LIFTS OF PARTIAL CHARACTERS WITH CYCLIC DEFECT GROUPS

JAMES P. COSSEY $^{\boxtimes}$ and MARK L. LEWIS

Abstract

We count the number of lifts of an irreducible π-partial character that lies in a block with a cyclic defect group.

1. Introduction

Throughout this paper, G is a finite group and p is a fixed prime. We use the standard notation of finite group theory, found, for instance, in [14]. In particular, we will write $\mathrm{Irr}(G)$ for the (ordinary) irreducible characters of G and $\mathrm{IBr}_p(G)$ for the irreducible p-Brauer characters of G.

Recall that the p-blocks of G are subsets of $\mathrm{Irr}(G) \cup \mathrm{IBr}_p(G)$ that describe the relationship between the ordinary (characteristic zero) and modular (characteristic p) representations of G. In particular, one is often interested in the decomposition matrix of the block B, which encodes how the characteristic-zero representations in B decompose when they are 'reduced modulo p'.

Moreover, much of the structure of a p-block B of G is determined by the associated defect group $D \subseteq G$ (see [15] for more details). The structure of a p-block B of G with a cyclic defect group D has long been well understood (see [6, Ch. 68]). These results, combined with an application of the Fong–Swan theorem, completely describe the structure of the decomposition matrix of a p-block B if G is assumed to be solvable or p-solvable (see Theorem 2.1 below).

Recall that a *lift* of a Brauer character φ is an ordinary character χ such that $\chi^o = \varphi$ where ^{o} represents restriction to the p-regular elements of G. When G is a p-solvable group and φ is an irreducible Brauer character of G whose defect group D is cyclic, one can use the above information about the block of φ to see that either φ has a

unique lift or φ has exactly $|D|$ lifts and the two possibilities are distinguished by whether $|\mathrm{IBr}_p(B)| > 1$ or $|\mathrm{IBr}_p(B)| = 1$.

For π-separable groups, it is possible to replace p by a set of primes π and obtain the Isaacs π-partial characters $\mathrm{I}_\pi(G)$ of the π-separable group G (see [12]), and many results that are true for Brauer characters of p-solvable groups generalize to the Isaacs partial characters of a π-separable group. The block theory of Brauer characters was generalized to the block theory of Isaacs partial characters by Slattery in [17, 18].

Thus, it makes sense to ask about the lifts of Isaacs partial characters in a block with cyclic defect group. If φ is an Isaacs partial character, then a lift of φ is an ordinary character χ whose restriction is φ. In a number of situations, the first author has been able to find bounds on the number of lifts of a given Isaacs partial character (see [2, 3]).

We develop exact counts for the number of lifts of an irreducible π-partial character that belongs to a π-block of a π-separable group with cyclic defect group. We will show that the count of lifts of Brauer characters with a cyclic defect group is actually a special case of a more general result that counts the number of lifts of an Isaacs π-partial character of a π-block of a π-separable group with cyclic defect group D. The definition and method of computation of the function l_S will be discussed in Section 8. For now, we point out that l_S takes values in $\{0, 1\}$.

THEOREM 1.1. *Suppose that G is a π-separable group and $\xi \in \mathrm{I}_\pi(B)$, where B is a π-block of G with cyclic defect group D. Then the number of lifts of ξ is given by the formula*

$$1 + \sum_S l_S(\xi) \prod_{p_j \in S} (|D|_{p_j} - 1),$$

where the sum is over the nonempty subsets S of the set of prime divisors of $|D|$.

We mention here that this count is clearly bounded above by $|D|$, and thus the number of lifts of ξ is bounded above by $|D|$. Also, if $l_S(\xi) = 1$ for all subsets S, then the above sum is equal to $|D|$. Though this bound can be obtained using other approaches, we know of no other way to obtain this exact count. Previously, only in specific examples was it possible to compute the exact number of lifts, and in most families of examples the best that we can hope for is an upper and lower bound on the number of lifts, so it is quite remarkable that in this case we can always determine precisely the number of lifts.

Recall that one can associate a (unique up to conjugacy) vertex subgroup $Q \subseteq G$ to each irreducible Brauer (or Isaacs π-partial) character ξ, and that the vertex subgroup is contained in the defect subgroup D of the block containing ξ. We will show in Section 9 that in the situation of Theorem 1.1, D is also a vertex subgroup for the character ξ. Therefore Theorem 1.1 is similar to the main result in [2], with the exception that here we do not require G to have odd order, but we do require the vertex subgroup to be cyclic.

To prove our result about lifts, we generalize the results of cyclic p-blocks in p-solvable groups to cyclic π-blocks in π-separable groups. The classical proofs about cyclic p-blocks (which apply to all finite groups) use results about modules

and some number theory. These results are not applicable when we replace the prime p with the complement of a set of primes π. Thus, we will provide a much simpler and self-contained proof of the structure of the decomposition matrix of a p-block of a p-solvable group G with cyclic defect group. We will then be able to generalize to obtain the decomposition matrix of a π-block of a π-separable group with cyclic defect group. Moreover, we will see that some of the results only require that the defect group of the π-block in question is abelian, and thus we will acquire new results under that assumption.

We conclude with an application. Let B be a block. The *Brauer graph* of B is the graph whose vertex set is $\text{Irr}(B)$, and an edge joins χ and ψ if their restrictions have a common irreducible Brauer (or Isaacs partial) constituent. It is well known [6] that in the classical case, the Brauer graph for B (when B has a cyclic defect group) has diameter at most 2. We will prove the following, as a corollary to Theorem 2.5.

THEOREM 1.2. *Suppose that G is a π-separable group and B is a π-block of G with cyclic defect group D. Then the diameter of the Brauer graph of B is at most 2.*

We will end with a discussion of the vertex subgroups of the π-blocks of the π-separable group G that have cyclic defect group.

2. Definitions and results

The following well-known theorem [6] describes the decomposition matrix of the p-block B of a p-solvable group G with cyclic defect group D. The subgroup E discussed below will be defined in Section 4.

THEOREM 2.1. *Let G be a p-solvable group and B a block of G with cyclic defect group D of order p^n. Then there exists a subgroup E of G such that $\mathbf{C}_G(D) \subseteq E \subseteq \mathbf{N}_G(D)$ and $e = |E : \mathbf{C}_G(D)|$, and further the decomposition matrix of B is the $(e + (p^n - 1)/e)$ by e matrix*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

We will generalize this result to the irreducible π-partial characters of a π-separable group G. Here we briefly mention that if π is a set of primes and G is a π-separable group, then one can define (see [12] for more details) a set of class functions $I_\pi(G)$ from the set G^o (which consists of the elements of G whose order is divisible by only

the primes in π) to $\mathbb{C}$ that plays the role of $\mathrm{IBr}_p(G)$, and in fact $\mathrm{I}_\pi(G) = \mathrm{IBr}_p(G)$ if $\pi = \{p'\}$, the complement of the prime p. In Sections 3 and 4 we will review the definition of π-blocks in this case and define the defect group and root of a π-block, the associated canonical character η, and the stabilizer subgroup E which will be used in Section 6 to prove our first main result, which we now state.

