K. Mimachi
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CONNECTION PROBLEM IN HOLONOMIC q-DIFFERENCE SYSTEM ASSOCIATED WITH A JACKSON INTEGRAL OF JORDAN-POCHHAMMER TYPE

KATSUHISA MIMACHI

§ 0. Introduction

Fix a complex number q with $|q| < 1$. Let $T_1, \cdots, T_n$ be n-commuting q-difference operators defined by

$$T_j f(x_1, \cdots, x_n) = f(x_1, \cdots, qx_j, \cdots, x_n)$$

for a function $f(x)$, $x = (x_1, \cdots, x_n) \in (C^*)^n$. Consider a system of linear q-difference equations in several variables for a matrix valued function $\Xi(x)$ on $(C^*)^n$ as follows:

$$T_i \Xi(x) = \Xi(x) A_i(x) \qquad (1 \leq i \leq n). \tag{1}$$

We assume that each $A_i(x)$ is a matrix valued rational function satisfying the following conditions:

$$A_i(x) T_i A_j(x) = A_j(x) T_j A_i(x) \qquad (1 \leq i, j \leq n). \tag{2}$$

Then (1) defines a *holonomic q-difference system*. It is known [3] that there is a solution of the system (1) characterized by asymptotic behavior at a boundary point of $(C^*)^n$. More precisely, we denote by $L_\nu(x) = \{(q^{\nu_1 m} x_1, \cdots, q^{\nu_n m} x_n) \,|\, m \in Z\}$ the trajectory through $x \in (C^*)^n$ of the transformation $(y_1, \cdots, y_n) \to (q^{\nu_1} y_1, \cdots, q^{\nu_n} y_n)$ which is determined by integers $\nu = (\nu_1, \cdots, \nu_n)$. Under Ass 1 and Ass 2 in the direction L_ν for $A_i(x)$ stated in [3] the above solution denoted by $\Xi_\nu(x)$ is characterized by the following asymptotic behavior along L_ν at $m = \infty$:

$$\Xi_\nu(x) \sim \bar{x}_1^{\Lambda_1} \bar{x}_2^{\Lambda_2} \cdots \bar{x}_n^{\Lambda_n} \cdot U_\nu,$$
$$\bar{x}_1 = q^{\nu_1 m} x_1, \cdots, \bar{x}_n = q^{\nu_n m} x_n, \qquad \text{as } m \to \infty,$$

where U_ν denotes a certain non-singular lower triangular matrix which

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is constant, and $\Lambda_1, \cdots, \Lambda_n$ denote constant diagonal matrices. We call $\Xi_\nu(x)$ the solution determined along the trajectory L_ν. We take two trajectories L_ν and L_μ for two sequences of integers ν and μ. Then there exists a linear relation between the corresponding solutions $\Xi_\nu(x)$ and $\Xi_\mu(x)$ along the trajectories L_ν and L_μ respectively as follows:

$$\Xi_\nu(x) = P_{\nu,\mu}(x)\Xi_\mu(x),$$

where $P_{\nu,\mu}(x)$ denotes a matrix valued function satisfying

$$T_j P_{\nu,\mu}(x) = P_{\nu,\mu}(x) \qquad (1 \le j \le n).$$

The matrix $P_{\nu,\mu}(x)$ is called a *connection matrix* between the two solutions determined by the trajectories L_ν and L_μ.

The main purpose of this paper is to solve the connection problem, namely to compute the matrices $P_{\nu,\mu}(x)$, in holonomic q-difference system associated with a *Jackson integral* of *Jordan-Pochhammer type* under a generic condition. Jordan-Pochhammer type is a natural extension of Heine's basic hypergeometric series.

The contents of this paper are as follows. Section 1 gives basic notation in the q-analysis and a short review of a system associated with a Jackson integral of Jordan-Pochhammer type. Section 2 is devoted to give relations among Jackson integrals over suitable q-intervals of the first kind, which play a key role in our argument. We remark that, as a bonus of these relations, a connection formula of the basic hypergeometric series is obtained. In Section 3 we compute asymptotic behavior of the solutions along generic trajectories, and solve the corresponding connection problem.

§ 1. Preliminaries

Fix a complex number q with $|q| < 1$. Following F.H. Jackson [12], for a nonzero complex number $c \in \boldsymbol{C}^*$, define on a half-line $[0, c]$

$$\int_{[0,c]} F(t)d_qt = \int_0^c F(t)d_qt = c(1-q)\sum_{n\ge 0} F(cq^n)q^n,$$

which is a q-analogue of the Riemann integral and is called a Jackson integral. We also consider a Jackson integral on a whole line $[0, \infty(s)]$

$$\int_{[0,\infty(s)]} F(t)d_qt = \int_0^{\infty(s)} F(t)d_qt = s(1-q)\sum_{-\infty\le n\le +\infty} F(sq^n)q^n,$$

for a complex number $s \in \boldsymbol{C}^*$. We shall call $[0, c]$ a *q-interval* of the first kind or of the second kind according as $c \in C^*$ or $c = \infty(s)$. The following is easily deduced.

