I)$ is the annihilator of a finite-dimensional R-module, by the definition of $Q(I)$ and Theorem 5.3. Therefore $Q(I) \in \operatorname{Maxspec}(R)$, and $Q = Q(I)$.

(ii) Recall Theorem 4.6. Suppose that $Z(R) = k$. Then by Theorem 6.6 (ii) there is a unique height one prime ideal P' of R for which $\varGamma_{P'} \neq \emptyset$ is not an impossibility. The result therefore holds in this case. Now suppose that $Z(R) \neq k$. By Theorem 4.6, $Z(R) = k[\varOmega]$, where $\varOmega = g^l z^N$ (only if $a \neq 1$) for some non-zero integer l and positive integer N, or $\varOmega = z^l$ for some positive integer l. Suppose that $\varGamma_P \cap \varGamma_Q \neq \emptyset$, and let $Q' \in \varGamma_P \cap \varGamma_Q$. Then, by part (i), $Q' = \operatorname{Ann}_R(L(M))$ for some $M \in \mathcal{D}_T$. By Schur's Lemma, $\varOmega$ acts on $L(M)$ as multiplication by a unique scalar $\delta \in k$. On inspection of the height one prime ideals of R listed in Theorem 6.6 (i) and (ii), Q' can only contain a unique height one prime ideal: zR (if $\delta = 0$) or $(\varOmega - \delta)R$ (if $\delta \neq 0$). Therefore P and Q must be equal, which is a contradiction. Hence $\varGamma_P \cap \varGamma_Q = \emptyset$. $\square$

Theorem 7.3. *Let $a, b, c \in k$, $ac \neq 0$, with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. When $a = 1$, set $R = \mathcal{U}_{1bc}$ (and $T = A$); when $a \neq 1$ set $R = \mathcal{L}_{abc}$ (and $T = S$).*

(i) $\varGamma_{zR} = \{\operatorname{Ann}_R(L(M)) : M \in \mathcal{H}_T\}$.

(ii) $0 \leqslant |\varGamma_{zR}| \leqslant r(w) - 1$.

Proof. (i) Suppose that $zR \notin \operatorname{Maxspec}(R)$. Let $M \in \mathcal{D}_T$. By [**9**, 2.13(i)], $z \in \operatorname{Ann}_R(L(M))$ if and only if $w \in M$, that is, $M \in \mathcal{H}_T$. Part (i) is now immediate from Theorem 7.2 (i).

(ii) By part (i), it will suffice to show that $0 \leqslant |\mathcal{H}_T| \leqslant r(w) - 1$. Recall that every maximal ideal of T is of infinite order under α. It is clear that for $M \in \operatorname{Maxspec}(T)$, $w \in M$ if and only if $M \in \{(t - \rho_i)T : 1 \leqslant i \leqslant r(w)\}$. Let $r(w) = 1$; so $w = \lambda(t - \rho_1)^m$ for some $\lambda \in k^*$. Suppose that $(t - \rho_1)T \in \mathcal{D}_T$. Then there exists $d > 0$ such that $w - c^d \alpha^d(w) \in (t - \rho_1)T$; therefore $\alpha^d(w) \in (t - \rho_1)T$, i.e. $w \in \alpha^{-d}((t - \rho_1)T)$. However, this means that $\alpha^{-d}((t - \rho_1)T) = (t - \rho_1)T$, a contradiction. Hence $(t - \rho_1)T \notin \mathcal{D}_T$. Now let $r(w) > 1$. Suppose that $|\mathcal{H}_T| = r(w)$. Then $(t - \rho_1)T \in \mathcal{D}_T$, and so there exists $d_1 > 0$ minimal such that $w - c^{d_1} \alpha^{d_1}(w) \in (t - \rho_1)T$. Therefore $\alpha^{d_1}(w) \in (t - \rho_1)T$, and so $w \in \alpha^{-d_1}((t - \rho_1)T)$. Hence we must have $\alpha^{-d_1}((t - \rho_1)T) = (t - \rho_i)T$, for some $2 \leqslant i \leqslant r(w)$. Without loss of generality we take $i = 2$. By hypothesis there exists $d_2 > 0$ minimal such that $w - c^{d_2} \alpha^{d_2}(w) \in (t - \rho_2)T$. Therefore $\alpha^{d_2}(w) \in (t - \rho_2)T$, and so $w \in \alpha^{-d_2}((t - \rho_2)T)$. Hence there exists $1 \leqslant i \leqslant r(w)$, $i \neq 2$, such that $(t - \rho_i)T = \alpha^{-d_2}((t - \rho_2)T) = \alpha^{-d_2 - d_1}((t - \rho_1)T)$. If $r(w) = 2$ we must have $i = 1$, which implies that $(t - \rho_1)T$ has finite order under α, a contradiction. Hence $0 \leqslant |\mathcal{H}_T| \leqslant 1$. Suppose that $r(w) > 2$. By continuing in this manner we have that