THEOREM 2.2. *Let G be a π-separable group and let B be a π-block of G with abelian defect group D. Let $(\mathbf{C}_G(D), C)$ be a root of the block B, let $\eta \in \mathrm{I}_\pi(C)$ be the canonical character of C, and let $E \subseteq \mathbf{N}_G(D)$ be the stabilizer of C. Then $E/\mathbf{C}_G(D)$ is a π-subgroup of $\mathrm{Aut}(D)$, and*

$$|\mathrm{I}_\pi(B)| = |\mathrm{I}_\pi(E|\eta)|.$$

We mention here that in the case that $\pi = \{p'\}$, this is a special case of the McKay conjecture, although we give an easy independent proof.

To discuss our second main result, we will need two rather technical definitions. However, we can first summarize this result (Theorem 2.5 below) by saying that the structure of the decomposition matrix of a π-block B of a π-separable group with cyclic defect group D is determined entirely by the action of the π-group $E/\mathbf{C}_G(D)$ on D and the characters of the subgroups of E lying over the canonical character of the root of B.

DEFINITION 2.3. Suppose that D is a cyclic group, and $D = D_{p_1} \times \cdots \times D_{p_k}$, where the D_{p_i} are the Sylow p_i-subgroups of D for the distinct primes $p_1, \ldots, p_k$. If $\lambda = \lambda_1 \times \cdots \times \lambda_k \in \mathrm{Irr}(D)$, then we define the support $S_\lambda \subseteq \{p_1, \ldots, p_k\}$ of λ by $p_i \in S_\lambda$ if and only if $\lambda_i \neq 1_{D_{p_i}}$.

We will show (as an easy consequence of Fitting's theorem) that if a π-group $\bar{E}$ acts coprimely on a cyclic π'-group D, and if $\lambda, \mu \in \mathrm{Irr}(D)$ have the same support S, then λ and μ have the same stabilizer in $\bar{E}$, which we will denote $\bar{E}_S$. With the above notation, define

$$n_S = \frac{1}{|\bar{E} : \bar{E}_S|} \prod_{p_i \in S} (|D|_{p_i} - 1)$$

if S is nonempty, and $n_\emptyset = 1$. Note that

$$\prod_{p_i \in S} (|D|_{p_i} - 1)$$

is the number of characters of D with support S if S is nonempty, and n_S is the number of orbits of the action of $\bar{E}$ on the characters of D with support S.

Now suppose that E is a π-separable group that acts on the cyclic π'-group D, with kernel C, and suppose that E/C is a π-group. Let $\eta \in \mathrm{I}_\pi(C)$ be invariant in E. We now work towards defining the matrix M_η, which will be key to our second main result.

As above, let $\{p_1, \ldots, p_k\}$ be the prime divisors of $|D|$, and let S be a (possibly empty) subset of $\{p_1, \ldots, p_k\}$. Let $E_S \subseteq E$ be the stabilizer of any (and hence

every, as discussed above) character of D with support S, and let $\varphi_1, \ldots, \varphi_{e_S}$ be the characters in $\mathrm{I}_\pi(E_S|\eta)$. Let $\theta_1, \ldots, \theta_e$ denote the characters of $\mathrm{I}_\pi(E|\eta)$. When $1 \leq i \leq e_S$, we define the n_S by e matrix $M(S, i)$ to be

$$\begin{pmatrix} [(\varphi_i)^E, \theta_1] & [(\varphi_i)^E, \theta_2] & \cdots & [(\varphi_i)^E, \theta_e] \\ [(\varphi_i)^E, \theta_1] & [(\varphi_i)^E, \theta_2] & \cdots & [(\varphi_i)^E, \theta_e] \\ \vdots & \vdots & & \vdots \\ [(\varphi_i)^E, \theta_1] & [(\varphi_i)^E, \theta_2] & \cdots & [(\varphi_i)^E, \theta_e] \end{pmatrix}.$$

We define the matrix $M(S)$ to be

$$\begin{pmatrix} M(S, 1) \\ M(S, 2) \\ \vdots \\ M(S, e_S) \end{pmatrix}.$$

DEFINITION 2.4. With the notation above, let $\{\emptyset = S_0, S_1, \ldots, S_l = S\}$ be the set of subsets of $\{p_1, \ldots, p_k\}$. Then the matrix M_η is defined to be

$$\begin{pmatrix} M(S_0) \\ M(S_1) \\ \vdots \\ M(S_l) \end{pmatrix}.$$

Notice that $M(S_0)$ is necessarily the e by e identity matrix. Our second main result describes the decomposition matrix of a π-block B of a π-separable group G with cyclic defect group D.

THEOREM 2.5. *Let G be a π-separable group and let B be a π-block of G with cyclic defect group D, and suppose that $\{p_1, \ldots, p_k\}$ is the set of prime divisors of $|D|$. Let $(\mathbf{C}_G(D), C)$ be a root of B with canonical character η, and let E be the stabilizer of C in $\mathbf{N}_G(D)$. Then, with the above notation, M_η is the decomposition matrix of B. Therefore the decomposition matrix of B is a $\sum_S e_S n_S \times e$ matrix, where the sum is over all subsets S of $\{p_1, \ldots, p_k\}$.*

Notice that by Theorem 2.5, each character $\xi \in \mathrm{I}_\pi(B)$ is 'labeled' by a character in $\varphi \in \mathrm{I}_\pi(\mathbf{N}_G(D)|\eta)$. We do not claim that this labeling is 'natural'.

As we will see, the decomposition matrices for π-blocks have many similarities and differences with the 'classical' case where $\pi = \{p'\}$. For instance, unlike in the classical case, the decomposition matrix does not have constant columns in the 'bottom' submatrix, though it is composed of submatrices that have constant columns. Note that in the classical case, there is only one nonempty subset to consider, namely $\{p\}$. We will see that $E_{\{p\}} = \mathbf{C}_G(D)$ in this case and that $E/\mathbf{C}_G(D)$ is cyclic, so that η extends to E, and thus that each entry in the matrix $M(\{p\})$ is 1, and we recover Theorem 2.1.

We began this section by mentioning that we will also be giving a new and simpler proof of Theorem 2.1. The arguments in the following sections are all done in the setting of π-blocks of π-separable groups. All of the basic lemmas that we develop in the next few sections are easy and well-known results in the classical case. Moreover, there are exactly two instances in the following proofs where we use very nontrivial facts about π-blocks. However, in the classical case these difficult arguments may be replaced by much simpler arguments that exploit the fact (discussed below) that the factor group $E/\mathbf{C}_G(D)$ is cyclic, and its order divides $p - 1$.

3. π-blocks of π-separable groups

This section and the next contain a lengthy discussion of certain results about covering and block induction for π-blocks, with the goal of defining roots of π-blocks. In the case where $\pi = \{p'\}$, so that $\mathrm{I}_\pi(G) = \mathrm{IBr}_p(G)$, all of these results are standard, and thus the reader only interested in the new proof of the values of the decomposition matrix for p-solvable groups may skip this section and the next. In the case where G is π-separable and a Hall π'-subgroup is nilpotent, the results in the next section (which define the root and canonical character of a π-block) are known (see, for instance, [10]). However, the standard proofs require the use of Brauer's first main theorem, which in turn requires that the Hall π'-subgroups of G are nilpotent. We would like to define roots and canonical characters without the assumption that the Hall π'-subgroups are nilpotent. While it is unclear whether this is true in general, it is possibly so if the defect group D of the block in question is abelian, as we prove in Section 4.