$$\int_0^c F(t)d_qt = c\int_0^1 F(ct)d_qt .$$

Here we define a *Jackson integral of Jordan-Pochhammer type* by

$$\int_{\mathscr{C}} t^{\alpha_0 - 1} \prod_{1\le j\le n} \frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty} d_qt , \tag{3}$$

where $(a)_\infty = \prod_{n\ge 0}(1 - aq^n)$, $\alpha_j \in C$ $(0 \le j \le n)$ and $\mathscr{C}$ denotes a suitable q-interval. The Jackson integral (3) tends to a Jordan-Pochhammer type integral:

$$\int t^{\alpha_0 - 1} \prod_{1\le j\le n} \left(1 - \frac{t}{x_j}\right)^{\alpha_j} dt ,$$

as $q \to 1$. In fact, if $(a)_n = (a)_\infty/(aq^n)_\infty$, then we have

$$\lim_{q\to 1} \frac{(t)_\infty}{(q^\alpha t)_\infty} = (1 - t)^\alpha ,$$

by the q-binomial theorem ([2], [7])

$$\sum_{n\ge 0} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty} . \tag{4}$$

The *holonomic q-difference system* associated with a Jackson integral of Jordan-Pochhammer type is given as follows. Set

$$\Phi(t) = t^{\alpha_0 - 1} \prod_{0\le j\le n} \frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty} ,$$
$$\Phi_j(t) = \Phi(t)/(1 - t/x_j) .$$

The following lemma has been communicated to the author by Prof. K. Aomoto ([4]):

LEMMA 1. 1) *A holonomic q-difference system for the function $\int \Phi d_qt$ can be derived in an explicit way*

$$T_k\left(\int \Phi_1 d_qt, \cdots, \int \Phi_n d_qt\right) = \left(\int \Phi_1 d_qt, \cdots, \int \Phi_n d_qt\right) A_k \qquad (1 \le k \le n) . \tag{5}$$

Here each $A_k = (a_{i,j}^{(k)})_{1\le i,j\le n}$ *is an* $n\times n$ *matrix valued function of* x *with the entries which are rational in* x

$$a_{i,j}^{(k)} = q^{\alpha_0}\frac{(1-q^{\alpha_k})\prod_{\substack{1\le l\le n\\ l\ne k}}\left(1-\frac{x_i}{x_l}q^{\alpha_l}\right)}{\left(q\frac{x_k}{x_j}-q^{\alpha_k}\right)\prod_{\substack{1\le l\le n\\ l\ne i}}\left(1-\frac{x_i}{x_l}\right)} \qquad (i,j\ne k, i\ne j),$$

$$a_{i,k}^{(k)} = q^{\alpha_0}\frac{\prod_{\substack{1\le l\le n\\ l\ne k}}\left(1-\frac{x_i}{x_l}q^{\alpha_l}\right)}{\prod_{\substack{1\le l\le n\\ l\ne i}}\left(1-\frac{x_i}{x_l}\right)} \qquad (i\ne k),$$

$$a_{i,i}^{(k)} = \frac{1-\frac{x_i}{qx_k}}{1-\frac{x_i}{x_k}q^{\alpha_k-1}} + q^{\alpha_0}\frac{(1-q^{\alpha_i})\prod_{\substack{1\le l\le n\\ l\ne k}}\left(1-\frac{x_i}{x_l}q^{\alpha_l}\right)}{\left(q\frac{x_k}{x_i}-q^{\alpha_k}\right)\prod_{\substack{1\le l\le n\\ l\ne i}}\left(1-\frac{x_i}{x_l}\right)} \qquad (i\ne k),$$

$$a_{k,j}^{(k)} = q^{\alpha_0}\frac{\prod_{1\le l\le n}\left(1-\frac{x_k}{x_l}q^{\alpha_l}\right)}{\left(q\frac{x_k}{x_j}-q^{\alpha_k}\right)\prod_{\substack{1\le l\le n\\ l\ne k}}\left(1-\frac{x_k}{x_l}\right)} \qquad (k\ne j),$$

$$a_{k,k}^{(k)} = q^{\alpha_0}\prod_{\substack{1\le l\le n\\ l\ne k}}\frac{1-\frac{x_k}{x_l}q^{\alpha_l}}{1-\frac{x_k}{x_l}}.$$

2) *A fundamental solution matrix is given by*

$$\Xi(x) = (\Xi_{i,j})_{1\le i,j\le n} = \left(\int_{w_i}\Phi_j(t)d_qt\right)_{1\le i,j\le n},$$

where w_i *denotes a q-interval* $[0, x_j]$ *for* $1\le i\le n$.

