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# On a Local Theory of Asymptotic Integration for Nonlinear Differential Equations

*Dedicated to the memory of Professor Cezar Avramescu*

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*Abstract.* We improve several recent results in the asymptotic integration theory of nonlinear ordinary differential equations via a variant of the method devised by J. K. Hale and N. Onuchic The results are used for investigating the existence of positive solutions to certain reaction-diffusion equations.

## 1 Introduction

Consider the Emden–Fowler like differential equation

$$x'' + q(t)x^{\lambda} = 0, \qquad t \geq t_0 \geq 1, \tag{1.1}$$

where $\lambda > 1$, the functional coefficient $q\colon [t_0, +\infty) \to [0, +\infty)$ is continuous and with (eventual) isolated zeros, and $x^{\lambda} = |x|^{\varepsilon}x$ for $\lambda = 1 + \varepsilon$.

In 1955, F. V. Atkinson published a spectacular result on the oscillation of equation (1.1) with a proof relying on asymptotic integration theory via the Picard iterations.

***Theorem 1.1*** ([4, Theorem 1]) *A necessary and sufficient condition for the oscillation of* (1.1) *is given by*

$$\int^{+\infty} tq(t)dt = +\infty. \tag{1.2}$$

The sufficiency part of Theorem 1.1 consisted of a demonstration of the following claim.

***Claim 1.2*** *If Atkinson's hypothesis* (1.2) *does not hold, meaning that we can take* ([4, p. 646])

$$\eta = \lambda \int_{t_0}^{+\infty} tq(t)dt < 1, \tag{1.3}$$

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*then the boundary value problem*

$$\begin{cases} x'' + q(t)x^{\lambda} = 0, & t \geq t_0, \\ \lim_{t\to+\infty} x(t) = 1, & \lim_{t\to+\infty} x'(t) = 0 \end{cases}$$

*has at least one solution.*

In recent years, Theorem 1.1 has been considered by several investigators ([7, 9, 14]). We would like hereafter to improve upon the conclusions of [14] and of some other recent work. Precisely, we shall give a different proof of Theorem 1.1 by controlling the behavior of the derivative $x'(t)$ for the solutions of (1.1). This type of analysis has been performed in different circumstances by Coffman and Wong [5, Theorems 1, 3] and in a general setting by Hale and Onuchic [12].

Some of the classical continuations of Atkinson's theorem are given in [13, 17, 18].

Our main motivation comes from a problem on the existence of positive smooth solutions to the semi-linear elliptic partial differential equation

$$\Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0, \qquad x \in G_A, \tag{1.4}$$

where $G_A = \{x \in \mathbb{R}^n : |x| > A\}$ and $n \geq 3$. One can find different conclusions regarding (1.4) in the contributions [1, 2, 6, 10].

## 2 Local Theory for the Asymptotic Integration of Equation (1.1)

We shall analyze in the following the asymptotic features of some of the (non-oscillatory) solutions of (1.1) using the Hale-Onuchic results [12].

By *local theory* we mean those results in whose hypotheses the behavior of the nonlinearity of a differential equation is described on a given family of functions (with these functions we compare either the solution we are looking for or other functional quantities associated with the solution) and not throughout its entire domain of existence.

The conclusions of such results are based on an observation made by Dubé and Mingarelli [7, Eq. (2.1)], according to which one can combine the Hale-Onuchic (difficult) hypotheses with a (local) Lipschitz restriction imposed on the nonlinearity of the differential equation. In this way, the functional analysis involved in the investigation is greatly simplified.

The first local theorem improves upon the results in [14, Theorem 1].

***Theorem 2.1*** *Set $M \in \mathbb{R}$ and let $\alpha, \beta \colon [t_0, +\infty) \to \mathbb{R}$ be continuous functions absolutely integrable over $[t_0, +\infty)$, with $\alpha(t) \leq \beta(t)$ for all $t \geq t_0$ and such that* $\lim_{t'\to+\infty} \alpha(t') = \lim_{t'\to+\infty} \beta(t') = 0$.