We briefly review the notion of a π-block of a π-separable group. For further discussion of the irreducible π-partial characters $\mathrm{I}_\pi(G)$, see [12], and for more discussion of the basic properties of π-blocks, see [17, 18]. For the moment, we need only the basic fact that if $\chi \in \mathrm{Irr}(G)$, then there is a unique decomposition

$$\chi^o = \sum_{\varphi \in \mathrm{I}_\pi(G)} d_{\chi\varphi}\varphi,$$

where the $d_{\chi\varphi}$ are nonnegative integers, and χ^o denotes the restriction of χ to the elements of G whose order is divisible only by primes in π.

DEFINITION 3.1. Let G be a π-separable group and let $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{I}_\pi(G)$. We say that χ and φ are linked if $d_{\chi\varphi} \neq 0$. A block B of G is a subset of $\mathrm{Irr}(G) \cup \mathrm{I}_\pi(G)$ that is a connected component under this linking.

We now begin a brief discussion of block covering, which will be followed by a discussion of defect groups and block induction for π-blocks of π-separable groups. It should be noted that the sets that we are calling here π-blocks would be called π'-blocks in [17, 18].

DEFINITION 3.2. Suppose that G is a π-separable group, $N \lhd G$ and b is a block of N. If $\chi \in \mathrm{Irr}(G)$ is such that some constituent of χ_N lies in b, then we say that

χ covers b. If B is a block of G such that some irreducible character in B covers b, then we say that B covers b.

Since all of the constituents of χ_N are conjugate, it is easy to see that the set of blocks of N covered by χ form a single G-orbit. Also, it is clear that G acts by conjugation on the set of blocks of N. The following result shows that in fact the set of blocks of N covered by B form a single conjugacy class.

LEMMA 3.3. *Suppose that G is a π-separable group and $N \lhd G$ and also that B is a block of G and b is a block of N covered by B. If b_1 is another block of N covered by B, then b and b_1 are conjugate. If χ is a character of B, then χ lies over some character φ in b. If θ is in b, then there is a character ψ in B such that ψ lies over θ.*

PROOF. The proofs of the statements in this lemma follow exactly as in the classical case (see [15], for instance), and we omit them here. □

Our next result—which is the Fong reduction for π-separable groups—is key, and is [17, Theorem 2.10].

THEOREM 3.4. *Let G be a π-separable group and N be a normal π-subgroup of G, such that $\alpha \in \mathrm{Irr}(N)$ and $T = G_\alpha$. Then Clifford induction of characters yields a bijection from the set of blocks of T that cover φ to the set of blocks of G that cover φ.*

The defect group of a π-block is defined in [18] by the Fong reduction. Let B be a π-block of a π-separable group, and let $N = \mathbf{O}_\pi(G)$ and suppose that $\alpha \in \mathrm{Irr}(N)$ is covered by B. If α is invariant in G, then B consists of all of the ordinary irreducible and irreducible π-partial characters of G lying above α, and the defect group of B is defined to be any Hall π'-subgroup of G. If α is not invariant in G, then let $T = G_\alpha$. Then there is a unique block B_1 of T that covers α such that the map $\psi \to \psi^G$ is a bijection from B_1 to B. One then recursively defines the defect group of B to be a defect group for B_1. It is shown in [18] that this is well defined and equivalent to the classical p-defect group in the case where $\pi' = \{p\}$. Note that an immediate consequence of the above definition is that $\mathbf{O}_{\pi'}(G)$ is contained in every defect group for the block B.

To define block induction, Slattery [18] uses the Glauberman correspondence along with the Fong reduction. Let D be a π'-subgroup of the π-separable group G, and suppose that b is a block of $\mathbf{N}_G(D)$ with defect group D. We define the block $B = b^G$ of G as follows. Let $N = \mathbf{O}_\pi(G)$, and let $C = \mathbf{C}_N(D) = \mathbf{N}_N(D)$. Let $\beta \in \mathrm{Irr}(C)$ be any character covered by b, and note that by the Glauberman correspondence, there is a unique character α of $\mathrm{Irr}_D(N)$ such that $\widehat{\alpha} = \beta$, where $\alpha \to \widehat{\alpha}$ is the Glauberman correspondence for the action of D on N. If α is invariant in G, then define $B = b^G$ to be the set of all characters of G lying above α. If $T = G_\alpha < G$, then let $I = T \cap \mathbf{N}_G(D)$, and note that by the uniqueness in the Glauberman correspondence, I is the stabilizer of β in $\mathbf{N}_G(D)$. Let $b_1 \in \mathrm{Bl}(I | \beta)$ be the Clifford correspondent for b, and we recursively define $B_1 \in \mathrm{Bl}(T)$ by $B_1 = (b_1)^T$. Slattery shows that B_1 lies over α, and lets $B = b^G$ be the block of G defined by Clifford induction of the

block B_1. Slattery also shows that this construction is independent of the choices made, and that, in the case where $\pi' = \{p\}$, this notion of block induction corresponds to the classical notion.

It would be nice to be able to say that if $b \in \text{Bl}(\mathbf{N}_G(D))$ has defect group D, then b^G has defect group D. Unfortunately, we would need to require that the Hall π'-subgroups of our π-separable group G are nilpotent for the above statement to be true. (In fact, Slattery [18] shows that the map $b \to b^G$ is a defect group preserving bijection for every π'-subgroup D if and only if the Hall π'-subgroups are nilpotent.) However, we will not need the full strength of Brauer's first main theorem. In fact, what we will need is a weak version of Brauer's first main theorem, which is Theorem 3.5 from [18].

THEOREM 3.5. *Let $D \subseteq G$ be a π'-subgroup of a π-separable group G. If B is a block of G with defect group D, then there is exactly one block b of $\mathbf{N}_G(D)$ with $b^G = B$. Moreover, b has defect group D.*

4. Roots of π-blocks

We will need to define roots of blocks, and we will need to prove some easy results about them. To do so we will need the following results, which are well known in the classical case, and are true in the π case, though the proofs do not seem to be written down anywhere. For the definition of the vertex of an irreducible π-partial character, see [13].

LEMMA 4.1. *Let G be a π-separable group, and let $\varphi \in \text{I}_\pi(G)$ be in the π-block B. If D is a defect group of B, then some vertex subgroup Q of φ is contained in D.*

PROOF. Let $N = \mathbf{O}_\pi(G)$, and let $\alpha \in \text{I}_\pi(N)$ be a constituent of φ_N. If α is invariant in G, then by the definition of the defect group of B, we see that D is a Hall π'-subgroup of G. Since any vertex Q of φ is a π'-subgroup and G is π-separable, D contains some vertex subgroup of φ.