We investigate the behavior of $\Xi(q^{\nu_1 m}x_1,\cdots,q^{\nu_n m}x_n)$ at $m=\infty$ along all generic trajectories L_ν determined by the $n!$ inequalities $\nu_{\sigma(1)}<\nu_{\sigma(2)}<\cdots<\nu_{\sigma(n)}$ where σ run over the symmetric group of order n. If we put $\bar{x}_1 = x_1q^{\nu_1 m},\cdots,\bar{x}_n = x_nq^{\nu_n m}$ for $x_j\in C^*$, then the condition $\nu_{\sigma(1)}<\cdots<\nu_{\sigma(n)}$ is equivalent to the condition $|\bar{x}_{\sigma(1)}|\gg\cdots\gg|\bar{x}_{\sigma(n)}|$ when m is sufficiently large. Therefore the connection problem is reduced to find a relation between $\Xi(x)$ in the region $|x_1|\gg\cdots\gg|x_n|\gg 1$ and $\Xi(x)$ in the region $|x_{\sigma(1)}|\gg\cdots\gg|x_{\sigma(n)}|\gg 1$, which will be denoted by $\Xi_e(x)$ and $\Xi_\sigma(x)$ respectively.

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§ 2. Relations among Jackson integrals of the first kind

In this section we give relations among Jackson integrals of Jordan-Pochhammer type, which will be essential in the sequel, and also give a connection formula for the basic hypergeometric series as its corollary. See [7], [15], [16] for related formulae.

We shall frequently use the theta function $\Theta(t) = (t)_\infty(q/t)_\infty(q)_\infty$.

LEMMA 2. *Let* $k = 1, \cdots, n-1$. *Under the condition* $|q^{-\alpha_k-\cdots-\alpha_n}| < |q^{\alpha_0}| < 1$ *and* $x_i/x_j \neq 1, q^{\pm 1}, q^{\pm 2}, \cdots$, *we have relations*

$$\int_{w_k} t^{\alpha_0-1}\prod_{j=k}^{n}\frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty}d_qt + \sum_{l=k+1}^{n} C_{k,l}\int_{w_l} t^{\alpha_0-1}\sum_{j=k}^{n}\frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty}d_qt \tag{6}$$
$$= x_k^{\alpha_0}\frac{\Theta(q^{1+\alpha_0+\alpha_k+\cdots+\alpha_n})}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n})}\prod_{j=k+1}^{n}\frac{\Theta(x_j/x_k)}{\Theta(x_j/x_x q^{\alpha_j})}$$
$$\times\int_{[0,1]} t^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n}\prod_{j=k}^{n}\frac{(x_jx_k^{-1}q^{\alpha_k-\alpha_j}t)_\infty}{(x_jx_k^{-1}q^{\alpha_k}t)_\infty}d_qt\,,$$

where

$$C_{k,l} = \left(\frac{x_k}{x_l}\right)^{\alpha_0+1}\frac{\Theta(x_k\cdot x_l^{-1}q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n+1})\prod_{j=k+1}^{n}\Theta(x_i/x_k)\Theta(x_lq^{\alpha_i+1}/x_i)}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n+1})\prod_{\substack{k\le i\le n\\ i\neq l}}\Theta(x_i/x_l)\prod_{i=k+1}^{n}\Theta(x_kq^{\alpha_i+1}/x_i)}\,.$$

Remark. If a function $F(x) = \prod_{1\le i\le n} x_i^{\lambda_i}f(x)$, where $f(x)$ is a meromorphic function on $(\boldsymbol{C}^*)^n$, satisfies $T_jF(x) = F(x)$ for $j = 1, \cdots, n$, then we say $F(x)$ to be pseudo-constant, which is also said to be q-periodic by C.R. Adams, G.D. Birkhoff, R.D. Carmichael, and W.J. Trjitzinsky ([1], [5], [6], [9]). The above functions $C_{k,l}$ $(1 \le k \le n-1,\ k+1 \le l \le n)$ are pseudo-constant.

Proof of Lemma 2. We show these relations by residue calculus, which is an extension of the method as in our previous paper [13]. Set

$$F(t) = \frac{(q^{-\alpha_0-\alpha_k-\cdots-\alpha_n-1}/t)_\infty(q^{\alpha_0+\alpha_k+\cdots+\alpha_n+2}t)_\infty}{(1/t)_\infty(q^{\alpha_k+1}t)_\infty}\prod_{l=k+1}^{n}\frac{(x_k^{-1}x_lq^{\alpha_k+1-\alpha_l}t)_\infty}{(x_k^{-1}x_lq^{\alpha_k+1}t)_\infty}\,,$$

and

$$\tilde{F}(t) = x_k^{\alpha_0}q^{\alpha_0+\alpha_k+1}\frac{(1-q)(q)_\infty^2}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n+1})}\prod_{i=k+1}^{n}\frac{\Theta(x_i/x_k)}{\Theta(x_i/x_kq^{\alpha_i})}F(t)\,.$$

Then $F(t)$ is a meromorphic function on $\boldsymbol{C}^*$. The residues of $\tilde{F}(t)$ at each point $q^{-1-\alpha_k-j}$, $x_kx_l^{-1}q^{-1-\alpha_k-j}$, q^i $(j = 0, 1, 2, \cdots)$ are expressed by the following Jackson integrals.