*Given the sets*

$$C_M = \Big\{ u \in C([t_0, +\infty), \mathbb{R}) : M - \int_t^{+\infty} \beta(s)ds \leq u(t) \leq M - \int_t^{+\infty} \alpha(s)ds \text{ for all } t \geq t_0 \Big\}$$

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*and*

$$D = \Big\{ v \in C\big([t_0, +\infty), \mathbb{R}\big) \,:\, \alpha(t) \le v(t) \le \beta(t) \text{ for all } t \ge t_0 \Big\},$$

*assume that*

$$\alpha(t) \le \int_t^{+\infty} f\big(s, u(s), v(s)\big)\, ds \le \beta(t), \qquad t \ge t_0, \tag{2.1}$$

*for all $u \in C_M$ and $v \in D$, where the function $f\colon [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. As a plus,*

$$\Big| f\big(t, u_2(t), v_2(t)\big) - f\big(t, u_1(t), v_1(t)\big) \Big| \le k_1(t)\big|u_2(t) - u_1(t)\big| + k_2(t)\big|v_2(t) - v_1(t)\big|$$

*for all $u_{1,2} \in C_M$, $v_{1,2} \in D$ and $t \ge t_0$. Here, the functions $k_{1,2}\colon [t_0, +\infty) \to [0, +\infty)$ are continuous and there exists a number $\zeta > 0$ such that*

$$\chi = \zeta \int_{t_0}^{+\infty} k_1(t)dt + \int_{t_0}^{+\infty} (t-t_0)k_1(t)dt + \int_{t_0}^{+\infty} k_2(t)dt + \frac{1}{\zeta}\int_{t_0}^{+\infty} (t-t_0)k_2(t)dt < 1.$$

*Then the boundary value problem*

$$\begin{cases} x'' + f(t, x, x') = 0, & t \ge t_0 \ge 0, \\ \lim_{t\to+\infty} x(t) = M \\ \alpha(t) \le x'(t) \le \beta(t), & t \ge t_0, \end{cases} \tag{2.2}$$

*has a unique solution.*

**Proof** Define the distance $d$ between the elements $v_1$ and $v_2$ of the set $D$ by the formula

$$d(v_1, v_2) = \|v_1 - v_2\|_{L^1((t_0,+\infty),\mathbb{R})} + \zeta \sup_{t \ge t_0}\{|v_1(t) - v_2(t)|\}.$$

The dominated convergence theorem (see [11, pp. 20–21]) ensures that the metric space $S = (D, d)$ is complete.

We introduce the operator $T\colon D \to C([t_0, +\infty), \mathbb{R})$ via the formula

$$T(v)(t) = \int_t^{+\infty} f\left(s, M - \int_s^{+\infty} v(\tau)d\tau, v(s)\right) ds, \qquad v \in D, t \ge t_0.$$

The restriction (2.1) shows that $T(D) \subseteq D$, since $M - \int_{(\cdot)}^{+\infty} v(s)ds \in C_M$ for all $v \in D$.

***Claim 2.2*** *The operator $T\colon D \to D$ is a contraction with coefficient $\chi$.*

In fact, we have

$$\begin{aligned} \zeta|T(v_2)(t) - T(v_1)(t)| &\le \zeta \int_t^{+\infty} k_1(s) \int_s^{+\infty} |v_2(\tau) - v_1(\tau)| d\tau ds \\ &\quad + \int_t^{+\infty} k_2(s)\big[\zeta|v_2(s) - v_1(s)|\big]\, ds \\ &\le \left[\zeta \int_{t_0}^{+\infty} k_1(s)ds + \int_{t_0}^{+\infty} k_2(s)ds\right] d(v_1, v_2) \end{aligned}$$