Thus we may assume that φ_N is not homogeneous. By the definition of the defect group of B, there is a constituent α of φ_N such that D is a defect group for B_0, where B_0 is the block of G_α containing the Clifford constituent $\psi \in \text{I}_\pi(G_\alpha | \alpha)$ for φ lying over α. By induction, a vertex subgroup Q for ψ is contained in D. Since $\psi^G = \varphi$, [13, Theorem B] implies that Q is also a vertex for φ. $\square$

If B is a π-block with defect group D, then we say that a character $\varphi \in \text{I}_\pi(B)$ has height zero if

$$\varphi(1)_{\pi'} = \frac{|G|_{\pi'}}{|D|}.$$

LEMMA 4.2. *Suppose that G is a π-separable group and B is a block of G. Suppose that $\varphi \in \text{I}_\pi(B)$ has height zero, and let Q be a vertex subgroup of φ. Then Q is a defect group of B.*

PROOF. By Lemma 4.1, Q is contained in a defect group of B. Write $N = \mathbf{O}_\pi(G)$ and let α be a constituent of φ_N. If α is invariant in G, then D is a Hall π'-subgroup of G, and since φ has height zero, we see that $\varphi(1)$ is a π-number and therefore D is a vertex subgroup of φ.

If α is not invariant in G, then we let $T = G_\alpha$. Let $\psi \in \mathrm{I}_\pi(T|\alpha)$ be the Clifford correspondent for φ. By conjugating if necessary, we may assume, by [13, Theorem B], that Q is a vertex subgroup for ψ, and thus by induction we see that Q is a defect subgroup for the block B_1 of T containing ψ. By the definition of the defect subgroup, however, Q is a defect subgroup for B, and we are done.   □

It is likely that a stronger version of the following result is true, but we will only need the weaker version.

COROLLARY 4.3. *Suppose that B is a π-block of a π-separable group G, and let D be an abelian normal defect group of B. Suppose that $D \subseteq M \lhd G$. If B covers the block b of M, then b has defect group D.*

PROOF. By [18, Theorem 2.18], we see that every character $\varphi \in \mathrm{I}_\pi(B)$ has height zero. By Lemma 4.2, D is a vertex subgroup of φ. By [3, Theorem 3.2], $D \cap M = D$ is a vertex subgroup for every constituent θ of φ_M. Thus for each character $\theta \in \mathrm{I}_\pi(b)$, we see that θ has vertex subgroup D. By [18, Lemma 2.15], we see that b must contain a character of height zero, and thus the vertex subgroup for this character must be D. Therefore b has defect group D.   □

We can now define the root of a π-block B of the π-separable group G. We point out that in the classical case, when $\pi = \{p'\}$, or even in the more general case where π is a set of primes such that G is π-separable and has a nilpotent Hall π'-subgroup, one can define roots using Brauer's first main theorem. However, since we are not assuming that Hall π'-subgroups of G are nilpotent, we cannot use Brauer's first main theorem. We get around this by only defining the root of the π-block B of the π-separable group G in the case where the defect group D of B is abelian.

DEFINITION 4.4. Suppose that B is a π-block of a π-separable group G with an abelian defect group D. Let b be the unique block of $\mathbf{N}_G(D)$ such that $b^G = B$, given by Theorem 3.5, and note that b has defect group D. By Lemma 3.3 applied to $\mathbf{C}_G(D) \lhd \mathbf{N}_G(D)$, there is a block C of $\mathbf{C}_G(D)$ covered by b, unique up to conjugacy in $\mathbf{N}_G(D)$. We call the pair $(\mathbf{C}_G(D), C)$ a root of B.

We will show in Corollary 4.6 that C has defect group D and there is exactly one character $\eta \in \mathrm{I}_\pi(C)$.

LEMMA 4.5. *Suppose that G is a π-separable group and D is a π'-subgroup contained in $\mathbf{Z}(G)$. If B is a block of G with defect group D, then $|\mathrm{I}_\pi(B)| = 1$.*

PROOF. We prove the lemma by induction on $|G|$. Let $N = \mathbf{O}_\pi(G)$ and suppose that $\alpha \in \mathrm{Irr}(N)$ is covered by B. If $T = G_\alpha < G$, then there is a block B_1 of T covering α with defect group D such that the map $\psi \to \psi^G$ is a bijection from B_1 to B. Since

$D \subseteq \mathbf{Z}(T)$, by induction,

$$|\mathrm{I}_\pi(B)| = |\mathrm{I}_\pi(B_1)| = 1.$$

If α is invariant in G, then D is a Hall π'-subgroup of G, and since D is central in G, we see that $G = H \times D$, where H is a Hall π-subgroup of G. Thus $|\mathrm{I}_\pi(B)| = 1$.   □

Of course in the classical case, we could replace the assumption that $D \subseteq \mathbf{Z}(G)$ by the assumption that $G = D\mathbf{C}_G(D)$ and get the same result in Lemma 4.5.

COROLLARY 4.6. *Suppose that B is a π-block of a π-separable group G and that B has abelian defect group D. If $(\mathbf{C}_G(D), C)$ is a root of B, then C has defect group D and $|\mathrm{I}_\pi(C)| = 1$.*

PROOF. By Corollary 4.3 applied to $\mathbf{C}_G(D) \lhd \mathbf{N}_G(D)$, we see that C has defect group D. Lemma 4.5 shows that $|\mathrm{I}_\pi(C)| = 1$.   □

Following the classical case, we call the unique character $\eta \in \mathrm{I}_\pi(C)$ in Corollary 4.6 the canonical character of B, and we note that η is uniquely defined by B up to conjugacy in $\mathbf{N}_G(D)$. Moreover, the stabilizer of η in $\mathbf{N}_G(D)$ is precisely the stabilizer of C in $\mathbf{N}_G(D)$. Finally, note that from the definition of block induction, the block b of $\mathbf{N}_G(D)$ lies over the Glauberman correspondent $\beta \in \mathrm{Irr}(\mathbf{C}_N(D))$ of $\alpha \in \mathrm{Irr}(N)$ lying under B, and thus the root of B can be chosen so that the root lies over β.

We next have an example that shows that the 'lower part' of the decomposition matrix need not have constant columns, and the entries in the decomposition matrix need not be bounded above by 1. Let $\pi = \{2\}$ and let D be a cyclic group of order 15, so that we may write $D = Z_3 \times Z_5$, where Z_3 has order three and Z_5 has order five. Let H be either $(Z_2)^3$ or Q_8, and let N be a normal subgroup of H of order two, and note that in either case H/N is a Klein 4-group. Let G be the semidirect product formed from the action of H on D, where N acts trivially on D and one factor of H/N acts trivially on Z_3 and inverts Z_5, and the other factor of H/N acts trivially on Z_5 and inverts Z_3. If α is the nontrivial character of N, then it can be easily shown that the decomposition matrix for the unique π-block of G that covers α is either

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
2 \\
2
\end{pmatrix},
$$

depending on whether H is $(Z_2)^3$ or Q_8.

We conclude this section with more detailed information on the structure of G when G has a π-block with an abelian defect group D.