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$$\sum_{j\geq 0}\operatorname*{Res}_{t=q^{-1-\alpha_k-j}}\tilde F(t)=-\int_{w_k}t^{\alpha_0-1}\prod_{i=k}^{n}\frac{(t/x_i)_\infty}{(q^{\alpha_i}t/x_i)_\infty}d_qt\,,$$

$$\sum_{j\geq 0}\operatorname*{Res}_{t=x_kx_l^{-1}q^{-1-\alpha_k-j}}\tilde F(t)=-C_{k,l}\int_{w_l}t^{\alpha_0-1}\prod_{j=k}^{n}\frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty}d_qt$$

and

$$\sum_{j\geq 0}\operatorname*{Res}_{t=q^j}\tilde F(t)=x_k^{\alpha_0}\frac{\Theta(q^{1+\alpha_0+\alpha_k+\cdots+\alpha_n})}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n})}\prod_{j=k+1}^{n}\frac{\Theta(x_j/x_k)}{\Theta(x_j/x_kq^{\alpha_j})}$$
$$\times\int_{[0,1]}t^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n}\sum_{j=k}^{n}\frac{(x_jx_k^{-1}q^{\alpha_k-\alpha_j}t)_\infty}{(x_jx_k^{-1}q^{\alpha_k}t)_\infty}d_qt\,,$$

where $\operatorname{Res}_{t=x}F(t)$ denotes the residue of a function $F(t)$ at $t=x$. Therefore it remains to prove

$$\sum_{j\geq 0}\operatorname*{Res}_{t=q^{-1-\alpha_k-j}}F(t)+\sum_{l=k+1}^{n}\sum_{j\geq 0}\operatorname*{Res}_{t=x_kx_l^{-1}q^{-1-\alpha_k-j}}F(t)+\sum_{j\geq 0}\operatorname*{Res}_{t=q^j}F(t)=0\,.$$

Here we set two circles $\mathscr{C}_m$, $\tilde{\mathscr{C}}_m$ for a natural number m as follows:

$$\mathscr{C}_m:=\left\{\rho_m\exp(\varphi\sqrt{-1})\,|\,\rho_m:=\frac{1}{2}(|q|^m+|q|^{m+1}),\ 0\leq\varphi\leq 2\pi\right\},$$

$$\tilde{\mathscr{C}}_m:=\left\{\tilde\rho_m\exp(\varphi\sqrt{-1})\,|\,\tilde\rho_m:=\frac{1}{2}|q^{-\alpha_k-1}|(|q|^{-m-1}+|q|^{-m}),\ 0\leq\varphi\leq 2\pi\right\},$$

with the counterclockwise direction. Then we have

$$\sum_{j=0}^{m}\operatorname*{Res}_{t=q^{-1-\alpha_k-j}}F(t)+\sum_{l=k+1}^{n}\sum_{j=0}^{m(l)}\operatorname*{Res}_{t=x_kx_l^{-1}q^{-1-\alpha_k-j}}F(t)+\sum_{j=0}^{m}\operatorname*{Res}_{t=q^j}F(t)$$
$$=\frac{1}{2\pi\sqrt{-1}}\int_{\tilde{\mathscr{C}}_m}F(t)dt-\frac{1}{2\pi\sqrt{-1}}\int_{\mathscr{C}_m}F(t)dt\,,$$

where each $m(l)$ $(l=k+1,\cdots,n)$ is a certain positive integer. And theer exists a positive number M such that

$$|F(\rho_me^{\sqrt{-1}\varphi})|\leq M|q^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n}|^m$$

for $0\leq\varphi\leq 2\pi$. Indeed

$$F(\rho_me^{\sqrt{-1}\varphi})=F(\rho_0|q|^me^{\sqrt{-1}\varphi})$$
$$=(q^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n})^m\frac{(q^{2-m+\alpha_0+\alpha_k+\cdots+\alpha_n}|q|^m\rho_0e^{\sqrt{-1}\varphi})_m}{(q^{1-m}|q|^m\rho_0e^{\sqrt{-1}\varphi})_m}$$
$$\times\frac{(q^{m-1-\alpha_0-\alpha_k-\cdots-\alpha_n}/|q|^m\rho_0e^{\sqrt{-1}\varphi})_\infty(q^{2+\alpha_0+\alpha_k+\cdots+\alpha_n}|q|^m\rho_0e^{\sqrt{-1}\varphi})_\infty}{(q^m/|q|^m\rho_0e^{\sqrt{-1}\varphi})_\infty(q^{1+\alpha_k}|q|^m\rho_0e^{\sqrt{-1}\varphi})_\infty}$$

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$$\times \prod_{l=k+1}^{n} \frac{(x_k^{-1}x_l q^{1+\alpha_k-\alpha_l}|q|^m \rho_0 e^{\sqrt{-1}\varphi})_\infty}{(x_k^{-1}x_l q^{1+\alpha_k}|q|^m \rho_0 e^{\sqrt{-1}\varphi})_\infty}.$$