$$(t - \rho_{r(w)})T = \alpha^{-d_{r(w)-1}}((t - \rho_{r(w)-1})T) = \cdots = \alpha^{-d_{r(w)-1} - \cdots - d_2 - d_1}((t - \rho_1)T),$$

where each $d_j > 0$. This means that for each $j = 1, \ldots, r(w) - 1$, there exists $0 < e_j \in \mathbb{Z}$ such that $(t - \rho_{r(w)})T = \alpha^{-e_j}((t - \rho_j)T)$. However, $(t - \rho_{r(w)})T \in \mathcal{D}_T$, by our hypothesis. So there exists an integer $d_{r(w)} > 0$ such that $w \in \alpha^{-d_{r(w)}}((t - \rho_{r(w)})T)$, that is $\alpha^{-d_{r(w)}}((t - \rho_{r(w)})T) = (t - \rho_l)T$ for some $1 \leqslant l \leqslant r(w)-1$. Hence $(t-\rho_{r(w)})T$ is equal to $\alpha^{-e_l-d_{r(w)}}((t - \rho_{r(w)})T)$, which is a contradiction. Therefore $0 \leqslant |\mathcal{H}_T| \leqslant r(w)-1$ when $r(w) > 2$, and this completes the proof. $\square$

Lemma 7.4. *Let $a, b, c \in k$, $ac \neq 0$, with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. When $a = 1$, set $R = \mathcal{U}_{1bc}$ (and $T = A$); when $a \neq 1$, set $R = \mathcal{L}_{abc}$ (and $T = S$). Choose $M \in \mathcal{D}_T$ and suppose that $\dim_k L(M) = d$. Recall from Theorem 5.3 (i) that $L(M) = R/I$, where $I = MR + xR + y^dR$ is a right ideal of R. Then, for each positive integer n, $z^n + I = (-1)^n w^n + I$.*

Proof. Recall that $z = xy - w$. Since $(xy)w = w(xy)$, we have $z^n = (xy - w)^n = xf + (-1)^n w^n$ for some $f \in R$. Therefore $z^n + I = (-1)^n w^n + I$. $\square$

We can now summarize our results. As in § 6, we consider the three cases (a), (b) and (c) of Remark 6.1 (ii) in turn.

Theorem 7.5. *Let $a, b, c \in k$, $ac \neq 0$, with a not a non-trivial root of unity. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. Suppose that c is a root of unity, of multiplicative order $l > 0$ in k^*. When $a = 1$, set $T = A$; when $a \neq 1$, set $T = S$. For every $\gamma \in k$, set $h_{l,\gamma}(t) = (-1)^l w^l - \gamma \in k[t]$ and $\mathcal{D}_T(\gamma) = \mathcal{D}_T \cap \{(t - \mu)T : h_{l,\gamma}(\mu) = 0\}$. When $a = 1$, set $R = \mathcal{U}_{1bc}$; when $a \neq 1$, set $R = \mathcal{L}_{abc}$.*

(i) *The zero ideal, the height one prime ideals of R listed in Theorem 6.6 (i), the set $\{\mathrm{Ann}_R(L(M)) : M \in \mathcal{D}_T \setminus \mathcal{H}_T\}$ together with, when they exist, the finite collection of prime ideals of R strictly containing zR, as in Theorem 7.3, form a complete and repetition-free list of the prime ideals of R.*

(ii) *For each $\gamma \in k$, let $P_\gamma = (z^l - \gamma)R$. Then $\Gamma_{P_\gamma} = \{\mathrm{Ann}_R(L(M)) : M \in \mathcal{D}_T(\gamma)\}$, and so $0 \leqslant |\Gamma_{P_\gamma}| \leqslant lm$.*

Proof. (i) That all the listed ideals of R are prime is clear from Theorem 6.6 (i) and Remark 6.3 (ii). That there are no other prime ideals is a consequence of Theorem 7.2.

(ii) Recall Theorem 7.2. Let $M \in \mathcal{D}_T$; so $M = (t - \mu)T$, for some $\mu \in k$. Recall that $L(M) = R/I$, where I is the right ideal of R as in the statement of Lemma 7.4. By Schur's Lemma, z^l acts on $L(M)$ as multiplication by some scalar $\gamma \in k$, and so $z^l - \gamma \in \mathrm{Ann}_R(L(M))$. In fact, by centrality, $z^l - \gamma \in \mathrm{Ann}_R(L(M))$ if and only $z^l - \gamma \in I$. By Lemma 7.4 this is equivalent to $(-1)^l w^l - \gamma \in I \cap T = M$, which can occur if and only if $h_{l,\gamma}(\mu) = 0$. That Γ_{P_γ} is as stated for each $\gamma \in k$ is now proved. Since the polynomial $h_{l,\gamma}(t)$ can have no more than lm distinct roots in k, the result follows. $\square$

