Canad. J. Math. Vol. 68 (1), 2016 pp. 67–87
http://dx.doi.org/10.4153/CJM-2015-032-8

A Lower Bound on the Euler–Poincaré Characteristic of Certain Surfaces of General Type with a Linear Pencil of Hyperelliptic Curves

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Abstract. Let S be a surface of general type. In this article, when there exists a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ whose slope is less than or equal to four, we give a lower bound on the Euler–Poincaré characteristic of S. Furthermore, we prove that our bound is the best possible by giving required hyperelliptic fibrations.

1 Introduction

Let S be a surface of general type defined over $\mathbb{C}$ and let $f: S \to C$ be a fibration over a nonsingular projective curve C of genus $g(C)$. We always assume that f is relatively minimal; that is, S has no (-1)-curves contained in a fiber of f. Denote the genus of a general fiber of f by $g(f)$. A fibration f is said to be *hyperelliptic* or *non-hyperelliptic* according to the type of a general fiber of f. Let K_f be the relative canonical bundle $K_S - f^*K_C$. We introduce the following numerical invariants associated with f:

$$\chi_f := \deg f_*K_f = \chi(\mathcal{O}_S) - \big(g(C)-1\big)\big(g(f)-1\big),$$
$$K_f^2 = K_S^2 - 8\big(g(C)-1\big)\big(g(f)-1\big).$$

It is well known that these numbers are non-negative integers. Moreover, f is locally trivial if and only if $\chi_f = K_f^2 = 0$. When f is locally trivial, we have $\chi_f > 0$ (*cf.* [2, III,Theorem 17.3]). In such a case, we can define the ratio $\lambda(f) = K_f^2/\chi_f$ and call it the *slope* of f.

The slope inequality $4 - 4/g(f) \le \lambda(f) \le 12$ was proved by Xiao [14, Theorem 2] (Horikawa [5, Theorem 2.1] and Persson [11, Proposition 2.12] proved it for a hyperelliptic fibration f). It shows that $4/(4-\lambda(f))$ is an upper bound on $g(f)$ in the case where $\lambda(f) < 4$. Furthermore, if f is non-hyperelliptic and the relative canonical bundle f_*K_f is semi-stable, then $5(g(f)-6)/g(f) \le \lambda(f)$ (see [8, Lemma 2.5]). Hence, if f is a hyperelliptic fibration with $\lambda(f) \ge 4$, then an upper bound on $g(f)$ may not exist. The author has studied hyperelliptic fibrations with slope four in [6] and proved the following theorem.

Received by the editors February 4, 2014.
Published electronically October 21, 2015.
AMS subject classification: **14D05**, 14J29, 14H30.
Keywords: hyperelliptic fibration, surface of general type, double cover.

Theorem 1.1 (Ishida [6, Théorème 0.1]) *Let S be a surface of general type and let $f\colon S \to C$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and satisfies that $\lambda(f) = 4, g(f) \ge 4$, then*

$$\chi_f \ge \Delta(g(f)) = \begin{cases} g(f)/2 - 1 & \text{if } g(f) \text{ is even,} \\ g(f) - 3 & \text{if } g(f) \text{ is odd.} \end{cases}$$

Furthermore, for any integer $g \ge 4$ there exists a surface of general type and a relatively minimal hyperelliptic fibration $f\colon S \to C$ with $\lambda(f) = 4, g(f) = g$, and $\chi_f = \Delta(g)$.

Remark 1.2 For any positive integer z, there exists a relatively minimal hyperelliptic fibration f with $\lambda(f) = 4$, $\chi_f = z$, and $g(f) = 2$ or 3 (see [12, Théorème 2.9] and [6, Theorem 0.2]).

By the above theorem, there exists no upper bound on $g(f)$; however, there exists the best possible lower bound $\Delta(g(f))$ on χ_f. The base curve of any fibration constructed in the proof of Theorem 1.1 is an elliptic curve. Hence, a lower bound on χ_f for a hyperelliptic fibration over $\mathbb{P}^1$ may not be the best. In this manuscript, we consider a hyperelliptic fibration $f\colon S \to \mathbb{P}^1$ with $\lambda(f) \le 4$ and prove the following theorem.

Theorem 1.3 *Let S be a surface of general type and let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and satisfies that $\lambda(f) \le 4$, then*

$$\chi_f \ge \Gamma(g(f)) = \begin{cases} 3g(f) - 9 & \text{if } g(f) \ge 6, \\ \left[\frac{3g(f)}{2} \right] & \text{if } 2 \le g(f) \le 5, \end{cases} \tag{1.1}$$

where $[\alpha]$ is the maximum integer not exceeding a real number α. In particular, $\chi(\mathcal{O}_S) \ge \Gamma(g(f)) - g(f) + 1$.

Furthermore, for any integer $g \ge 2$, there exists a surface of general type and a relatively minimal hyperelliptic fibration $f\colon S \to \mathbb{P}^1$ with $\lambda(f) = 4$, $g(f) = g$, and $\chi_f = \Gamma(g)$.

By Theorem 1.3, we have the best possible lower bound on χ_f for a hyperelliptic fibration $f\colon S \to \mathbb{P}^1$ with $\lambda(f) \le 4$. On the other hand, since the base curve of any fibration constructed in the proof of the following theorem is a projective line, we see that there exists no upper bound on χ_f.

Theorem 1.4 (Ishida [7, Theorem 0.3]) *Let g and z be integers satisfying either of the following conditions*

(i) *g is an even integer that is greater than 4 and $z \ge g^2 + \frac{g}{2} - 2$;*
(ii) *g is an odd integer that is greater than 5 and $z \ge g^2 - 1$.*

Then there exists a relatively minimal hyperelliptic fibration f with $\lambda(f) = 4$, $g(f) = g$, and $\chi_f = z$.

The paper is organized as follows. In Section 2, we recall the structure of a hyperelliptic fibration and the canonical resolution of a double cover. The relative canonical map of a hyperelliptic fibration $f: S \to C$ is a generically two-to-one map, and its proper image is a birationally ruled surface over C. Hence, S is birationally equivalent to a double cover of a ruled surface over C (*cf.* [1, Theorem III. 4]). For resolving the singularities of this double cover, we use the canonical resolution. Then we have formulas for computing $\chi(\mathcal{O}_S)$ and K_S^2 according to [3, Lemma 6]. We introduce these formulas to calculate χ_f and K_f^2, which are used in Sections 3–6. In Sections 3 and 4, we prove inequality (1.1) in the cases where $g(f) \geq 6$ and $2 \leq g(f) \leq 5$, respectively. In Section 3, by using formulas for computing χ_f and K_f^2 given in Section 2, we show inequality $\chi_f \geq 12(g(f)-3)/(3\lambda(f)-8)$, which is stronger than (1.1). Even if $f: S \to \mathbb{P}^1$ is relatively minimal, S may have a (-1)-curve not contained in a fiber of f. In Section 4, by considering the number of exceptional curves on S, we show that $\chi_f \geq [3g(f)/2]$. In Sections 5 and 6, in order to prove that the inequality (1.1) is the best possible, we give the required fibrations by constructing double coverings of Hirzebruch surfaces. Furthermore, we shall characterize a surface S of general type with a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ satisfying the numerical properties $6 \leq g(f) \leq 10$, $\lambda(f) \leq 4$, and $\chi_f = 3(g(f)-3)$.

2 Invariants of Hyperelliptic Fibrations

In this section, we recall the terminology and formulas for computing invariants of a hyperelliptic fibration (*cf.* [3, 10]). Let $f: S \to C$ be a hyperelliptic fibration from a complex surface of general type onto a nonsingular projective curve C of genus $g(C)$.

Since f is hyperelliptic, the image of the relative canonical map of f is isomorphic to a birationally ruled surface over C. Hence, we can take a geometrically ruled surface $pr: W \to C$ that is birationally equivalent to this image (see [1, Theorem III 4] and [6, Lemma 1.1]).