LEMMA 4.7. *Suppose that G is a π-separable group and B is a π-block of G with an abelian defect group D. Let $N = \mathbf{O}_\pi(G)$ and suppose that $\alpha \in \mathrm{Irr}(N)$ is covered by B and invariant in G. Then the following hold.*

(a)   *D is a Hall π'-subgroup of G.*

(b)   *If $M = DN$, then $M \lhd G$.*

(c)   *$N\mathbf{N}_G(D) = G$.*

(d)   *If $\bar{G} = G/N$, then $\mathbf{C}_{\bar{G}}(\bar{M}) = \bar{M}$.*

(e)   *$M \cap \mathbf{N}_G(D) = \mathbf{C}_G(D) = D\mathbf{C}_N(D)$.*

PROOF. Statement (a) follows immediately from the fact that α is invariant in G and the definition of the defect group of B. Since $\mathbf{O}_\pi(\bar{G}) = 1$, the Hall–Higman lemma and the fact that D is abelian imply that

$$\bar{M} = \mathbf{C}_{\bar{G}}(\bar{M}) = \mathbf{O}_{\pi'}(\bar{G}),$$

proving statements (b) and (d). An easy 'Frattini argument', applied to $M \lhd G$, shows that $M\mathbf{N}_G(D) = G$, and thus

$$ND\mathbf{N}_G(D) = N\mathbf{N}_G(D) = G,$$

proving (c). Finally, since D is abelian,

$$M \cap \mathbf{N}_G(D) \subseteq \mathbf{C}_G(D),$$

and by (d) we see that

$$\mathbf{C}_G(D) \subseteq M \cap \mathbf{N}_G(D),$$

and therefore $\mathbf{C}_G(D) = M \cap \mathbf{N}_G(D)$. Clearly $D\mathbf{C}_N(D) \subseteq \mathbf{C}_G(D)$, and again (d) shows that $\mathbf{C}_G(D) = D\mathbf{C}_N(D)$.   □

5. Over the Glauberman correspondence

In this section we discuss some rather technical lemmas regarding the Glauberman correspondence that will be used repeatedly in the proof of our main results. Notice that the character $\alpha \in \mathrm{Irr}(\mathbf{O}_\pi(G))$ in the statement of the following theorem is necessarily invariant in D.

LEMMA 5.1. *Let B be a π-block of a π-separable group G with an abelian defect group D. Suppose that B covers the character α of $N = \mathbf{O}_\pi(G)$, and let $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ be such that $b^G = B$ and b has defect group D, so that b covers the Glauberman correspondent β of α in $\mathbf{C}_N(D)$. Set $T = G_\alpha$. If $B_0 \in \mathrm{Bl}(T \,|\, \alpha)$ is the Clifford correspondent for B, then there is a root $(\mathbf{C}_G(D), C)$ of B covering β and a root $(\mathbf{C}_T(D), C_0)$ of B_0 covering β that satisfy the following conditions.*

(a)   *The canonical character ψ in C_0 induces irreducibly to the canonical character η in C.*

(b)   *If E is the stabilizer of C in $\mathbf{N}_G(D)$ and F is the stabilizer of C_0 in $\mathbf{N}_T(D)$, then $E = \mathbf{C}_G(D)F$.*

(c)   *Induction is a bijection from $\mathrm{I}_\pi(F|\psi)$ to $\mathrm{I}_\pi(E|\eta)$.*

PROOF. Let $b_0 \in \mathrm{Bl}(\mathbf{N}_T(D))$ be the Clifford correspondent for $b \in \mathrm{Bl}(\mathbf{N}_G(D))$, so that b_0 covers β and by the definition of block induction, $(b_0)^T = B_0$. Note that b_0 has defect group D. By the definition of block induction, we see that $(b_0)^{\mathbf{N}_G(D)} = b$. Let C_0 be any block of $\mathbf{C}_T(D)$ with defect group D that is covered by b_0 and covers β. Thus $(\mathbf{C}_T(D), C_0)$ is a root of B_0, and by Corollary 4.6 we see that there is a unique character $\psi \in \mathrm{I}_\pi(C_0)$.

Note that ψ lies over β, and that due to the uniqueness of the Glauberman correspondence, $\mathbf{C}_T(D)$ is precisely the stabilizer of β in $\mathbf{C}_G(D)$. Therefore ψ induces irreducibly to $\eta \in \mathrm{I}_\pi(\mathbf{C}_G(D))$. Now b_0 covers C_0 and induction is a bijection from b_0 to b, so C_0 induces to a block of $\mathbf{C}_G(D)$ covered by b, and therefore, if we define C to be the induced block of $\mathbf{C}_G(D)$ obtained from Clifford induction of C_0, then we see that $(\mathbf{C}_G(D), C)$ is a root of B, that C covers β, and that $\mathrm{I}_\pi(C) = \{\eta\}$ by Corollary 4.6, and we have proved (a).

To prove (b), note that since F stabilizes C_0, and the canonical character in C_0 induces to the canonical character in C, it is clear that F stabilizes C and thus $F\mathbf{C}_G(D) \subseteq E$. To prove the reverse inclusion, let $x \in E$. Then x stabilizes η, thus $\beta^x = \beta^c$ for some element $c \in \mathbf{C}_G(D)$. Therefore $xc^{-1} \in \mathbf{N}_G(D)$ stabilizes β, and thus xc^{-1} stabilizes α by the Glauberman correspondence, and therefore $xc^{-1} \in T = G_\alpha$. Since xc^{-1} stabilizes β and η, it stabilizes the Clifford correspondent ψ, so xc^{-1} stabilizes C_0. Therefore $xc^{-1} \in F$, and thus $x \in \mathbf{C}_G(D)F$, and we have proved part (b).

Finally, to prove (c), note that $E \cap T = F$, so that F is the stabilizer of β in E. Thus we are done by the Clifford correspondence.   □

We will not prove our next result as it has been proved in the literature. However, the story of its proof is somewhat complicated. First, the results in the literature are stated in different terminology. The result was announced by Dade in [5] and a sketch of the proof was outlined. However, the proof of Step 12 in that paper was much more difficult than he expected, and its proof was never published. The proof of that step is in a preprint by Dade [4] and in a paper by Puig [16]. Recently, a complete proof was published by Turull who proved a stronger result in [19].

THEOREM 5.2. *Suppose that G is a π-separable group and N is a normal π-subgroup of G. Suppose that D is a cyclic Hall π'-subgroup of G and $M = DN \lhd G$. Let $\alpha \in \mathrm{Irr}(N)$ be invariant in G and let $\beta \in \mathrm{Irr}(\mathbf{C}_N(D))$ be the Glauberman correspondent of α, so that β is invariant in $\mathbf{N}_G(D)$. Let $\widehat{\alpha}$ be the unique π-special extension of α to M, and let $\widehat{\beta}$ be the unique π-special extension of β to $\mathbf{C}_G(D)$. Then $\widehat{\alpha}$ and $\widehat{\beta}$ are invariant in G and $\mathbf{N}_G(D)$ respectively, and $(G, N, \widehat{\alpha})$ is character triple isomorphic to $(\mathbf{N}_G(D), \mathbf{C}_G(D), \widehat{\beta})$.*

6. Proofs of the main theorems

We can now prove Theorem 2.2.