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and

$$\begin{aligned}
&\int_t^{+\infty} |T(v_2)(s) - T(v_1)(s)|ds \\
&\quad \leq \int_t^{+\infty} (s-t)k_1(s) \times \int_s^{+\infty} |v_2(\tau) - v_1(\tau)|d\tau ds \\
&\qquad + \frac{1}{\zeta} \int_t^{+\infty} (s-t)k_2(s)\big[\zeta|v_2(s) - v_1(s)|\big]\, ds \\
&\quad \leq \left[\int_{t_0}^{+\infty} (s-t_0)k_1(s)ds + \frac{1}{\zeta}\int_{t_0}^{+\infty} (s-t_0)k_2(s)ds\right] d(v_1, v_2).
\end{aligned}$$

Now,

$$\int_t^{+\infty} \big|T(v_2)(s) - T(v_1)(s)\big|\, ds + \zeta\big|T(v_2)(t) - T(v_1)(t)\big| \leq \chi d(v_1, v_2)$$

for all $v_{1,2} \in D$ and $t \geq t_0$, which validates the claim.

Denote with $v_0(t)$ the fixed point of operator $T$ in $D$. Then the function $x(t) \equiv M - \int_t^{+\infty} v_0(s)ds$ is the solution we are looking for. ∎

The next result is a variant of [7, Theorem 2.1].

***Theorem 2.3*** *Let $g\colon [t_0, +\infty) \to [0, +\infty)$ be a continuous function, integrable over $[t_0, +\infty)$ and such that $\lim_{t\to+\infty} g(t) = 0$. Assume that $f(t, x, x') = f(t, x)$ is continuous, nonnegative-valued and such that*

$$\int_t^{+\infty} f\big(s, u(s)\big)\, ds \leq g(t), u \in X_M, t \geq t_0, \qquad \int_{t_0}^{+\infty} g(t')dt' \leq M,$$

*where*

$$X_M = \Big\{ u \in C\big([t_0, +\infty), \mathbb{R}\big) : 0 \leq u(t) \leq M \text{ for all } t \geq t_0 \Big\}.$$

*Suppose further that there exists the continuous function $k\colon [t_0, +\infty) \to [0, +\infty)$ subjected to $\eta = \int_{t_0}^{+\infty} (t - t_0)k(t)dt < 1$ and*

$$\big|f\big(t, u_2(t)\big) - f(t, u_1(t))\big| \leq k(t)\big|u_2(t) - u_1(t)\big|, \qquad t \geq t_0, u_{1,2} \in X_M.$$

*Then, the equation from (2.2) will have a solution $x(t)$ with the asymptotic profile given by*

$$\lim_{t\to+\infty} x(t) = M, \qquad 0 \leq x'(t') \leq g(t'), t' \geq t_0.$$

**Proof** Take $\alpha = 0$, $\beta = g$, $k_1 = k$, $k_2 = 0$ and $\zeta \in (0, 1)$ with the property that

$$\zeta < (1 - \eta)\left(\int_{t_0}^{+\infty} k(t)dt\right)^{-1}. \tag{2.3}$$

Since $\chi = \zeta \int_{t_0}^{+\infty} k(t)dt + \eta < 1$, the conclusion follows readily from Theorem 2.1. ∎

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In its essence, the Hale–Onuchic "philosophy" of asymptotic integration of nonlinear differential equations reduces to transforming the boundary value problem into the existence problem of a fixed point to an integral operator $T$, a common fact in this field, and to identifying an invariant set ($D$, $X_M$, *etc.,*) in which one can use, with minimal effort, a fixed point theorem. This is why in a Hale–Onuchic type of approach, the verification of the hypotheses of such a (fixed point) theorem is quite easy, the weight leaning upon the formula of the integral operator (associated with an intermediate integro-differential problem), respectively upon the detection of invariant sets. Some authors, maybe too drastic in this respect, exclude the verification of the hypotheses of the fixed point theorem from the investigation; see *e.g.,* [8, p. v].

Atkinson's original claim is improved in the following.

***Theorem 2.4*** *Assume that* (1.3) *holds. Then, there exists* $p > 1$ *such that, for all* $c \in (0, 1]$, *equation* (1.1) *has at least one solution* $x(t)$ *defined in* $[t_0, +\infty)$ *with the property that*

$$\frac{c}{p} \le x(t) \le c, \qquad t \ge t_0. \tag{2.4}$$

*The asymptotic profile of the solution is given by* $x(t) = c + o(1)$, *respectively* $x'(t) = o(t^{-1})$ *when* $t \to +\infty$.