Hence we get the following estimates.

$$\left|\frac{1}{2\pi\sqrt{-1}}\int_{\mathscr{C}_m} F(t)dt\right| \leq \left|\frac{\rho_m}{2\pi}\int_0^{2\pi} F(\rho_m e^{\sqrt{-1}\varphi})d\varphi\right| \leq \rho_m \operatorname*{Max}_{0\leq\varphi\leq 2\pi}|F(\rho_m e^{\sqrt{-1}\varphi})|$$
$$\leq \rho_m M|q^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n}|^m \leq \rho_0 M|q^{-\alpha_0-\alpha_k-\cdots-\alpha_n}|^m .$$

Thus we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{\mathscr{C}_m} F(t)dt \longrightarrow 0 \qquad (m\to\infty),$$

when $|q^{-\alpha_0-\alpha_k-\cdots-\alpha_n}|$ is less than one. By the same argument we can show

$$\frac{1}{2\pi\sqrt{-1}}\int_{\tilde{\mathscr{C}}_m} F(t)dt \longrightarrow 0 \qquad (m\to\infty),$$

when $|q^{\alpha_0}|$ is less than one. This completes the proof. □

As a corollary of Lemma 2, a connection formula of the basic hypergeometric series, which tends to that of hypergeometric series as $q\to 1$, can be deduced. To state the corollary, we recall the definitions of the basic hypergeometric series ${}_2\varphi_1$ and a q-analogue of the gamma function Γ_q:

$${}_2\varphi_1(\alpha,\beta,\gamma;x) = \sum_{n\geq 0}\frac{(q^\alpha)_n(q^\beta)_n}{(q^\gamma)_n(q)_n}x^n ,$$

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}.$$

Refer to [2], [5], [6], [11] for details.

COROLLARY.

$${}_2\varphi_1(\alpha,\beta,\gamma;x) = \frac{\Gamma_q(\gamma)\Gamma_q(\beta-\alpha)}{\Gamma_q(\beta)\Gamma_q(\gamma-\alpha)}\frac{\Theta(q^\alpha x)}{\Theta(x)}{}_2\varphi_1(\alpha,\alpha-\gamma+1,\alpha-\beta+1;q^{\gamma+1-\alpha-\beta}x^{-1})$$
$$+\frac{\Gamma_q(\gamma)\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)\Gamma_q(\gamma-\beta)}\frac{\Theta(q^\beta x)}{\Theta(x)}{}_2\varphi_1(\beta,\beta-\gamma+1,\beta-\alpha+1;q^{\gamma+1-\alpha-\beta}x^{-1}).$$

Proof. Consider the case of $k=1$ and $n=2$ in (6). Putting $\alpha_0 = 1+\beta-\gamma$, $\alpha_1=\gamma-1-\alpha$, $\alpha_2=-\beta$ and $x_1\cdot x_2^{-1} = q^{\gamma-\alpha}x^{-1}$, one has

$$\int_0^1 t^{\alpha_0-1}\frac{(qt)_\infty(q^\beta xt)_\infty}{(q^{\gamma-\alpha}t)_\infty(qxt)_\infty}d_qt$$

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$$= \frac{\Theta(q^{\beta+1-\gamma})\Theta(q^{\beta}x)}{\Theta(q^{\alpha-\beta})\Theta(x)} \int_0^1 t^{\beta-\gamma} \frac{(qt)_\infty (q^{\gamma-\alpha+1}x^{-1}t)_\infty}{(q^{\gamma-\alpha}t)_\infty (q^{\gamma+1-\alpha-\beta}x^{-1}t)_\infty} d_q t$$
$$+ \frac{\Theta(q^{1-\beta})\Theta(q^{\alpha}x)}{\Theta(q^{1+\alpha-\beta})\Theta(x)} \int_0^1 t^{\alpha-1} \frac{(qt)_\infty (q^{2-\beta}x^{-1}t)_\infty}{(q^{1-\beta}t)_\infty (q^{\gamma+1-\alpha-\beta}x^{-1}t)_\infty} d_q t .$$

Thanks to the Jackson integral representation of the basic hypergeometric series

$${}_2\varphi_1(\alpha, \beta, \gamma; x) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\alpha)\Gamma_q(\gamma-\alpha)} \int_0^1 t^{\alpha-1} \frac{(qt)_\infty (q^{\beta}xt)_\infty}{(q^{\gamma-\alpha}t)_\infty (xt)_\infty} d_q t ,$$

we obtain the required relation.

§ 3. Solution to the connection problem

Let $s = \max\{|x_2/x_1|, |x_3/x_2|, \cdots, |x_n/x_{n-1}|\}$. For $k = 1, 2, \cdots, n$, we have the following estimates, which are easily shown by the q-binomial theorem (4).