PROOF OF THEOREM 2.2. Assume first that B covers a character α of $N = \mathbf{O}_\pi(G)$ that is not invariant in G, and assume the notation of Lemma 5.1. By induction, we see that $F/\mathbf{C}_T(D)$ is a π-subgroup of $\mathrm{Aut}(D)$ and that $|\mathrm{I}_\pi(B_0)| = |\mathrm{I}_\pi(F|\psi)|$. Note that $|\mathrm{I}_\pi(B_0)| = |\mathrm{I}_\pi(B)|$ and part (c) of Lemma 5.1 shows that $|\mathrm{I}_\pi(F|\psi)| = |\mathrm{I}_\pi(E|\eta)|$, and we are done in this case.

Thus we may assume that α is invariant in G, and we assume the notation of Lemma 4.7. Since α is invariant in G, the Glauberman correspondent $\beta \in \mathbf{C}_N(D)$ of α is invariant in $\mathbf{N}_G(D)$. By Lemma 4.7, we see that $M/N \cong \mathbf{C}_G(D)/\mathbf{C}_N(D)$ is a π'-group, and thus β extends to a unique π-special character $\widehat{\beta} \in \mathrm{I}_\pi(\mathbf{C}_G(D))$, and therefore $\widehat{\beta}$ is the canonical character of the root of B and $E = \mathbf{N}_G(D)$. Thus $E/\mathbf{C}_G(D) = \mathbf{N}_G(D)/\mathbf{C}_G(D)$ is a π-group. By [13, Theorem 4.3] (which applies because D here is abelian, thus certainly solvable), we see that

$$|\mathrm{I}_\pi(B)| = |\mathrm{I}_\pi(G|\alpha)| = |\mathrm{I}_\pi(\mathbf{N}_G(D)|\beta)| = |\mathrm{I}_\pi(\mathbf{N}_G(D)|\widehat{\beta})|,$$

and we are done. □

Note that in the above proof, we invoked the rather nontrivial [13, Theorem 4.3]. In the classical case, however, we see that the above argument shows that $E/\mathbf{C}_G(D)$ is a cyclic group of order dividing $p - 1$, and thus the canonical character extends to E and it is immediate that

$$|\mathrm{I}_\pi(B)| = |G/M| = |\mathbf{N}_G(D)/\mathbf{C}_G(D)| = |\mathrm{I}_\pi(\mathbf{N}_G(D)|\widehat{\beta})|,$$

and thus there is no need to invoke [13, Theorem 4.3] in the classical case where $\pi = \{p'\}$.

Before we can prove our second main result, Theorem 2.5, we need the following easy lemmas.

LEMMA 6.1. *Suppose that G is a π-separable group, N is a normal π-subgroup of G, and B is a π-block of G that covers the character α of N. Set $T = G_\alpha$. Let $B_0 \in \mathrm{Bl}(T)$ be the Clifford correspondent for B lying over α. Suppose that $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{I}_\pi(B)$ are such that $\chi = \chi_0^G$ and $\varphi = \varphi_0^G$ for some $\chi_0 \in \mathrm{Irr}(B_0)$ and $\varphi_0 \in \mathrm{I}_\pi(B_0)$. Then $d_{\chi\varphi} = d_{\chi_0\varphi_0}$.*

PROOF. Notice that since α is invariant in T, every character of B_0 lies over α. Now

$$d_{\chi\varphi} = d_{\chi_0^G \varphi_0^G} = d_{\chi_0\varphi_0} + d_{\chi_0\Xi},$$

where Ξ is a π-partial character of T (or zero) that does not lie above α. Therefore no constituent of Ξ lies in B_0, and thus $d_{\chi_0\Xi} = 0$ and we are done. □

Recall that if the cyclic group D is decomposed as $D_{p_1} \times \cdots \times D_{p_k}$ for Sylow subgroups D_{p_i} for the distinct primes $p_1, \ldots, p_k$, then we say that the character λ of D is supported on the subset S of $\{p_1, \ldots, p_k\}$ if the nontrivial factors of λ occur precisely in the factors $\{D_{p_s} \mid p_s \in S\}$. The following is the result mentioned following Definition 2.3.

LEMMA 6.2. *Suppose that the group H acts coprimely on the cyclic group D. Write $D = D_{p_1} \times \cdots \times D_{p_k}$, where the D_{p_i} are the Sylow p-subgroups of D for the distinct primes dividing $|D|$. If λ and μ are nontrivial characters of the factor D_i, then λ and μ have the same stabilizer in H. Thus if δ and ϵ are characters of D that are supported on the same subset of $\{p_1, \ldots, p_k\}$, then $H_\delta = H_\epsilon$.*

PROOF. Let H_λ be the stabilizer of λ in H, and let $p_i \in \{p_1, \ldots, p_k\}$ be the support of λ. Since H_λ acts coprimely on the cyclic group D_{p_i}, by Fitting's lemma, $\mathbf{C}_{D_{p_i}}(H_\lambda)$ is 1 or D_{p_i}. Now H_λ fixes a nontrivial character of D_{p_i}, so H_λ fixes a nontrivial element of D_{p_i}, and thus $\mathbf{C}_{D_{p_i}}(H_\lambda) = D_{p_i}$. Therefore H_λ fixes μ. Similarly H_μ fixes λ, and $H_\lambda = H_\mu$. The final statement of the lemma then follows immediately. $\square$

We now prove our second main theorem regarding the decomposition matrix of a π-block of a π-separable group G with a cyclic defect group. We will use some of the basic properties of π-special and π-factorable characters. For proofs and further discussion of these characters, see [9].

PROOF OF THEOREM 2.5. We must prove that the matrix M_η, which is determined by the action of $E/\mathbf{C}_G(D)$ on D and the multiplicities of characters induced from the subgroups E_S to E, is actually the decomposition matrix for the block B.

Let $N = \mathbf{O}_\pi(G)$, and let $\alpha \in \mathrm{Irr}(N)$ be covered by B. Suppose that $T = G_\alpha$ is proper in G, and adopt the notation of Lemma 5.1. By Lemma 5.1, we see that $E = \mathbf{C}_G(D)F$, where E is the stabilizer of the root of B (and hence also the canonical character η) and F is the stabilizer of a root of B_0 for which the canonical character ψ of the root of B_0 induces irreducibly to η. Thus for any subset S, we see that $E_S = \mathbf{C}_G(D)F_S$. Since F is also precisely the stabilizer of β in E, we see that the matrix M_η is the same as the matrix M_ψ, defined by the action of $F/\mathbf{C}_T(D)$ on D and the characters of the subgroups F_S lying over ψ. By induction, M_ψ is the decomposition matrix for B_0, and, by Lemma 6.1, the decomposition matrix for the block B_0 of T is the same as the decomposition matrix for B. Thus we are done in the case where α is not invariant in G.