Let K be the rational function field of S and let $\phi: S' \to W$ be the K-normalization of W. Let $\mu: \widetilde{S} \to S'$ be a resolution of singularities of S'. If we assume that f is relatively minimal, then there exists a contraction $\rho: \widetilde{S} \to S$ of exceptional curves in fibers of $pr \circ \phi \circ \mu$ such that the following diagram commutes (*cf.* [4, Lemma 4] and [6, p. 470]) :

$$\begin{array}{ccccc} \widetilde{S} & \xrightarrow{\rho} & S & \xrightarrow{f} & C \\ \Big\downarrow{\scriptstyle \mu} & & & & \Big\| \\ S' & \xrightarrow[\phi]{} & W & \xrightarrow[pr]{} & C. \end{array}$$

Moreover, we can choose the canonical resolution of $\phi: S' \to W$ as μ. The canonical resolution of $\phi: S' \to W$ is determined by the following process; refer to [3] for details.

Denote the branch locus of the double cover ϕ by $B(\phi)$. Let $\nu_1: W_1 \to W = W_0$ be the blow-up at a singular point of $B(\phi)$ and let $\phi_1: S_1 \to W_1$ be the K-normalization of W_1. Then we obtain the natural birational morphism $\mu_1: S_1 \to S' = S_0$ and the following commutative diagram:

$$\begin{array}{ccc} S_1 & \xrightarrow{\mu_1} & S_0 = S' \\ \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_0=\phi} \\ W_1 & \xrightarrow{\nu_1} & W_0 = W. \end{array}$$

Continuing this process until $B(\phi_n)$ has no singularities, we obtain the sequence of birational morphisms μ_k $(k = 1, 2, \ldots, n)$ and the following diagram:

$$\begin{array}{ccccccccc} \widetilde{S} = S_n & \xrightarrow{\mu_n} & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_1 & \xrightarrow{\mu_1} & S_0 = S' \\ \Big\downarrow{\scriptstyle \phi_n} & & \Big\downarrow{\scriptstyle \phi_{n-1}} & & & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_0=\phi} \\ W_n & \xrightarrow{\nu_n} & W_{n-1} & \longrightarrow & \cdots & \longrightarrow & W_1 & \xrightarrow{\nu_1} & W = W \end{array}$$

Then $\widetilde{S} = S_n$ is a smooth surface; *i.e.*, $\mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_n : \widetilde{S} \to S'$ is a resolution of singularities of S'.

We now introduce the formulas for computing invariants of f by [3, Lemma 6] and [10, Corollary 2.2] (see also [6, Lemma 1.3]). We assume that W is the $\mathbb{P}^1$-bundle associated with a vector bundle of degree d. Let H be a tautological divisor of pr and F a fiber of pr. Since f is hyperelliptic, we can assume that $B(\phi)$ is linearly equivalent to $2(g(f) + 1)H + 2NF$, where N is an integer. Then we have the following lemma.

Lemma 2.1 (Horikawa [3, Lemma 6], Persson [10, Corollary 2.2]) *Let $f : S \to C$ be a relatively minimal hyperelliptic fibration. Under the same notation as above, denote the multiplicity of $B(\phi_{k-1})$ at the center of the blow-up ν_k by m_k. Then we obtain the following numerical properties*:

$$\chi_f = \frac{dg(f)(g(f)+1)}{2} + Ng(f) - \sum_{k=1}^{n} \frac{1}{2}\Big[\frac{m_k}{2}\Big]\Big(\Big[\frac{m_k}{2}\Big] - 1\Big), \tag{2.1}$$

$$\begin{aligned} K_f^2 = {} & 2d(g(f)^2 - 1) + 4N(g(f) - 1) - \sum_{k=1}^{n} 2\Big(\Big[\frac{m_k}{2}\Big] - 1\Big)^2 \\ & + (\textit{the number of curves contracted by } \rho), \end{aligned} \tag{2.2}$$

where $[\alpha]$ is the maximum integer not exceeding a real number α.

3 The Lower Bound on χ_f in the Case Where $g(f) \geq 6$

Let f be a relatively minimal hyperelliptic fibration. Assume that f is not locally trivial. We employ the same notation as in Section 2. Let $M(f) = d(g(f) + 1) + 2N$. In this section, we will prove that $\chi_f \geq 3(g(f) - 3)$ in the case where $g(f) \geq 6$. First, we prove the following lemma, which will play an important role in the proof of this inequality.

Lemma 3.1 *Let $f : S \to C$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and there exists a positive integer m such that $[m_k/2] \leq m$ for any k, then* *the following holds:*

$$\frac{2M(f)(g(f)-m)}{m\lambda(f)-4(m-1)} \le \chi_f \le \frac{g(f)}{2}M(f). \tag{3.1}$$

Proof By (2.2) and the assumption that $[m_k/2] \le m$, we have

$$\begin{aligned} 2(g(f)-1)M(f)-K_f^2 &\le \sum_{k=1}^{n} 2\Big(\Big[\frac{m_k}{2}\Big]-1\Big)^2 \\ &= \sum_{k=1}^{n} \frac{4([m_k/2]-1)}{[m_k/2]}\frac{1}{2}\Big[\frac{m_k}{2}\Big]\Big(\Big[\frac{m_k}{2}\Big]-1\Big) \\ &\le \sum_{k=1}^{n} \frac{4(m-1)}{m}\frac{1}{2}\Big[\frac{m_k}{2}\Big]\Big(\Big[\frac{m_k}{2}\Big]-1\Big) \\ &= \frac{4(m-1)}{m}\sum_{k=1}^{n}\frac{1}{2}\Big[\frac{m_k}{2}\Big]\Big(\Big[\frac{m_k}{2}\Big]-1\Big). \end{aligned}$$

It follows from (2.1) and $K_f^2 = \lambda(f)\chi_f$ that

$$2(g(f)-1)M(f)-\lambda(f)\chi_f \le \frac{4(m-1)}{m}\Big(\frac{g(f)}{2}M(f)-\chi_f\Big).$$

Hence, we obtain

$$2M(f)(g(f)-m) \le \{m\lambda(f)-4(m-1)\}\chi_f.$$

It is clear from (2.1) that $\chi_f \le g(f)M(f)/2$. Therefore, we can show inequality (3.1). ■

We assume that S is a surface of general type and $C = \mathbb{P}^1$. Note that W is isomorphic to a Hirzebruch surface. We consider all possible values of $M(f)$. If $d = 0$, then W is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; *i.e.,* W has another projection onto $\mathbb{P}^1$. This projection also induces a fibration from S to $\mathbb{P}^1$. Since S is a surface of general type, the genus of its general fiber is greater than one. Hence, we must have $N \ge 3$, which implies that $M(f)$ is an even integer that is greater than or equal to six. On the other hand, if $d > 0$, then the branch locus of ϕ may contain the minimal section of the Hirzebruch surface W, *i.e.,* $2N \ge -d$. If $g(f)$ is odd, then $M(f)$ is an even integer that is greater than $g(f)$. If $g(f)$ is even, then $M(f)$ is an odd integer that is greater than $g(f)$ or an even integer that is greater than $2g(f)-1$. In summary, we have

$$M(f) = \begin{cases} 6, 8, \dots, g(f)-2, g(f), g(f)+1, g(f)+2, \dots & \text{if } g(f) \text{ is even,} \\ 6, 8, 10, \dots & \text{if } g(f) \text{ is odd.} \end{cases}$$

In particular, if $6 \le M(f) \le g$, then we have $d = 0$. By using Lemma 3.1, we give the region of χ_f as follows.

Lemma 3.2 *Let S be a surface of general type and let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial, then the following hold.*

(i) *If* $6 \le M(f) \le g(f)$, *then* M *is even and*

$$A\big(g(f),\lambda(f),M(f)\big) := \frac{2M(f)(2g(f)-M(f))}{\lambda(f)M(f)-4(M(f)-2)} \le \chi_f \le \frac{g(f)}{2}M(f), \tag{3.2}$$

(ii) *If* $M(f) \ge g(f)+1$, *then*

$$B(g(f),\lambda(f),M(f)) := \frac{2M(f)(g(f)-[g(f)/2]-1)}{\lambda([g(f)/2]+1)-4[g(f)/2]} \le \chi_f \le \frac{g(f)}{2}M(f). \tag{3.3}$$

Proof By suitable elementary transformations of W, we can assume that the multiplicity of $B(\phi_{k-1})$ at the center of the blow-up ν_k is less than or equal to $g(f)+2$; that is, $[m_k/2] \le [g(f)/2]+1$ for every k (*cf.* [5, p. 746]). Applying Lemma 3.1, putting $m = [g(f)/2]+1$, we obtain inequality (3.3) in any case. In particular, assertion (ii) is proved.