1) For $l = k, \cdots, n$,

$$\int_{w_k} \Phi(t) d_q t = \int_{w_l} t^{\alpha_0 - 1} \prod_{j=k}^{n} \frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty} d_q t (1 + O(s)) \qquad (s \to 0) .$$

2)

$$\frac{\Theta(q^{1+\alpha_0+\alpha_k+\cdots+\alpha_n})}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n})} \int_{[0,1]} t^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n} \prod_{j=k}^{n} \frac{(x_j x_k^{-1} q^{\alpha_k-\alpha_j} t)_\infty}{(x_j x_k^{-1} q^{\alpha_k} t)_\infty} d_q t$$
$$= \frac{\Theta(q^{1+\alpha_0+\alpha_k+\cdots+\alpha_n})}{\Theta(q^{\alpha_0+\alpha_{k+1}+\cdots+\alpha_n})} \int_{[0,1]} t^{-1-\alpha_0-\alpha_k-\cdots-\alpha_n} \frac{(t)_\infty}{(q^{\alpha_k}t)_\infty} d_q t (1 + O(s))$$
$$= \frac{\Gamma_q(1+\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n)}{\Gamma_q(1+\alpha_0+\alpha_k+\cdots+\alpha_n)} q^{-\alpha_0-\alpha_k-\cdots-\alpha_n}(1 + O(s) \quad (s \to 0) .$$

By the above estimates 1), the left hand side of (6) is

$$\left(\int_{w_k} \Phi d_q t + \sum_{l=k+1}^{n} C_{k,l} \int_{w_l} \Phi d_q t\right)(1 + O(s)) \qquad (s \to 0) ,$$

and by 2) the right hand side of (6) is

$$\frac{\Gamma_q(1+\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n)}{\Gamma_q(1+\alpha_0+\alpha_k+\cdots+\alpha_n)} q^{-\alpha_0-\alpha_k-\cdots-\alpha_n}$$
$$\times x_k^{\alpha_0} \prod_{j=k+1}^{n} \frac{\Theta(x_j/x_k)}{\Theta(x_j/x_k q^{\alpha_j})} (1 + O(s)) \qquad (s \to 0) .$$

Therefore, by Lemma 2, we obtain the following.

LEMMA 3. *Let $k=1,\cdots,n-1$ and $\tilde{W}_k$ be the q-interval $w_k+\sum_{l=k+1}^{n}C_{k,l}w_l$. Then we have*

$$\int_{\tilde{W}_k}\Phi(t)d_qt=\tilde{C}_k\frac{\Gamma_q(1+\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n)}{\Gamma_q(1+\alpha_0+\alpha_k+\cdots+\alpha_n)}q^{-\alpha_0-\alpha_k-\cdots-\alpha_n}$$
$$\times x_k^{\alpha_0}\prod_{j=k+1}^{n}\left(\frac{x_k}{x_j}\right)^{\alpha_j}(1+O(s))\qquad(s\to 0),$$

where

$$\tilde{C}_k=\prod_{j=k+1}^{n}\left(\frac{x_j}{x_k}\right)^{\alpha_j}\frac{\Theta(x_j/x_k)}{\Theta(x_j/x_kq^{\alpha_j})}.$$

Remark. The functions $\tilde{C}_k$ $(1\leq k\leq n-1)$ are pseudo-constant.

By using Lemma 3, we give the asymptotic behavior of fundamental solutions of the system (5) in the region $|x_1|\gg\cdots\gg|x_n|\gg 1$. In fact, if the variables α_j and x_j are changed by α_j-1 and x_jq^{-1} respectively, then $\int_{w_l}\Phi(t)d_qt$ is transformed into $\int_{w_l}\Phi_j(t)d_qt$ for $1\leq j,\ l\leq n$, and $\tilde{C}_k$, $C_{k,l}$ remain invariant. Hence if we put $\tilde{W}_n=w_n$, we have

$$(7)\qquad \begin{bmatrix}\int_{\tilde{W}_1}\Phi_1d_qt, & \cdots\cdots, & \int_{\tilde{W}_1}\Phi_nd_qt\\ \cdots & \cdots & \cdots\\ \int_{\tilde{W}_n}\Phi_1d_qt, & \cdots\cdots, & \int_{\tilde{W}_n}\Phi_nd_qt\end{bmatrix}$$
$$=\begin{bmatrix}1 & C_{1,2}\cdots\cdots & C_{1,n}\\ & 1 & \vdots\\ & \ddots & C_{n-1,n}\\ 0 & & 1\end{bmatrix}\begin{bmatrix}\int_{w_1}\Phi_1d_qt, & \cdots\cdots, & \int_{w_1}\Phi_nd_qt\\ \cdots & \cdots & \cdots\\ \int_{w_n}\Phi_1d_qt, & \cdots\cdots, & \int_{w_n}\Phi_nd_qt\end{bmatrix},$$