Therefore we assume that α is invariant in G, and we adopt the notation of Lemma 4.7. Note that the π-special character α is invariant in G and thus extends to a unique π-special character $\widehat{\alpha} \in \mathrm{Irr}(M)$. Similarly, the Glauberman correspondent $\beta \in \mathrm{Irr}(\mathbf{C}_N(D))$ of α extends to a π-special character $\widehat{\beta}$ of $M \cap \mathbf{N}_G(D) = \mathbf{C}_G(D)$ (where the equality here is from Lemma 4.7). If b is the unique block of $\mathbf{N}_G(D)$ with defect group D such that $b^G = B$, then, by the definition of block induction, we see that b must cover β, and therefore $\widehat{\beta}^o$ is the canonical character of a root of B. Thus we must show that $M_{\widehat{\beta}^o}$ is the decomposition matrix of B.

Notice that restriction is a bijection from $\mathrm{Irr}(M/N)$ to $\mathrm{Irr}(D)$, and that $G_\lambda \cap \mathbf{N}_G(D)$ is precisely the stabilizer of λ_D in $\mathbf{N}_G(D)$ when $\lambda \in \mathrm{Irr}(M/N)$. By Theorem 5.2, we see there is a character triple isomorphism from $(G, M, \widehat{\alpha})$ to $(\mathbf{N}_G(D), \mathbf{C}_G(D), \widehat{\beta})$, and thus the matrix $M_{\widehat{\beta}^o}$ is equal to the matrix $M_{\widehat{\alpha}^o}$, which is defined by the action of G on M/N (where the action has kernel M), and the character $\widehat{\alpha}^o$. Therefore it is enough to show that the matrix $M_{\widehat{\alpha}^o}$ is the decomposition matrix for B.

Thus suppose that $\chi \in \mathrm{Irr}(B)$ lies over $\widehat{\alpha}\lambda$, where $\lambda \in \mathrm{Irr}(M/N)$ has support S, and let G_S be the stabilizer of λ, and hence the stabilizer of $\widehat{\alpha}\lambda$. Let $\mu \in \mathrm{Irr}(G_S)$ be the Clifford correspondent for χ, and note that $\mu = \varphi\widehat{\lambda}$, where $\widehat{\lambda}$ is the unique π'-special extension of λ to G_S and φ is π-special and lies over $\widehat{\alpha}$. Also $\chi^o = ((\varphi\widehat{\lambda})^o)^G = (\varphi^o)^G$, since $\widehat{\lambda}^o = 1^o_{G_S}$. Thus for $\theta \in \mathrm{I}_\pi(B) = \mathrm{I}_\pi(G|\widehat{\alpha}^o)$, we see that $d_{\chi\theta} = [(\varphi^o)^G, \theta]$.

Thus it remains only to compute the dimension of the decomposition matrix. For a given π-special character φ of G_S lying over $\widehat{\alpha}$, the number of ordinary irreducible characters of G of the form $(\varphi\widehat{\lambda})^G$, where $\lambda \in \mathrm{Irr}(M/N)$ has support S, is equal to the number of orbits of the action of G/M on the linear characters of M/N with support S, which is n_S. Thus $M_{\widehat{\alpha}^o}$ is the decomposition matrix for B and we are done. $\square$

We point out here that we invoked Theorem 5.2 in the above proof, which is a nontrivial result about coprime actions. However, in the classical case this can be avoided, as again $\mathbf{N}_G(D)/\mathbf{C}_G(D)$ is cyclic (of order dividing $p-1$) in this case and thus all of the characters in $\mathrm{Irr}(\mathbf{N}_G(D)|\widehat{\beta})$ and all of the characters in $\mathrm{Irr}(G|\widehat{\alpha})$ are extensions of $\widehat{\beta}$ and $\widehat{\alpha}$ respectively, and thus Theorem 5.2 is easy in this case.

7. Brauer graphs

We now have applications of Theorem 2.5 in this section and the next. We begin by examining the Brauer graph of the block B.

DEFINITION 7.1. Suppose that G is a π-separable group and let B be a block of G. The Brauer graph of B has vertices labeled by $\chi \in \mathrm{Irr}(B)$, and χ and ψ are connected if there exists a character $\varphi \in \mathrm{I}_\pi(B)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$.

We now prove Theorem 1.2.

PROOF OF THEOREM 1.2. Since B is a π-block of the π-separable group G and B has a cyclic defect group, Theorem 2.5 describes the decomposition matrix of B. Let $\lambda \in \mathrm{Irr}(D)$ be a faithful character of D, and suppose that $x\mathbf{C}_G(D) \in E/\mathbf{C}_G(D)$ fixes λ. Then $x\mathbf{C}_G(D)$ fixes a nontrivial character of each Sylow subgroup of D, and thus $x\mathbf{C}_G(D)$ fixes a nontrivial element of each Sylow subgroup of D. Thus by Fitting's lemma, we see that $x \in \mathbf{C}_G(D)$. Therefore if S is the set of prime divisors of $|D|$, then $E_S = \mathbf{C}_G(D)$, and therefore the decomposition matrix M_η of B has at least one row in which every entry is nonzero. Let $\chi \in \mathrm{Irr}(B)$ be the character corresponding to that row of the decomposition matrix. Then it is clear that every character $\eta \in \mathrm{Irr}(B)$ is linked to χ, and thus the Brauer graph of B has diameter at most 2. $\square$

Note that the examples in Section 4 show that the Brauer graph may have diameter 1 or 2.

8. Lifts

Recall that we say that $\chi \in \mathrm{Irr}(G)$ is a lift of $\xi \in \mathrm{I}_\pi(G)$ if $\chi^o = \xi$. In [2], the first author gave a lower bound for the number of lifts of $\xi \in \mathrm{I}_\pi(G)$, and gave an upper bound in the classical case (that is, where $\pi = \{p'\}$) for groups of odd order. In the latter case, the bound was

$$|\{\chi \in \mathrm{Irr}(G) \mid \chi^o = \xi\}| \le |Q : Q'|,$$

where Q is the vertex subgroup associated to ξ. We now give an exact count of the number of lifts (without any odd order assumption, although of course we still require G to be π-separable) if the defect group D is cyclic.

If $\xi \in \mathrm{I}_\pi(B)$, then let $\theta \in \mathrm{I}_\pi(\mathbf{N}_G(D)|\eta)$ be the character that labels the column of ξ in the decomposition matrix of B (see the proof of Theorem 2.2), and for a nonempty subset S of the prime divisors of $|D|$, define $l_S(\xi)$ to be 1 if θ is induced irreducibly from E_S and $l_S(\xi)$ to be 0 if θ is not induced irreducibly from E_S. We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let S be a nonempty subset of the primes that divide $|D|$. Suppose that $\chi \in \mathrm{Irr}(B)$ is a lift of $\xi \in \mathrm{I}_\pi(B)$, and that χ belongs to the submatrix $M(S)$ of the decomposition matrix of B. Then by Theorem 2.5, there is a character $\theta \in \mathrm{I}_\pi(E)$ such that θ is induced irreducibly from a character $\varphi_i \in \mathrm{I}_\pi(E_S)$. Here $\theta \in \mathrm{I}_\pi(E)$ is the 'label' of the column of the decomposition matrix of B corresponding to $\xi \in \mathrm{I}_\pi(B)$. Note that since $E/\mathbf{C}_G(D)$ is abelian, $E_S \lhd E$. Also, by Theorem 2.5, every character in $\mathrm{Irr}(B)$ that corresponds to the $M(S, i)$ submatrix of the decomposition matrix is of this form and thus is a lift of ξ. Notice that if θ is induced from φ_i, then there are exactly $|E : E_S|$ characters of E_S that induce irreducibly to θ. Thus, if there is at least one lift of ξ in the submatrix $M(S, i)$, then there are exactly

$$|E : E_S| n_S = \prod_{p_j \in S}(|D|_{p_j} - 1)$$

characters in $M(S)$ that are lifts of ξ. Summing over all nonempty subsets S of the prime divisors of $|D|$ gives the desired sum. $\square$

To get the upper bound mentioned previously, note that the number of lifts of ξ corresponding to nontrivial characters of D is bounded above by

$$\sum_S \prod_{p_j \in S}(|D|_{p_j} - 1),$$

where the sum is over all nonempty subsets S of the prime divisors of $|D|$. However, this is exactly the number of nontrivial characters of D. There is of course one lift of ξ corresponding to the trivial character of D, and thus we obtain the aforementioned upper bound.