We next consider the case where $6 \le M(f) \le g(f)$. By the argument just before this lemma, we have $d = 0$ and $M(f)$ is even. In particular, W is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For any point $P \in W$, there exists a section that is linearly equivalent to H and passes through P; that is, $m_k \le B\cdot H = 2N = M(f)$ for every k. By applying Lemma 3.1 putting $m = M(f)/2$, we have inequality (3.2). Note that inequality (3.3) is satisfied in the case where $6 \le M(f) \le g(f)$. However, it follows from $[M(f)/2] \le [g(f)/2]+1$ that inequality (3.2) is stronger than (3.3). Therefore, we obtain assertion (ii). ∎

We consider the minimum value of $A(g,\lambda,M)$ in the case where $6 \le M \le g$.

Lemma 3.3 *For integers* g, M *and a rational number* λ *such that*

$$6 \le M \le g, \quad 4(g-1)/g \le \lambda \le 4,$$

we have $A(g,\lambda,M) \ge A(g,\lambda,6)$.

Proof Since $M/2 < g$, we have $\lambda \ge 4(g-1)/g > (2M-4)/(M/2)$, that is, $\lambda M - 4(M-2) > 0$. Then it follows from $6 \le M \le g$ that

$$\begin{aligned} A(g,\lambda,M) - A(g,\lambda,6) &= \frac{2M(2g-M)}{\lambda M - 4(M-2)} - \frac{12(g-3)}{3\lambda - 8} \\ &= \frac{2(M-6)\{8g+(8-3\lambda)M-24\}}{\{\lambda M - 4(M-2)\}(3\lambda-8)} \\ &\ge \frac{36(M-6)(4-\lambda)}{\{\lambda M - 4(M-2)\}(3\lambda-8)}. \end{aligned}$$

Since $\lambda \le 4$ and $6 \le M$, we obtain $A(g,\lambda,M) \ge A(g,\lambda,6)$. ∎

When $6 \le M(f) \le g(f)$, by Lemmas 3.2 and 3.3, we see that

$$\chi_f \ge A\big(g(f),\lambda(f),6\big) = 12\big(g(f)-3\big)/\big(3\lambda(f)-8\big).$$

We next compare $A(g,\lambda,6)$ with $B(g,\lambda,M)$.

Lemma 3.4 *For integers g, M and a rational number λ such that*

$$6 \leq g \leq M - 1, \qquad 4(g-1)/g \leq \lambda \leq 4$$

and g is even, the following hold.

(i) *If $g \geq 6$ and $M \geq g+3$, then $B(g,\lambda,M) \geq A(g,\lambda,6)$.*
(ii) *If $g \geq 8$, then $B(g,\lambda,g+2) \geq A(g,\lambda,6)$.*
(iii) *If $g \geq 10$, then $B(g,\lambda,g+1) \geq A(g,\lambda,6)$. In particular, if $B(g,\lambda,M) < A(g,\lambda,6)$, then we have $(g,M) = (6,7),(6,8),(8,9)$.*

Proof For any $M \geq g+3$, it is clear that $B(g,\lambda,g+3) \leq B(g,\lambda,M)$. We consider $B(g,\lambda,g+3) - A(g,\lambda,6)$. Then we have

$$\begin{aligned} B(g,\lambda,g+3) - A(g,\lambda,6) &= \frac{2(g+3)(g-2)}{\lambda(g+2)-4g} - \frac{12(g-3)}{3\lambda-8} \\ &= \frac{6\lambda\{-(g-3/2)^2+33/4\}+32(g^2-5g+3)}{\{\lambda(g+2)-4g\}(3\lambda-8)}. \end{aligned}$$

By assumption, we have $-(g-3/2)^2+33/4 < 0$, *i.e.*, $6\lambda\{-(g-3/2)^2+33/4\} \geq -24g^2+72g+144$. Hence,

$$\begin{aligned} B(g,\lambda,g+3) - A(g,\lambda,6) &= \frac{6\lambda\{-(g-3/2)^2+33/4\}+32(g^2-5g+3)}{\{\lambda(g+2)-4g\}(3\lambda-8)} \\ &\geq \frac{8(g-5)(g-6)}{\{\lambda(g+2)-4g\}(3\lambda-8)}. \end{aligned}$$

This implies that assertion (i) holds.

Similarly as in the preceding case, we have

$$B(g,\lambda,g+2) - A(g,\lambda,6) \geq \frac{8(g-4)(g-8)}{\{\lambda(g+2)-4g\}(3\lambda-8)},$$
$$B(g,\lambda,g+1) - A(g,\lambda,6) \geq \frac{8(g^2-13g+34)}{\{\lambda(g+2)-4g\}(3\lambda-8)}.$$

Hence, $B(g,\lambda,g+2) \geq A(g,\lambda,6)$ if $g \geq 8$, and $B(g,\lambda,g+1) \geq A(g,\lambda,6)$ if $g \geq 10$. The last assertion follows immediately from (i)–(iii). ■

Lemma 3.5 *For integers g, M and a rational number λ such that*

$$7 \leq g \leq M - 1, \quad 4(g-1)/g \leq \lambda \leq 4,$$

and g is odd, we have $B(g,\lambda,M) \geq A(g,\lambda,6)$.

Proof Similarly as in Lemma 3.4, we consider $B(g,\lambda,g+1) - A(g,\lambda,6)$. Then we have

$$\begin{aligned} B(g,\lambda,g+1) - A(g,\lambda,6) &= \frac{2(g+1)(g-1)}{\lambda(g+1)-4(g-1)} - \frac{12(g-3)}{3\lambda-8} \\ &= \frac{6\lambda\{-(g-2)^2+9\}+32(g^2-6g+5)}{\{\lambda(g+1)-4(g-1)\}(3\lambda-8)}. \end{aligned}$$

Since $g \geq 7$ and $\lambda \leq 4$, we see that $6\lambda\{-(g-2)^2+9\} \geq -24g^2+96g+120$. Hence,

$$B(g,\lambda,g+1) - A(g,\lambda,6) \geq \frac{8(g-5)(g-7)}{\{\lambda(g+1)-4(g-1)\}(3\lambda-8)};$$

that is, $B(g,\lambda,M) \geq A(g,\lambda,6)$. ∎

Proposition 3.6 *Let S be a surface of general type and let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and satisfies that $g(f) \geq 6$, $\lambda(f) \leq 4$, then $\chi_f \geq 12(g(f)-3)/(3\lambda(f)-8)$.*

Proof By Lemmas 3.2–3.5, we have

$$\chi_f \geq A\big(g(f),\lambda(f),6\big) = 12\big(g(f)-3\big)/\big(3\lambda(f)-8\big)$$

if $g(f) = 7, 9$ or $g(f) \geq 10$. If $\chi_f < A(g(f),\lambda(f),6)$, then, by the argument in Lemma 3.2, we have $B(g(f),\lambda(f),M(f)) \leq \chi_f$, that is,

$$B\big(g(f),\lambda(f),M(f)\big) < A\big(g(f),\lambda(f),6\big).$$

Hence, by Lemma 3.4, we have three possibilities:

(a) $g(f) = 6, M(f) = 7$,
(b) $g(f) = 6, M(f) = 8$,
(c) $g(f) = 8, M(f) = 9$.

In order to prove the assertion, we only have to show that cases (a), (b), and (c) do not occur.