**Proof** We shall use Theorem 2.1. So, fix the numbers $p > 1$, $\zeta \in (0, 1)$ in order for (2.3) and

$$\int_{t_0}^{+\infty} tq(t)dt < \frac{1}{\lambda} \le \frac{p-1}{p} < 1$$

to hold.

Define the functions

$$\alpha(t) = \left(\frac{c}{p}\right)^{\lambda} \int_t^{+\infty} q(s)ds, \qquad \beta(t) = c^{\lambda} \int_t^{+\infty} q(s)ds, \qquad t \ge t_0.$$

Also, introduce $k_1 = k = \lambda q$, $k_2 = 0$.

We have

$$\int_{t_0}^{+\infty} \beta(t)dt = c^{\lambda} \int_{t_0}^{+\infty} (t - t_0)q(t)dt \le c \int_{t_0}^{+\infty} tq(t)dt < c\left(1 - \frac{1}{p}\right). \tag{2.5}$$

According to [14, p. 183], and taking into account (2.5), the double inequality (2.1) follows from the estimates

$$\begin{aligned}
\alpha(t) &= \int_t^{+\infty} q(s)\left(\frac{c}{p}\right)^{\lambda} ds \le \int_t^{+\infty} q(s)\left(c - \int_s^{+\infty} \beta(\tau)d\tau\right)^{\lambda} ds \\
&\le \int_t^{+\infty} q(s)\big[u(s)\big]^{\lambda} ds = \int_t^{+\infty} f\big(s, u(s)\big)\, ds, \quad u \in C_c, \big(M = c \in (0, 1]!\big) \\
&\le \int_t^{+\infty} q(s)\left(c - \int_s^{+\infty} \alpha(\tau)d\tau\right)^{\lambda} ds \le c^{\lambda} \int_t^{+\infty} q(s)ds = \beta(t)
\end{aligned}$$

for all $t \geq t_0$.

Consequently, by applying Theorem 2.1 we establish the existence of a solution with the formula $x(t) \equiv c - \int_t^{+\infty} v_0(s)ds$ of the boundary value problem (2.2), where $v_0$ is the fixed point of operator $T$ in $D$.

The estimate (2.4) is a by-product of (2.5). More precisely,

$$\frac{c}{p} \leq c\left[1 - \int_{t_0}^{+\infty}(t-t_0)q(t)dt\right] \leq c - c^{\lambda}\int_{t_0}^{+\infty}\int_{t'}^{+\infty} q(s)dsdt'$$
$$\leq c - \int_t^{+\infty}\beta(t')dt' \leq c - \int_t^{+\infty} v_0(t')dt' = x(t) \leq c, \qquad t \geq t_0.$$

The proof is complete. ■

***Remark 2.5*** We impose this restriction upon $c$ just to make use of (1.3). Elsewhere, in the spirit of [5], fix $C$ such that

$$C > \lambda(C - |c|) > 0, \qquad C^{\lambda}\int_{t_0}^{+\infty} t|q(t)|dt \leq C - |c|.$$

Then, the operator $T\colon \mathcal{C} \to X_2(t_0;1)$ (see [3, p. 4]), where

$$T(u)(t) = c - \int_t^{+\infty}(s-t)q(s)[u(s)]^{\lambda}ds, \text{ and}$$
$$\mathcal{C} = \left\{ u \in X_2(t_0;1) : |u(t)| \leq C \text{ for all } t \geq t_0 \right\},$$

is a contraction of coefficient $\lambda\frac{C-|c|}{C}$. Its fixed point in $\mathcal{C}$ is the solution we are looking for.

The following theorem has been announced in [16, Theorem 2.3].