We will show in the next section that if the vertex subgroup of $\xi \in I_\pi(G)$ is cyclic, then the vertex subgroup of ξ is equal to the defect group of the block containing ξ. Thus we will be able to replace the defect subgroup with the vertex subgroup of ξ in Theorem 1.1.

9. Vertex subgroups and some open questions

Vertex subgroups arose in Section 4 as a tool for defining the root of a π-block. Vertex subgroups are interesting in their own right, however, and we now prove the π-generalization of a result of Cliff [1] about p-solvable groups. Cliff showed that if G is a p-solvable group and $\varphi \in IBr_p(G)$ has a cyclic vertex subgroup Q, then Q is also a defect subgroup for the p-block of G containing φ. Soon after, Erdmann [7] proved the same result for arbitrary finite groups. We will prove that if G is π-separable and if $\varphi \in I_\pi(G)$ has cyclic vertex subgroup Q, then Q is a defect subgroup for the π-block of G containing φ.

We first need to discuss some basic properties of irreducible π-partial characters of π-separable groups. One fundamental result (see [11, Proposition 3.4]) is that if $\varphi \in I_\pi(G)$ does not have π-degree, then there exists a subgroup $M \lhd G$ such that the irreducible constituents of φ_M have π-degree and are not invariant in G. Moreover, M can be chosen to be maximal with the property that the constituents of φ_M have π-degree. Thus if $\varphi \in I_\pi(G)$ does not have π-degree, then we let M be as above, let $\theta \in I_\pi(M)$ be a constituent of φ_M and let $\psi \in I_\pi(G_\theta | \theta)$ be the Clifford correspondent for φ. If ψ does not have π-degree, then we may repeat this process until we have a subgroup U of G and a character $\eta \in I_\pi(U)$ of π-degree that induces irreducibly to φ. We define a vertex subgroup Q for φ to be a Hall π'-subgroup of U, and [13, Theorem B] shows that Q is uniquely defined up to conjugacy in G.

We are now ready to prove the π-generalization of Cliff's result. We thank Isaacs for suggesting a simplified proof of the following theorem.

THEOREM 9.1. *Suppose that G is a π-separable group and $\varphi \in I_\pi(G)$ has cyclic vertex subgroup Q. If φ is in the π-block B, then Q is a defect group of B.*

PROOF. Let $N = \mathbf{O}_\pi(G)$ and let $\alpha \in Irr(N)$ be a constituent of φ_N. If $T = G_\alpha < G$, then let $\psi \in I_\pi(T|\alpha)$ be the Clifford correspondent of φ, and note that any defect subgroup of the block of T containing ψ is a defect subgroup for B. By [13, Theorem B], we see that any vertex subgroup for ψ is a vertex subgroup of φ, and thus in this case we are done by induction.

Therefore we may assume that α is invariant in G, and thus that the defect subgroup D of B is a Hall π'-subgroup of G. Thus it is enough to show that the vertex subgroup Q is a Hall π'-subgroup of G. Let $K/N = \mathbf{O}_{\pi'}(G/N)$. Since α is invariant in G, α extends to a character $\widehat{\alpha} \in I_\pi(K)$, and thus K is contained in the unique largest normal subgroup M of G for which the constituents of φ_M have π-degree. Therefore each Hall π'-subgroup of K is contained in a vertex subgroup for φ, and it is enough to show that a Hall π'-subgroup of K is a Hall π'-subgroup of G.

A vertex subgroup of φ is cyclic, and thus K/N is cyclic, and thus, by the Hall–Higman lemma, G/K is isomorphic to a subgroup of $\mathrm{Aut}(K/N)$ and therefore is abelian. Therefore G/N has a normal Hall π'-subgroup and therefore K/N is a Hall π'-subgroup of G/N and any Hall π'-subgroup of K is a Hall π'-subgroup of G, and we are done. $\square$

We now immediately have the following corollary, which was mentioned at the end of Section 8.

COROLLARY 9.2. *Let G be a π-separable group and suppose that $\varphi \in \mathrm{I}_\pi(G)$ has a cyclic vertex subgroup Q. Then*

$$|\{\chi \in \mathrm{Irr}(G) \mid \chi^o = \varphi\}| \le |Q|.$$

PROOF. By Theorem 9.1, Q is a defect subgroup for the π-block of G containing φ, and thus Theorem 1.1 gives the desired bound. $\square$

We now discuss some related questions that we think deserve further study. In the classical case [6] there is more known about the blocks of G with cyclic defect group (even if G is an arbitrary finite group) than just the structure of the decomposition matrix. For instance, there is much that can be said about the generalized decomposition numbers. Slattery has shown [18] that Brauer's second main theorem holds for π-blocks of π-separable groups, and thus it seems we should be able to prove results about the generalized decomposition numbers of π-blocks of π-separable groups with cyclic defect group.

The problem that originally motivated this work was classifying and studying the lifts of a character $\varphi \in \mathrm{I}_\pi(G)$, and Theorem 1.1 is a useful result along those lines. However, it still remains to be determined how the lifts of $\varphi \in \mathrm{I}_\pi(G)$ behave with respect to normal subgroups, as it is known that in some cases the constituents of χ_N are lifts and in some cases they are not, where $\chi \in \mathrm{Irr}(G)$ is a lift of φ and $N \lhd G$. If φ is in a block with cyclic defect group, then it is easy to show that the constituents of φ_N lie in blocks with cyclic defect group, and thus this should be useful for settling the problem of lifts restricting to normal subgroups when the block has cyclic defect group.

Finally, it would be nice to have applications of the cyclic defect group theory to solvable groups in general. For instance, Feit and Thompson [8] proved an application of the classical theory to the study of the p-subgroups of linear groups. It is hoped that methods discussed in this paper, combined with a better understanding of the behavior of the generalized decomposition numbers, will lead to new results regarding solvable or π-separable linear groups.

Acknowledgements

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JAMES P. COSSEY, Department of Theoretical and Applied Mathematics,
University of Akron, Akron, OH 44325, USA
e-mail: cossey@uakron.edu

MARK L. LEWIS, Department of Mathematical Sciences,
Kent State University, Kent, OH 44242, USA
e-mail: lewis@math.kent.edu