In case (a), it follows from (3.3) that

$$B(6,\lambda(f),7) = \frac{7}{\lambda(f)-3} \leq \chi_f < \frac{36}{3\lambda(f)-8} = A\big(6,\lambda(f),6\big). \tag{3.4}$$

In particular, we have $\lambda(f) > 52/15$. By the assumption that $\lambda(f) \leq 4$ and (3.4), we obtain

$$3\chi_f + 7 \leq K_f^2 \leq \min\Big\{\frac{8}{3}\chi_f + 12,\ 4\chi_f + 1\Big\},$$
$$7 \leq \frac{7}{\lambda(f)-3} \leq \chi_f < \frac{36}{3\lambda(f)-8} < 15.$$

Then possible pairs of values of $g(f)M(f)/2 - \chi_f$ and $2(g(f)-1)M(f) - K_f^2$ are as follows:

$$\begin{aligned}(g(f)M(f)/2-\chi_f, 2(g(f)-1)M(f)-K_f^2) = {} & (14,42), (13,39), (13,38),\\ & (12,36), (12,35), (11,33),\\ & (11,32), (10,30), (10,29),\\ & (9,27), (8,24), (7,21).\end{aligned}$$

https://doi.org/10.4153/CJM-2015-032-8 Published online by Cambridge University Press

By (2.1) and (2.2), we have

$$\frac{g(f)M(f)}{2} - \chi_f = \sum_{k=1}^{n} \frac{1}{2}[m_k/2]\big([m_k/2]-1\big),$$
$$2\big(g(f)-1\big)M(f) - K_f^2 \le \sum_{k=1}^{n} 2\big([m_k/2]-1\big)^2.$$

Furthermore, since $m_k \le g(f)+2$ from the proof of Lemma 3.2, it follows from $m_k \le 8$ that $([m_k/2]([m_k/2]-1)/2, 2([m_k/2]-1)^2)$ must coincide with one of three pairs $(1,2)$, $(3,8)$, and $(6,18)$. Therefore, we have $gM/2-\chi_f = 12$; that is, $m_1 = m_2 = 8$ and $m_k \le 3$ $(k = 3,4,\dots,n)$. Since $m_k \ne 7$, an octuple point of B lies on a fiber contained in B. On the other hand, since $g(f) = 6$ and $M(f) = 7$, we have $d = 1$ and $N = 0$. Since W is isomorphic to the Hirzebruch surface of degree one, if B contains a fiber F of pr, then B must contain the minimal section of W and F is only one fiber contained in B. But B has two octuple points. This is impossible, since B does not contain two fibers.

In case (b), similarly as in case (a), it follows from

$$B(6,\lambda(f),8) < \chi_f < A(6,\lambda(f),6)$$

that

$$11/3 < \lambda(f) \le 4, \quad 3\chi_f + 8 \le K_f^2 < \frac{8}{3}\chi_f + 12, \quad 8 \le \chi_f \le 12.$$

Then, pairs of values of $g(f)M(f)/2-\chi_f$ and $2(g(f)-1)M(f)-K_f^2$ satisfying these inequalities are as follows:

$$\begin{aligned}(g(f)M/2-\chi_f, 2(g(f)-1)M-K_f^2) = &(16,48),(16,47),(15,45),\\ &(14,42),(13,39).\end{aligned}$$

But since $([m_k/2]([m_k/2]-1)/2, 2([m_k/2]-1)^2) = (1,2),(3,8),(6,18)$, these are impossible.

In case (c), it follows from $B(8,\lambda(f),9) < \chi_f < A(6,\lambda(f),6)$ that

$$88/23 < \lambda(f) \le 4, \quad \frac{16}{5}\chi_f + \frac{54}{5} \le K_f^2 < \frac{8}{3}\chi_f + 20, \quad 14 \le \chi_f \le 17.$$

Then pairs of $g(f)M(f)/2-\chi_f$ and $2(g(f)-1)M(f)-K_f^2$ satisfying above inequalities are as follows:

$$\big(g(f)M/2-\chi_f, 2(g(f)-1)M-K_f^2\big) = (22,70),(22,69),(21,67),(20,64).$$

Since $m_k \le 10$, we have $([m_k/2]([m_k/2]-1)/2, 2([m_k/2]-1)^2)) = (1,2),(3,8),(6,18),(10,32)$. Therefore, we have the following numerical properties:

$$g(f)M(f)/2-\chi_f = 20, \quad m_1 = m_2 = 10, \quad m_k \le 3\ (k = 3,4,\dots,n).$$

On the other hand, since $g(f) = 8$ and $M(f) = 9$, we have $d = 1$ and $N = 0$. By the same reasoning as in case (a), we see that this is impossible. ∎

By Proposition 3.6, we immediately obtain the following corollaries.

Corollary 3.7 *Let S be a surface of general type and let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and satisfies that $g(f) \geq 6, \lambda(f) \leq 4$, then $\chi_f \geq 3(g(f)-3)$. If $\chi_f = 3(g(f)-3)$, then we have $\lambda(f) = 4$.*

Corollary 3.8 *Let S be a surface of general type and let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If $g(f) \geq 6$ and $\chi_f = 3(g(f)-3)$, then $\lambda(f) \geq 4$.*

4 The Lower Bound on χ_f in the Case Where $2 \leq g(f) \leq 5$

Let $f\colon S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. We assume that f is not locally trivial and satisfies that $2 \leq g(f) \leq 5$. In this section, we prove the inequality $\chi_f \geq [3g(f)/2]$. Since the base curve is $\mathbb{P}^1$, the surface S may not be minimal. Let $\overline{\rho}\colon S \to \overline{S}$ be the contraction of exceptional curves on S and ϵ the number of exceptional curves contracted by $\overline{\rho}$. We now prove the following two lemmas regardless of the value of $g(f)$.

Lemma 4.1 *Let S be a surface of general type and $f\colon S \to \mathbb{P}^1$ a relatively minimal hyperelliptic fibration. Let ϵ be as above. Then we have*

$$8\big(g(f)-1\big) - \epsilon < K_f^2 \quad \text{and} \quad \big\{K_f^2 - 4(g(f)-1)\big\}\epsilon \leq 4\big(g(f)-1\big)^2.$$

In particular, if $\epsilon \neq 0$, then the following inequality holds:

$$8(g(f)-1) - \epsilon < K_f^2 \leq 4(g(f)-1) + \frac{4(g(f)-1)^2}{\epsilon}.$$

Proof Since S is a surface of general type, we have

$$0 < K_{\overline{S}}^2 = K_S^2 + \epsilon = K_f^2 - 8\big(g(f)-1\big) + \epsilon;$$

that is, $8(g(f)-1) - \epsilon < K_f^2$.

Note that $\overline{S}$ has the linear pencil that consists of the proper images of fibers of f by $\overline{\rho}$. Let T be a member of this linear pencil. Denote the arithmetic genus of T by $p_a(T)$. Moreover, $\overline{\rho}$ is the composition of blow-ups at ϵ points (possibly infinitely near). Let $E_1, E_2, \dots, E_\epsilon$ be the exceptional curves of these blow-ups. Denote the intersection number of the proper transform of T by $\overline{\rho}$ and the total transform of E_i on S by l_i. Then we have $T^2 = \sum_{i=1}^{\epsilon} l_i^2 \geq \epsilon$ and $p_a(T) = g(f) + \sum_{i=1}^{\epsilon} l_i(l_i - 1)/2$. Hence, we obtain

$$\begin{aligned} K_{\overline{S}} \cdot T &= 2p_a(T) - 2 - T^2 = 2g(f) - 2 + \sum_{i=1}^{\epsilon} l_i(l_i-1) - T^2 \\ &= 2g(f) - 2 - \sum_{i=1}^{\epsilon} l_i \leq 2g(f) - 2 - \epsilon. \end{aligned}$$

By Hodge's index theorem, we see that $K_{\overline{S}}^2 \cdot T^2 \leq (K_{\overline{S}} \cdot T)^2$ (*cf.* [13, p. 127]). By using this inequality, we have $(K_f^2 - 8(g(f)-1) + \epsilon)\epsilon \leq \{2(g(f)-1) - \epsilon\}^2$; that is,

$$\big\{K_f^2 - 4\big(g(f)-1\big)\big\}\epsilon \leq 4\big(g(f)-1\big)^2. \qquad \blacksquare$$

The assumption that $2 \le g(f) \le 5$ is not used in Lemma 4.1. In the next section, we use this lemma for the case where $g(f) \ge 6$. By Lemma 4.1, we obtain the following lemma.