Theorem 7.6. *Let $a, b, c \in k$, $ac \neq 0$, with a not a non-trivial root of unity and c not a root of unity, such that, for each $i, j \in \mathbb{Z}$ with $i > 0$ and $j \neq 0$, we have $c^i \neq a^j$. Suppose that $\mathcal{U}_{abc}$ is J-conformal with respect to $w \in A$, of degree $m > 0$. When $a = 1$,* *set* $R = \mathcal{U}_{1bc}$ *(and* $T = A$*); when* $a \neq 1$*, set* $R = \mathcal{L}_{abc}$ *(and* $T = S$*). Then the zero ideal, the height one prime ideals listed in Theorem 6.4 (ii) and, when they exist, the finite collection of prime ideals of* R *strictly containing* zR*, as in Theorem 7.3, form a complete and repetition-free list of the prime ideals of* R*.*

Proof. That all the listed ideals of R are prime is clear from Theorem 6.6 (ii) and Remark 6.3 (ii). That there are no other prime ideals is a consequence of Theorem 7.2. □

Theorem 7.7. *Let* $a, b, c \in k$*,* $ac \neq 0$*, with* a *and* c *non-roots of unity. Suppose that* $\mathcal{U}_{abc}$ *is J-conformal with respect to* $w \in A$*, of degree* $m > 0$*. Suppose that there exists an integer* $N \geqslant 1$ *minimal such that* $c^N = a^l$ *for some (necessarily unique) non-zero integer* $l > 0$*. For every* $\gamma \in k$*, set* $h^+_{N,l,\gamma}(t) = (-1)^N g^l w^N - \gamma \in k[t]$ *and* $\mathcal{D}^+(\gamma) = \mathcal{D}_S \cap \{(t-\mu)S : h^+_{N,l,\gamma}(\mu) = 0\}$*.*

(i) *The zero ideal, the height one prime ideals of* $\mathcal{L}_{abc}$ *listed in Theorem 6.6 (iii), the set* $\{\mathrm{Ann}_{\mathcal{L}_{abc}}(L(M)) : M \in \mathcal{D}_S \setminus \mathcal{H}_S\}$ *and, when they exist, the finite collection of prime ideals of* $\mathcal{L}_{abc}$ *strictly containing* $z\mathcal{L}_{abc}$*, as in Theorem 7.3, form a complete and repetition-free list of the prime ideals of* $\mathcal{L}_{abc}$*.*

(ii) *For each* $\gamma \in k^*$*, let* $P_\gamma = (g^l z^N - \gamma)\mathcal{L}_{abc}$*. Then we have that*

$$\Gamma_{P_\gamma} = \{\mathrm{Ann}_{\mathcal{L}_{abc}}(L(M)) : M \in \mathcal{D}^+(\gamma)\},$$

and $0 \leqslant |\Gamma_{P_\gamma}| \leqslant l + mN$*.*

Proof. Similar to the proof of Theorem 7.5. □

Theorem 7.8. *Let* $a, b, c \in k$*,* $ac \neq 0$*, with* a *and* c *non-roots of unity. Suppose that* $\mathcal{U}_{abc}$ *is J-conformal with respect to* $w \in A$*, of degree* $m > 0$*. Suppose that there exists an integer* $N \geqslant 1$ *minimal such that* $c^N = a^l$ *for some (necessarily unique) non-zero integer* $l < 0$*. For every* $\gamma \in k$*, set* $h^-_{N,l,\gamma}(t) = (-1)^N w^N - \gamma g^{-l} \in k[t]$ *and* $\mathcal{D}^-(\gamma) = \mathcal{D}_S \cap \{(t-\mu)S : h^-_{N,l,\gamma}(\mu) = 0\}$*.*

(i) *The zero ideal, the height one prime ideals of* $\mathcal{L}_{abc}$ *listed in Theorem 6.6 (iii) and* $\{\mathrm{Ann}_{\mathcal{L}_{abc}}(L(M)) : M \in \mathcal{D}_S \setminus \mathcal{H}_S\}$ *and, when they exist, the finite collection of prime ideals of* $\mathcal{L}_{abc}$ *strictly containing* $z\mathcal{L}_{abc}$*, as in Theorem 7.3, form a complete and repetition-free list of the prime ideals of* $\mathcal{L}_{abc}$*.*

(ii) *For each* $\gamma \in k$*, let* $P_\gamma = (g^l z^N - \gamma)S$*. Then* $\Gamma_{P_\gamma} = \{\mathrm{Ann}_{\mathcal{L}_{abc}}(L(M)) : M \in \mathcal{D}^-(\gamma)\}$*, and so* $0 \leqslant |\Gamma_{P_\gamma}| \leqslant \max\{-l, mN\}$*.*

Proof. Similar to the proof of Theorem 7.5. □

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