***Theorem 2.6*** *Consider $t_0 \geq 1$, $a, b \geq 0$, $c \in (0,1]$ and the bounded continuous functions $\alpha, \beta\colon [t_0, +\infty) \to [0, +\infty)$ such that $\alpha(t') \leq \beta(t')$ for all $t' \geq t_0$.*

*Introduce the set $F_{a,b,c}$ by the formula*

$$F_{a,b,c} = \left\{ u \in C([t_0,+\infty),\mathbb{R}) : at + b + t\int_t^{+\infty}\frac{\alpha(s)}{s^{1+c}}ds \leq u(t) \right.$$
$$\left. \leq at + b + t\int_t^{+\infty}\frac{\beta(s)}{s^{1+c}}ds \text{ for all } t \geq t_0 \right\}$$

*and assume that*

$$\alpha(t) \leq \frac{1}{t^{1-c}}\int_{t_0}^{t} f(s,u(s))ds \leq \beta(t), \qquad u \in F_{a,b,c},\ t \geq t_0,$$

*where the function* $f\colon [t_0,+\infty)\times\mathbb{R}\to[0,+\infty)$ *is continuous. As a plus,*

$$\Big|f\big(t,u_2(t)\big)-f\big(t,u_1(t)\big)\Big|\le\frac{k(t)}{t}\big|u_2(t)-u_1(t)\big|$$

*for all* $u_{1,2}\in F_{a,b,c}$ *and* $t\ge t_0$. *Here, the function* $k\colon[t_0,+\infty)\to[0,+\infty)$ *is continuous and such that* $\varsigma=\frac{1}{c}\int_{t_0}^{+\infty}k(t)dt<1$.

*Then the boundary value problem*

$$\begin{cases} x''+f(t,x)=0, & t\ge t_0,\\ x(t)\ge b, & t\ge t_0\\ x(t)=at+O(t^{1-c}), & \text{when } t\to+\infty\end{cases}\tag{2.6}$$

*has a unique solution* $x(t)$ *with the property that*

$$\alpha(t)\le t^c\Big[\frac{x(t)-b}{t}-x'(t)\Big]\le\beta(t),\qquad t\ge t_0.\tag{2.7}$$

**Proof** Introduce the set $G$ by the formula

$$G=\Big\{v\in C\big([t_0,+\infty),\mathbb{R}\big)\,:-t^{-c}\beta(t)\le v(t)\le -t^{-c}\alpha(t)\text{ for all }t\ge t_0\Big\}.$$

The distance between the elements $v_1$ and $v_2$ of the set $D$ has the formula

$$d(v_1,v_2)=\sup_{t\ge t_0}\big\{t^c|v_1(t)-v_2(t)|\big\},$$

and the metric space $S=(G,d)$ is complete.

We define the operator $T\colon G\to C([t_0,+\infty),\mathbb{R})$ taking into account the comments in [15, Theorem 8]. Precisely,

$$T(v)(t)=-\frac{1}{t}\int_{t_0}^{t}sf\Big(s,as+b-s\int_s^{+\infty}\frac{v(\tau)}{\tau}d\tau\Big)ds,\qquad v\in G,t\ge t_0.$$

It is easy to notice that $T(G)\subseteq G$, since the mapping $t\mapsto at+b-t\int_t^{+\infty}\frac{v(s)}{s}ds$ belongs to $F_{a,b,c}$ for all $v\in G$.

The estimates given by

$$\begin{aligned} t^c\big|T(v_2)(t)-T(v_1)(t)\big| &\le \frac{1}{t^{1-c}}\int_{t_0}^{t}sk(s)\int_s^{+\infty}\frac{|v_2(\tau)-v_1(\tau)|}{\tau}d\tau ds\\ &\le\frac{1}{ct^{1-c}}\int_{t_0}^{t}s^{1-c}k(s)ds\cdot d(v_1,v_2)\le\frac{1}{c}\int_{t_0}^{t}k(s)ds\cdot d(v_1,v_2)\end{aligned}$$

show that the operator $T$ is a contraction of coefficient $\varsigma$ in $S$.