Lemma 4.2 *Let S be a surface of general type and let $f: S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. Let ϵ be as above. Then $\epsilon \le 2g(f) - 3$.*

Proof Assume that $\epsilon \ne 0$. Thus, $\epsilon \le 2g(f) - 2$. Since $K_{\overline{S}}$ is nef, we have $0 \le K_{\overline{S}} \cdot T \le 2g(f) - 2 - \epsilon$. It follows from Lemma 4.1 that $8(g(f)-1)\epsilon - \epsilon^2 < 4(g(f)-1)\epsilon + 4(g(f)-1)^2$, *i.e.*, $\{\epsilon - 2(g(f)-1)\}^2 > 0$. Hence, $\epsilon \ne 2g(f) - 2$. ∎

We now obtain the lower bound on χ_f in the case where $2 \le g(f) \le 5$.

Proposition 4.3 *Let S be a surface of general type and let $f: S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If f is not locally trivial and satisfies that $\lambda(f) \le 4, 2 \le g(f) \le 5$, then we have $\chi_f \ge [3g(f)/2]$.*

Proof Assume contrarily that $\chi_f \le [3g(f)/2] - 1$. Then by Lemma 4.1, we have

$$8\big(g(f)-1\big) - \epsilon < K_f^2 \le \lambda(f)\big([3g(f)/2]-1\big) \le 6g(f) - 4;$$

that is, $\epsilon > 2g(f) - 4$. It follows from Lemma 4.2 that $\epsilon = 2g(f) - 3$. Hence, by Lemma 4.1, we have

$$6g(f) - 5 < K_f^2 \le 4\big(g(f)-1\big) + \frac{4(g(f)-1)^2}{2g(f)-3}.$$

If $g(f) \ne 2$, then we have

$$4\big(g(f)-1\big) + \frac{4(g(f)-1)^2}{2g(f)-3} - (6g(f)-5) - 1 = \frac{2(2-g(f))}{2g(f)-3} < 0;$$

that is, there exists no integer that is greater than $6g(f)-5$ and not exceeding $4(g(f)-1) + 4(g(f)-1)^2/(2g(f)-3)$. Hence, we can assume that $g(f) = 2$. Then we obtain $\chi(\mathcal{O}_{\overline{S}}) = 1$ and $K_{\overline{S}}^2 = 1$; *i.e.*, $\overline{S}$ is a numerical Godeaux surface. Let T be as in the proof of Lemma 4.1. Since $K_{\overline{S}} \cdot T \le 1$, we see that $\dim H^0(\overline{S}, \mathcal{O}_{\overline{S}}(T)) \le 1$ by [9, Lemma 5]. It contradicts that T is a member of linear pencil of $\overline{S}$. ∎

By Corollary 3.7 and Proposition 4.3, we have the inequality $\chi_f \ge \Gamma(g(f))$ in Theorem 1.3. By Theorem 1.3, we see the following corollary.

Corollary 4.4 *Let S be a surface of general type and let $f: S \to C$ be a relatively minimal hyperelliptic fibration with $\lambda(f) = 4$. If $\chi_f < \Gamma(g(f))$, then C is a irrational curve.*

5 The Existence in the Case Where $g(f) \ge 6$

In this section, we shall show that there exists a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ with $g(f) = g, \lambda(f) = 4$ and $\chi_f = 3(g-3)$ for any integer g that is greater than or equal to six. As we have seen in Section 2, a hyperelliptic fibration onto $\mathbb{P}^1$ is induced by a double cover of a Hirzebruch surface. Let $pr_i: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection map onto the i-th factor. Put $\Delta_t = pr_1^{-1}(t)$ and $F_t = pr_2^{-1}(t)$ for any point $t \in \mathbb{P}^1$. Note that $A(g,4,6) = 3(g-3)$. Moreover, if $M(f) = 6$, then we have $d = 0$ and $N = 3$. Therefore, we construct an effective divisor B on $\mathbb{P}^1 \times \mathbb{P}^1$ that is linearly equivalent to $2(g+1)\Delta_0 + 6F_0$ for giving required fibrations. By (2.1) and (2.2), we have

$$3g - 9 = 3g - \sum_{k=1}^{n} [m_k/2]\big([m_k/2] - 1\big)/2,$$

$$12g - 36 = 12g - 12 - \sum_{k=1}^{n} 2\big([m_k/2] - 1\big)^2$$
$$+ \text{(the number of curves contracted by } \rho).$$

This implies that there exist conditions for the singularities of B and the number of curves contracted by ρ. The simplest conditions for B are as follows:

- B has three sextuple points (including infinitely near points);
- B has at worst double points except for these sextuple points;
- there exists no exceptional curve contracted by ρ.

Denote by $f_B\colon S \to C$ the hyperelliptic fibration induced by the double cover branched along an effective divisor B. Let the notation about the structure of f_B and the canonical resolution be as in Section 2.

A singular point P is called a 2-*fold* m-*ple point* of a curve B, if and only if it turns into an ordinary m-ple point after the blow-up at P.

Proposition 5.1 *For any integer $g \geq 6$, there exists a surface S of general type and a relatively minimal hyperelliptic fibration $f\colon S \to \mathbb{P}^1$ with $g(f) = g$, $\lambda(f) = 4$, and $\chi_f = 3(g-3)$.*

Proof We first give a required branch locus. Let (x,y) be local coordinates of $\mathbb{P}^1 \smallsetminus \{\infty\} \times \mathbb{P}^1 \smallsetminus \{\infty\}$. Denote the closure of the zero set of a polynomial Φ in x, y on $\mathbb{P}^1 \times \mathbb{P}^1$ by $D(\Phi)$.

Let $\Phi_{\alpha,\beta}(x,y) = y + \alpha xy + \beta x^2$, where $\alpha, \beta \in \mathbb{C} \smallsetminus \{0\}$. Then, $D(\Phi_{\alpha,\beta})$ satisfies the following properties:

- $D(\Phi_{\alpha,\beta})$ is linearly equivalent to $2\Delta_0 + F_0$;
- $D(\Phi_{\alpha,\beta})$ is tangent to F_0 at $(0,0)$ with order two;
- $D(\Phi_{\alpha,\beta})$ passes through (∞,∞) and meets F_∞ transversally.

We choose six distinct members $D_1, \dots, D_6$ of $\{D(\Phi_{\alpha,\beta}) \mid \alpha, \beta \in \mathbb{C} \smallsetminus \{0\}\}$ and $2g-10$ points $t_1, t_2, \dots, t_{2g-10}$ on $\mathbb{P}^1$. We set $B_g = \sum_{i=1}^{6} D_i + \sum_{j=1}^{2g-10} \Delta_{t_j}$. Then by a suitable choice of the above curves, B_g has the following properties:

(a) B_g is linearly equivalent to $2(g+1)\Delta_0 + 6F_0$;
(b) $(0,0)$ is the 2-fold sextuple point of B_g and the singular point infinitely near $(0,0)$ lies on the proper transform of F_0 after the blow-up at $(0,0)$;
(c) (∞,∞) is the ordinary sextuple point of B_g;
(d) B_g has at worst double points except for these sextuple points.

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We next consider the numerical properties of f_{B_g}. In Figure 1, we describe the canonical resolution of the double cover branched along B_g. To illustrate the canonical resolution, thick lines denote the branch locus B_g. Broken lines are used to represent curves not contained in B_g and thin lines denote rational curves on W_n. Double lines are used to represent irrational curves on W_n. The self-intersection number is written near the curve. When the number near a curve is omitted, it means that the self-intersection number of this curve is -2. In Figures 2, 3, 5, and 7, curves are represented in a similar manner as Figure 1.