where

$$\begin{cases}\tilde{C}_k\dfrac{\Gamma_q(1+\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n)}{\Gamma_q(1+\alpha_0+\alpha_k+\cdots+\alpha_n)}q^{1-\alpha_0-\alpha_k-\cdots-\alpha_n}\\ \qquad\times x_k^{\alpha_0}\displaystyle\prod_{l=k+1}^{n}\left(\frac{x_k}{x_l}\right)^{\alpha_l}\quad(1\leq k\leq n-1,\ 1\leq j\leq k-1),\\ \tilde{C}_k\dfrac{\Gamma_q(\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n)}{\Gamma_q(\alpha_0+\alpha_k+\cdots+\alpha_n)}q^{1-\alpha_0-\alpha_k-\cdots-\alpha_n}\\ \qquad\times (q^{-1}x_k)^{\alpha_0}\displaystyle\prod_{l=k+1}^{n}\left(\frac{x_k}{qx_l}\right)^{\alpha_l}\quad(1\leq k\leq n-1,\ j=k),\end{cases}$$

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$$\int_{\tilde{W}_k}\Phi_j d_q t \sim \begin{cases} \tilde{C}_k \dfrac{\Gamma_q(1+\alpha_k)\Gamma_q(\alpha_0+\alpha_{k+1}+\cdots+\alpha_n-1)}{\Gamma_q(\alpha_0+\alpha_k+\cdots+\alpha_n)} q^{(-\alpha_0-\alpha_k-\cdots-\alpha_n)+\alpha_j} \\ \quad \times x_j x_k^{\alpha_0-1}\prod_{l=k+1}^{n}\left(\dfrac{x_k}{x_l}\right)^{\alpha_l} \quad (1\le k\le n-1,\ k+1\le j\le n), \\ \dfrac{\Gamma_q(\alpha_0)\Gamma_q(\alpha_n+1)}{\Gamma_q(\alpha_0+\alpha_n+1)}(qx_n)^{\alpha_0} \quad (k=n,\ 1\le j\le n-1), \\ \dfrac{\Gamma_q(\alpha_0)\Gamma_q(\alpha_n)}{\Gamma_q(\alpha_0+\alpha_n)}x_n^{\alpha_0} \quad (k=n,\ j=n). \end{cases}$$

The matrix on the left hand side of (7), which will be denoted by Ξ_e, may be taken as a fundamental solution of system (5) in the region $|x_1| \gg \cdots \gg |x_n| \gg 1$. Then the matrix expressed by

$$(8) \qquad \Xi_\sigma(x) = \begin{bmatrix} 1 & \sigma(C_{1,2}) & \cdots & \sigma(C_{1,n}) \\ & 1 & \ddots & \vdots \\ & & \ddots & \sigma(C_{n-1,n}) \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} \int_{w_{\sigma(1)}}\Phi_1 d_q t, & \cdots, & \int_{w_{\sigma(1)}}\Phi_n d_q t \\ \cdots & \cdots & \cdots \\ \int_{w_{\sigma(n)}}\Phi_1 d_q t, & \cdots, & \int_{w_{\sigma(n)}}\Phi_n d_q t \end{bmatrix}$$

defines a fundamental solution in the region $|x_{\sigma(1)}| \gg \cdots \gg |x_{\sigma(n)}| \gg 1$, where σ is an element of the symmetric group of order n. By the expressions (7) and (8), we can get the connection matrix $P_{\sigma,e}(x)$ between the solutions $\Xi_e(x)$ and $\Xi_\sigma(x)$. Indeed,

$$\begin{aligned} \Xi_\sigma(x) &= \begin{bmatrix} 1 & \sigma(C_{1,2}) & \cdots & \sigma(C_{1,n}) \\ & 1 & \ddots & \vdots \\ & & \ddots & \sigma(C_{n-1,n}) \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} \int_{w_{\sigma(1)}}\Phi_1 d_q t, & \cdots, & \int_{w_{\sigma(1)}}\Phi_n d_q t \\ \cdots & \cdots & \cdots \\ \int_{w_{\sigma(n)}}\Phi_1 d_q t, & \cdots, & \int_{w_{\sigma(n)}}\Phi_n d_q t \end{bmatrix} \\ &= \begin{bmatrix} 1 & \sigma(C_{1,2}) & \cdots & \sigma(C_{1,n}) \\ & 1 & \ddots & \vdots \\ & & \ddots & \sigma(C_{n-1,n}) \\ 0 & & & 1 \end{bmatrix} \cdot S_\sigma \cdot \Xi(x), \\ &= \begin{bmatrix} 1 & \sigma(C_{1,2}) & \cdots & \sigma(C_{1,n}) \\ & 1 & \ddots & \vdots \\ & & \ddots & \sigma(C_{n-1,n}) \\ 0 & & & 1 \end{bmatrix} \cdot S_\sigma \cdot \begin{bmatrix} 1 & C_{1,2} & \cdots & C_{1,n} \\ & 1 & \ddots & \vdots \\ & & \ddots & C_{n-1,n} \\ 0 & & & 1 \end{bmatrix}^{-1} \Xi_e(x), \end{aligned}$$

where S_σ is a permutation defined by

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$$S_\sigma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix},$$

and the action of σ on $C_{i,j}$ is defined by

$$\sigma(C_{i,j}) = C_{i,j}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)};\ x_{\sigma(1)}, \cdots, x_{\sigma(n)}).$$

Hence we finally come to the main theorem of this paper.