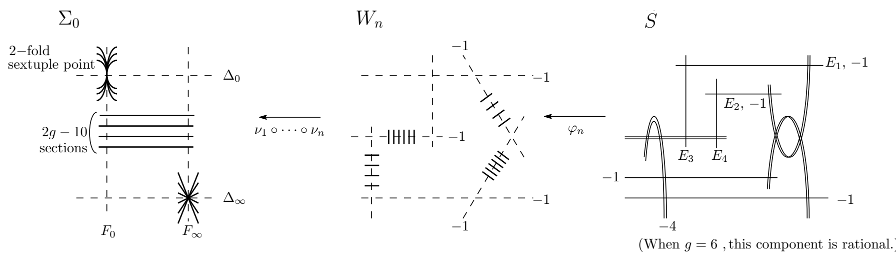

Figure 1: Double covering branched along B_g

By properties (b), (c), and (d), we may assume that $m_1 = m_2 = m_3 = 6$ and $m_k = 2$ $(k = 4, 5, \ldots, n)$. It is clear that there exists no (-1)-curve contracted by ρ (see [7, Lemma 2.2]). Hence, by (2.1) and (2.2), we have

$$g(f_{B_g}) = g, \qquad \chi_{f_{B_g}} = 3g - 9, \qquad K_{f_{B_g}} = 12(g-1) - 24 = 12g - 36.$$

We next consider the number of exceptional curves contracted by $\overline{\rho}$. Assume that $(0,0)$ is the center of the blow-up ν_1. Since $B_g \cdot \Delta_0 = 6$, it follows from property (b) that the center of ν_k does not lie on Δ_0 for $k = 2, 3, \ldots, n$. Thus, the inverse image of the proper transform of Δ_0 by ϕ_n consists of two (-1)-curves, say E_1 and E_2. After the blow-up at the ordinary sextuple point infinitely near $(0,0)$, we see that the proper transform of the exceptional curve of ν_1 is a (-2)-curve and does not meet the branch locus. It follows that its inverse image by ϕ_n consists of two (-2)-curves, say E_3 and E_4. Since E_3 (resp. E_4) meets E_1 (resp. E_2) in one point, these four curves are contracted by $\overline{\rho}$. By the property (c), there exists only one blow-up in $\{\nu_k\}_{k=1}^n$ whose center lies on Δ_∞. Then the inverse image of the proper transform of Δ_∞ by ϕ_n also consists of two (-1)-curves. It follows that there exist six exceptional curves contracted by $\overline{\rho}$ (see Figure 1). We have

$$K_{\overline{S}}^2 = 12g - 36 - 8(g-1) + 6 = 4g - 22 > 0.$$

This implies that S is a surface of general type. Hence, $f_{B_g}\colon S \to \mathbb{P}^1$ is a required fibration. ■

Remark 5.2 The surface $\overline{S}$ constructed in the proof of the above proposition has the following numerical properties:

$$\chi(\mathcal{O}_{\overline{S}}) = 3g - 9 - (g-1) = 2g - 8, \quad K_{\overline{S}}^2 = 4g - 22,$$

i.e., $\overline{S}$ is the minimal surface on the Noether line $K_{\overline{S}}^2 = 2\chi(\mathcal{O}_{\overline{S}}) - 6$.

Considering possible values of ϵ, our surfaces with $6 \le g(f) \le 10$ can be characterized by the value of $g(f)$.

Corollary 5.3 *Let S be a surface of general type and let $f: S \to \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration. If $6 \le g(f) \le 10$, $\chi_f = 3(g(f)-3)$ and $\lambda(f) \le 4$, then the minimal model $\overline{S}$ of S has the following properties:*

$$\chi(\mathcal{O}_{\overline{S}}) = 2g(f) - 8, \quad K_{\overline{S}}^2 = 4g(f) - 22.$$

Proof By Corollary 3.7, we have $\lambda(f) = 4$, *i.e.,* we have following numerical properties:

$$\chi(\mathcal{O}_{\overline{S}}) = \chi_f - \big(g(f) - 1\big) = 2g(f) - 8,$$
$$K_{\overline{S}}^2 = K_f^2 - 8\big(g(f) - 1\big) + \epsilon = 4g(f) - 28 + \epsilon.$$

Hence, it suffices to show that $\epsilon = 6$. By Lemma 4.1, we have $\epsilon \le (g(f)-1)^2/(2g(f)-8)$. Since $6 \le g(f) \le 10$, the maximum value of $(g(f)-1)^2/(2g(f)-8)$ is equal to $27/4$, *i.e.,* $\epsilon < 7$. On the other hand, by using Noether's inequality, we have

$$K_{\overline{S}}^2 \ge 2\chi(\mathcal{O}_{\overline{S}}) - 6 = 4g(f) - 22,$$

from which follows $\epsilon \ge 6$. Therefore, we obtain $\epsilon = 6$. ∎

6 Existence in the Case Where $2 \le g(f) \le 5$

In this section, we show that there exist relatively minimal hyperelliptic fibrations with $2 \le g(f) \le 5$, $\lambda(f) = 4$ and $\chi_f = [3g(f)/2]$. For this purpose, we use a similar method as in Section 5. We now introduce some notations. Let $p_d: \Sigma_d = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$ be the d-th Hirzebruch surface. Put

$$\Delta_0^{(d)} = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) \subset \Sigma_d \quad \text{and} \quad \Delta_\infty^{(d)} = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(d)) \subset \Sigma_d.$$

Note that $\Delta_0^{(d)}$ is linearly equivalent to the tautological divisor H.

Proposition 6.1 *There exists a surface S of general type and a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ with $g(f) = 2$, $\lambda(f) = 4$, and $\chi_f = 3$.*

Proof We first give the required fibration by constructing the branch locus. By [7, Lemma 2.1], we have an effective divisor D_2 on Σ_3 satisfying the following properties:

(a) D_2 is linearly equivalent to $6\Delta_0^{(3)}$;
(b) D_2 has six 2-fold triple points on $\Delta_0^{(3)}$, say $Q_1, Q_2, \dots, Q_6$;
(c) for each i, the singular point R_i infinitely near Q_i lies on the proper transform of $p_3^{-1}(p_3(Q_i))$ after the blow-up at Q_i;

(d) D_2 has at worst double point except for singular points on $\Delta_0^{(3)}$.

We put $B_2 = D_2 + p_3^{-1}(p_3(Q_1)) + p_3^{-1}(p_3(Q_2))$ and consider the hyperelliptic fibration $f_{B_2}: S \to \mathbb{P}^1$. Note that Q_1 and Q_2 are 2-fold quadruple points of B_2. In Figure 2, we describe the canonical resolution of the double cover branched along B_2.

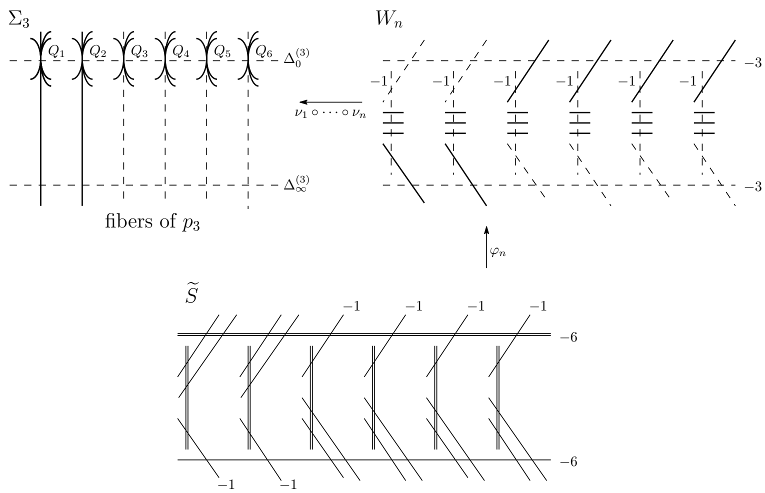

Figure 2: B_2 and the branch divisor after the canonical resolution

We next count the number of exceptional curves on $\widetilde{S}$. By properties (b) and (c), the set $\{\nu_k\}_{k=1}^n$ contains blow-ups at Q_i's and R_i's. Since $B_2 \cdot F = 6$, it follows again from properties (b) and (c) that there exists no other blow-up at a point on $p_d^{-1}(p_d(Q_i))$ in $\{\nu_k\}_{k=1}^n$. Hence, the proper transform of each $p_d^{-1}(p_d(Q_i))$ is a (-2)-curve. Since B_2 contains both $p_d^{-1}(p_d(Q_1))$ and $p_d^{-1}(p_d(Q_2))$, the inverse image of the proper transform of each $p_d^{-1}(p_d(Q_i))$ by ϕ_n is a (-1)-curve.

On the other hand, we see that inverse images of the proper transforms of exceptional curves introduced by blow-ups at $Q_3, \dots, Q_6$ are (-1)-curves on $\widetilde{S}$ (see [11, p. 13]). Hence, six exceptional curves are contracted by ρ (see Figure 2). It follows from (2.1) and (2.2) that

$$g(f_{B_2}) = 2, \quad \chi_{f_{B_2}} = 11 - 8 = 3, \quad K_{f_{B_2}}^2 = 22 - 16 + 6 = 12, \quad K_S^2 \geq 4;$$

that is, f_{B_2} is a required fibration. ■

Proposition 6.2 *There exists a surface S of general type and a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ with $g(f) = 3$, $\lambda(f) = 4$, and $\chi_f = 4$.*

Proof In order to construct the required fibration, we give two kinds of effective divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. Put $\Psi_\alpha(x, y) = y^2 - 1 + (y^2 + 1)x + \alpha(y^2 + 1)x(x - 1)$. If $\alpha \neq 0$, then $D(\Psi_\alpha)$ satisfies the following properties:

- $D(\Psi_\alpha)$ is linearly equivalent to $2\Delta_0 + 2F_0$;
- $D(\Psi_\alpha)$ is tangent to $F_{\sqrt{-1}}$ (resp. $F_{-\sqrt{-1}}$) at $(\infty, \sqrt{-1})$ (resp. $(\infty, -\sqrt{-1})$);
- $D(\Psi_\alpha)$ is tangent to Δ_1 at $(1,0)$;
- $D(\Psi_\alpha)$ passes through $(0, \pm 1)$ and meets F_1 and F_{-1} transversally.

Let $D_{\alpha,\beta} = D((x-1)\Psi_\alpha + \beta x y^2)$. If $\alpha, \beta \neq 0$, then $D_{\alpha,\beta}$ satisfies the following properties:

- $D_{\alpha,\beta}$ is linearly equivalent to $3\Delta_0 + 2F_0$;
- $D_{\alpha,\beta}$ is tangent to $F_{\sqrt{-1}}$ (resp. $F_{-\sqrt{-1}}$) at $(\infty, \sqrt{-1})$ (resp. $(\infty, -\sqrt{-1})$);
- $(1,0)$ is the simple double point of $D_{\alpha,\beta}$;
- $D_{\alpha,\beta}$ passes through $(0,\pm1)$ and meets F_1 and F_{-1} transversally.

We choose four distinct nonzero complex numbers $\alpha_1, \alpha_2, \alpha_3, \beta$. Put

$$B_3 = \sum_{i=1}^{2} D(\Psi_{\alpha_i}) + D(\Psi_{\alpha_3,\beta}) + \Delta_0.$$

Under an appropriate choice of β, we can take B_3 satisfying the following properties:

(a) B_3 is linearly equivalent to $8\Delta_0 + 6F_0$;
(b) $(\infty, \pm\sqrt{-1})$ are 2-fold triple points of B_3 and the singular points infinitely near $(\infty, \pm\sqrt{-1})$ lie on the proper transform of fibers after the blow-up at Q_i;
(c) $(1,0)$ is a quadruple point of B_3 that decomposes into one ordinary double point after the blow-up at $(1,0)$;
(d) $(0,\pm1)$ are ordinary quadruple points of B_3;
(e) B_3 has at worst double points except for these singularities.

We now consider the numerical properties of the hyperelliptic fibration $f_{B_3}: S \to \mathbb{P}^1$ and S. In Figure 3, we describe the canonical resolution of the double cover branched along B_3. The symbol $\circ$ means that two curves passing it do not intersect each other.

Since B_3 has two 2-fold triple points, we see that there exists two exceptional curves contracted by ρ (*cf.* [11, p. 13]). Hence, by (2.1) and (2.2), we obtain

$$g(f_{B_3}) = 3, \qquad \chi_{f_{B_3}} = 9 - 5 = 4, \qquad K^2_{f_{B_3}} = 24 - 10 + 2 = 16.$$

Since there exists no singular point on Δ_0 except for $(0,\pm1)$, the proper transform of Δ_0 is a (-2)-curve. Moreover, since Δ_0 is contained in B_3, the inverse image of the proper transform of Δ_0 by ϕ_n must be a (-1)-curve (see Figure 3). Thus, we have $K_S^2 = 16 - 8(3-1) + 1 > 0$; *i.e.*, S is a surface of general type. It follows that f_{B_3} is a required fibration. ∎

Proposition 6.3 *There exists a surface S of general type and a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ with $g(f) = 4$, $\lambda(f) = 4$, and $\chi_f = 6$.*

Proof We first give a required fibration by constructing the branch locus on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $a_k(y)$ $(k = 1,2,3)$ be polynomials of degree two satisfying the following conditions:

- $a_k(0) \neq 0$ for every k;
- if $k \neq l$, then $a_k(y)$ and $a_l(y)$ are coprime and $a_k(0) \neq a_l(0)$.

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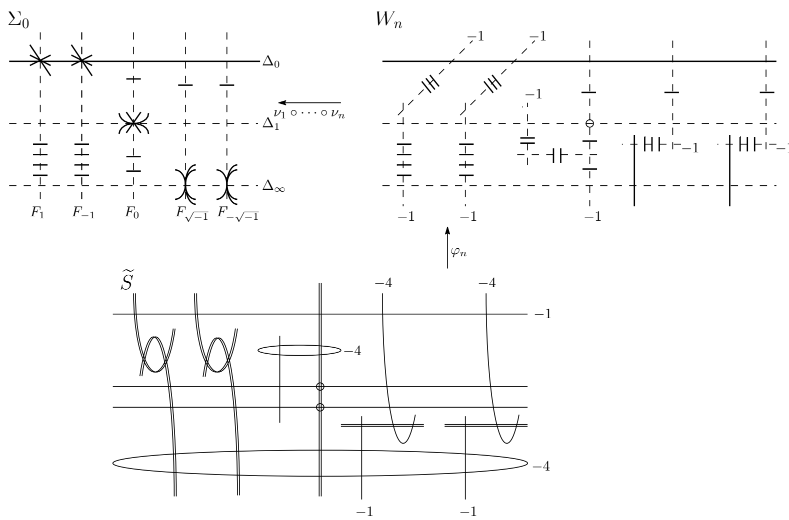

Figure 3: B_3 and the branch divisor after the canonical resolution

Put $P_1 = (0,0)$, $P_2 = (0,\infty)$ and $D_4 = \sum_{k=1}^{3} D(y + a_k(y)x)$. Denote the elementary transformation centered $P \in \Sigma_d$ by ι_P. Let $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the automorphism that exchanges two factors. In order to simplify the notation, in cases where no confusion can arise, the image of $P \in \Sigma_d$ by birational maps is denoted by the same letter P. By appropriate choices of $a_k(y)$'s, we may assume that P_1 and P_2 are ordinary triple points of D_4 and that D_4 has six ordinary double points, say $Q_1, Q_2, \dots, Q_6$. Furthermore, we can assume that $pr_2(Q_i)$'s are mutually distinct points on $\mathbb{P}^1$. Then $\iota = \sigma \circ \iota_{Q_4} \circ \iota_{Q_3} \circ \iota_{Q_2} \circ \iota_{Q_1} \circ \sigma$ is a birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself. Let $\iota[D_4]$ be the proper image of D_4 by ι. In Figure 4, we describe $\sigma(D_4)$ and $\sigma(\iota[D_4])$. Thick lines denote $\sigma(D_4)$ and $\sigma(\iota[D_4])$. Broken lines are used to represent curves except for $\sigma(D_4)$ and $\sigma(\iota[D_4])$.

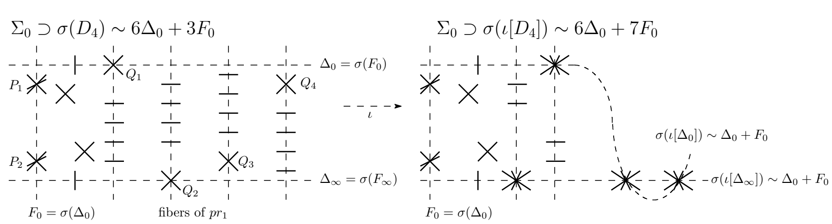

Figure 4: $\sigma(D_4)$ and $\sigma(\iota[D_4])$

Since Q_i is an ordinary double point of D_4, the elementary transformation centered Q_i induces a simple quadruple point on $\iota[D_4]$. Thus, $\iota[D_4]$ has four simple quadruple points. Since P_1 and P_2 lie on the fiber $\sigma(\Delta_0)$, the images of P_1 and P_2 by ι are also ordinary triple points of $\iota[D_4]$ contained in Δ_0. Then $\iota[D_4]$ is linearly equivalent to $7\Delta_0 + 6F_0$ and has two ordinary triple points and four simple quadruple points (see Figure 4).