THEOREM. *The connection matrix between the solution* Ξ_e *and* Ξ_σ *is given by*

$$P_{\sigma,e} = \begin{bmatrix} 1 & \sigma(C_{1,2}) & \cdots & \sigma(C_{1,n}) \\ & 1 & \ddots & \vdots \\ & & \ddots & \sigma(C_{n-1,n}) \\ 0 & & & 1 \end{bmatrix} \cdot S_\sigma \cdot \begin{bmatrix} 1 & C_{1,2} & \cdots & C_{1,n} \\ & 1 & \ddots & \vdots \\ & & \ddots & C_{n-1,n} \\ 0 & & & 1 \end{bmatrix}^{-1}.$$

Remark. The connection matrices $P_{\sigma,e}$ satisfy the following cocycle conditions: $P_{\sigma,\tau} = P_{\sigma,e}P^{-1}_{\tau,e}$, $P_{\sigma\nu,e} = \sigma(P_{\tau,e})P_{\sigma,e}$ for arbitrary two permutations σ and τ.

§ 4. Appendix

During the preparation of this paper K. Aomoto suggested to the author the following question: to evaluate the integral (3) over an arbitrary q-interval $\mathscr{C} = [0, \infty(s)]$ of the second kind in terms of the integrals (3) over w_i $(1 \leq i \leq n)$. The answer is the following.

THEOREM. *Under the condition* $|q^{-\alpha_1-\cdots-\alpha_n}| < |q^{\alpha_0}| < 1$, *we have*

$$\int_{[0,\infty(s)]} \Phi(t)d_qt = \sum_{l=1}^{n} R_l \int_{[0,x_l]} \Phi(t)d_qt,$$

where

$$R_l = \left(\frac{s}{qx_l}\right)^{\alpha_0-1} \frac{\Theta(q^{\alpha_l+1})\Theta(q^{\alpha_0+\cdots+\alpha_n-2}sx_l^{-1})}{\Theta(q^{\alpha_0+\cdots+\alpha_n-1})\Theta(q^{-1}sx_l^{-1})} \prod_{j=1}^{n} \frac{\Theta(s/x_j)}{\Theta(q^{\alpha_j}s/x_j)} \prod_{\substack{j=1\\ j\neq l}}^{n} \frac{\Theta(x_jx_l^{-1}q^{-\alpha_j})}{\Theta(x_jx_l^{-1})},$$

and

$$\Phi(t) = \prod_{j=1}^{n} \frac{(t/x_j)_\infty}{(q^{\alpha_j}t/x_j)_\infty}. \qquad \square$$

Proof. Set

$$F(t)=\frac{(q^{\alpha_0+\alpha_1+\cdots+\alpha_n-1}/t)_\infty(q^{2-\alpha_0-\alpha_1-\alpha_n}t)_\infty}{(1/t)_\infty(qt)_\infty}\prod_{j=1}^{n}\frac{(qx_j/q^{\alpha_j}st)_\infty}{(qx_j/st)_\infty}$$

and

$$\tilde{F}(t)=\frac{s^{\alpha_0}(1-q)(q)_\infty^2}{(q^{\alpha_0+\alpha_1+\cdots+\alpha_n-1})_\infty(q^{-\alpha_0-\alpha_1-\cdots-\alpha_n+2})_\infty}\prod_{j=1}^{n}\frac{\Theta(s/x_j)}{\Theta(sq^{\alpha_j}/x_j)}F(t).$$

Then the residues of $\tilde{F}(t)$ are expressed by the following Jackson integrals.

$$\sum_{-\infty<i<+\infty}\operatorname*{Res}_{t=q^i}\tilde{F}(t)=\int_{[0,\infty(s)]}\Phi(t)d_qt,$$

$$\sum_{i=0}^{\infty}\operatorname*{Res}_{t=qx_l/sq^i}\tilde{F}(t)=-R_l\int_{[0,x_l]}\Phi(t)d_qt,$$

where $\operatorname*{Res}_{t=x}F(t)$ denotes the residue of a function $F(t)$ at $t=x$. Therefore it remains to prove

$$\sum_{-\infty<i<+\infty}\operatorname*{Res}_{t=q^i}F(t)+\sum_{l=1}^{n}\sum_{i=0}^{+\infty}\operatorname*{Res}_{t=qx_l/sq^i}F(t)=0.$$

We can show it in a similar way to that of Lemma 2; i.e. estimates of the integration of the function $F(t)$ on suitable cycles.

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Department of Mathematics
School of Science
Nagoya University
Chikusa-ku, Nagoya 464-01, Japan

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