We take Δ_{t_l} $(l = 1, 2)$ not meeting $\iota[D_4]$ at its singularities. Put $B_4 = \iota[D_4] + \Delta_0 + \sum_{l=1}^{2} \Delta_{t_l}$. Then B_4 satisfies the following properties:

(a) B_4 is linearly equivalent to $10\Delta_0 + 6F$;
(b) B_4 has six simple quadruple points;
(c) two of these quadruple points lie on Δ_0;
(d) B_4 has at worst double points except for these singularities.

Considering the hyperelliptic fibration $f_{B_4}: S \to \mathbb{P}^1$, it is clear that there exists no (-1)-curve contracted by ρ.

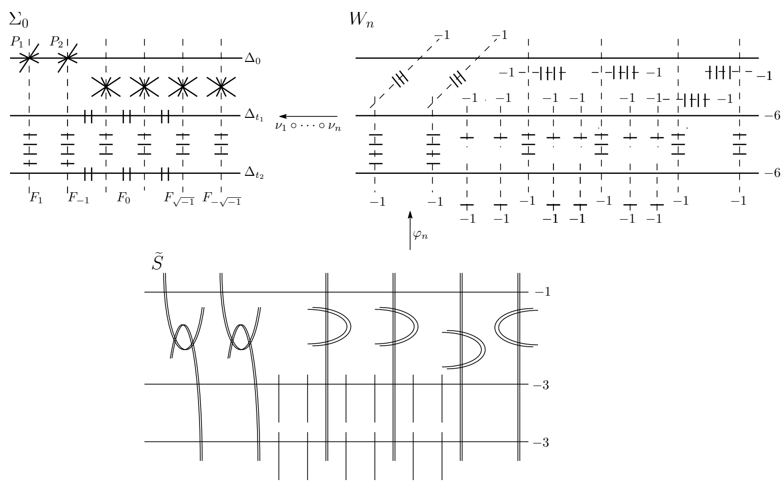

Figure 5: B_4 and the branch divisor after the canonical resolution

Hence, by (2.1), (2.2), and the above properties, we have

$$g(f_{B_4}) = 4, \qquad \chi_{f_{B_4}} = 12 - 6 = 6, \qquad K_{f_{B_4}}^2 = 36 - 12 = 24.$$

Furthermore, by the same argument as in the proof of Proposition 6.2, the inverse image of the proper transform of Δ_0 by ϕ_n must be a (-1)-curve, *i.e.*, $K_{\widetilde{S}}^2 = 24 - 8(4 - 1) + 1 = 1 > 0$ (see Figure 5). It implies that S is a surface of general type. Hence, f_{B_4} is a required fibration. ∎

Proposition 6.4 *There exists a surface S of general type and a relatively minimal hyperelliptic fibration $f: S \to \mathbb{P}^1$ with $g(f) = 5$, $\lambda(f) = 4$ and $\chi_f = 7$.*

https://doi.org/10.4153/CJM-2015-032-8 Published online by Cambridge University Press

Proof Let D_4 be as in the proof of Proposition 6.3. Let R_1 and R_2 be points on D_4 such that $pr_i(R_1) \neq pr_i(R_2)$ $(i = 1, 2)$. Then $\iota' = \sigma \circ \iota_{R_1} \circ \iota_{R_2} \circ \sigma$ is a birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself. In Figure 6, using a similar manner as in Figure 4, we describe $\sigma(D_4)$ and $\sigma(\iota'[D_4])$.

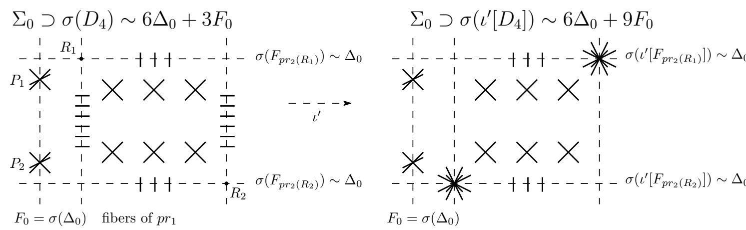

Figure 6: $\sigma(D_4)$ and $\sigma(\iota'[D_4])$

Since R_i does not lie on D_4, the elementary transformation centered R_i induces a simple sextuple point on $\iota'[D_4]$. Hence, $\iota'[D_4]$ has two simple sextuple points. Furthermore, the image of P_1 and P_2 by ι' are also ordinary triple points on Δ_0.

We take Δ_{s_l} $(l = 1, 2)$ not meeting $\iota'[D_4]$ at its singularities. Put $B_5 = \iota'[D_4] + \Delta_0 + \sum_{l=1}^{2} \Delta_{s_l}$. Then B_5 satisfies the following properties:

(a) B_5 is linearly equivalent to $12\Delta_0 + 6F$;
(b) B_5 has two simple sextuple points and two ordinary quadruple points;
(c) two quadruple points lie on Δ_0;
(d) B_5 has at worst double points except for these singularities.

We now consider the numerical properties of the hyperelliptic fibration $f_{B_5}: S \to \mathbb{P}^1$. It is easy to see that there exists no exceptional curve contracted by ρ. By (2.1), (2.2), and the above properties, we have

$$g(f_{B_5}) = 5, \qquad \chi_{f_{B_5}} = 15 - 8 = 7, \qquad K^2_{f_{B_5}} = 48 - 20 = 28.$$

By the same argument as in the proof of Proposition 6.2, the inverse image of the proper transform of Δ_0 by ϕ_n must be a (-1)-curve. Furthermore, since each $\Delta_{pr_2(R_i)}$ meets B_5 in only one sextuple point, the proper transform of $\Delta_{pr_2(R_i)}$ by ϕ_n consists of two (-1)-curves (see Figure 7). Hence, we have $K_S^2 = 28 - 8(5-1) + 5 > 0$, which implies that S is a surface of general type. Thus, f_{B_5} is a required fibration. ■

By Propositions 5.1 and 6.1–6.4, we complete the proof of Theorem 1.3. ■

Acknowledgment The author expresses his thanks to Professor Tadashi Ashikaga for valuable discussions and advice. Thanks are also due to Professor Kazuhiro Konno for encouragements and advice. The author is also grateful to the referee for suggesting improvements to this paper.

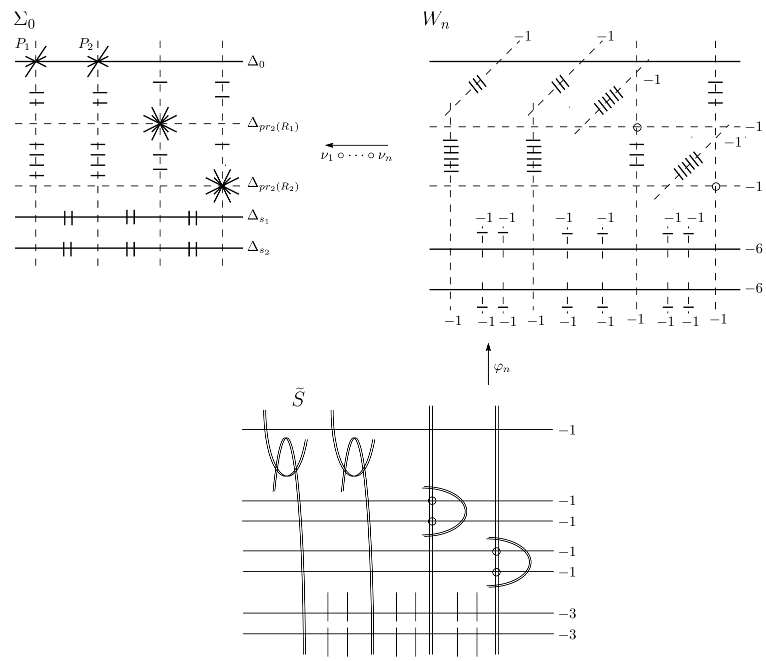

Figure 7: B_5 and the branch divisor after the canonical resolution

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