By denoting with $v_0$ its fixed point, where $v_0\in G$, the solution we are looking for has the formula $x(t)\equiv at+b-t\int_t^{+\infty}\frac{v_0(s)}{s}ds$. ■

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***Remark 2.7*** We notice that, in the circumstances of Theorem 2.6, if we assume that $\lim_{t\to+\infty} t^{1-c}\alpha(t) = \lim_{t\to+\infty} t^{1-c}\beta(t) = d \in [0,+\infty)$, all the elements of the set $F_{a,b,c}$, and, in particular, the solutions of problems (2.6) and (2.7), have the asymptotic profile $u(t) = At + B + o(1)$ as $t \to +\infty$, where $A = a$ and $B = b + d$. Moreover, $\int_{t_0}^{+\infty} sf(s,u(s))ds = d$ for all $u \in F_{a,b,c}$.

***Corollary 2.8*** *Set $t_0, \lambda \geq 1$, $a, b \geq 0$, $c \in (0,1]$ and $\varepsilon \in (0,1)$. Assume that the continuous function $q\colon [t_0,+\infty) \to [0,+\infty)$ satisfies the conditions*

$$\lambda(a+\varepsilon)^{\lambda-1} I_c < c \quad \text{and} \quad \frac{b}{t_0} + (a+\varepsilon)^{\lambda}\frac{I_c}{ct_0^c} < \varepsilon,$$

*where $I_c = \int_{t_0}^{+\infty} t^{\lambda+c}q(t)dt$.*

*Then, equation (1.1) admits the solution $x\colon [t_0,+\infty) \to [b,+\infty)$ with the asymptotic profile $x(t) = at + O(t^{1-c})$ as $t \to +\infty$ for which*

$$a^{\lambda} \cdot \frac{1}{t}\int_{t_0}^{t} s^{\lambda+1}q(s)ds \leq \frac{x(t)-b}{t} - x'(t) \leq (a+\varepsilon)^{\lambda} \cdot \frac{1}{t^c}\int_{t_0}^{t} s^{\lambda+c}q(s)ds$$

*in $[t_0,+\infty)$.*

**Proof** Introduce the functions

$$\alpha(t) = \frac{a^{\lambda}}{t^{1-c}}\int_{t_0}^{t} s^{\lambda+1}q(s)ds, \beta(t) = (a+\varepsilon)^{\lambda}\int_{t_0}^{t} s^{\lambda+c}q(s)ds, \qquad t \geq t_0.$$

Obviously, $\beta(t) \leq (a+\varepsilon)^{\lambda} I_c$ in $[t_0,+\infty)$.

We have that

$$\begin{aligned}
\alpha(t) &\leq \frac{1}{t^{1-c}}\int_{t_0}^{t} sq(s)\left[as + b - s\int_{s}^{+\infty}\frac{v(\tau)}{\tau}d\tau\right]^{\lambda} ds\\
&\leq \frac{1}{t^{1-c}}\int_{t_0}^{t} sq(s)\left(as + b + s\int_{s}^{+\infty}\frac{\|\beta\|_{\infty}}{\tau^{1+c}}d\tau\right)^{\lambda} ds\\
&\leq \frac{1}{t^{1-c}}\int_{t_0}^{t} sq(s)\left[\left(a + \frac{b}{t_0} + \frac{(a+\varepsilon)^{\lambda} I_c}{ct_0^c}\right)s\right]^{\lambda} ds\\
&\leq \frac{1}{t^{1-c}}\int_{t_0}^{t} s^{\lambda+1}q(s)(a+\varepsilon)^{\lambda}ds \leq \beta(t), \qquad v \in D, t \geq t_0,
\end{aligned}$$

and respectively

$$\begin{aligned}
&\Big| f\big(t,u_2(t)\big) - f\big(t,u_1(t)\big)\Big|\\
&\leq \lambda t^{\lambda}q(t)\left\{\frac{1}{t}\left[at + b + t\int_{t}^{+\infty}\frac{\beta(s)}{s^{1+c}}ds\right]\right\}^{\lambda-1} \times \frac{|u_2(t)-u_1(t)|}{t}\\
&\leq \lambda t^{\lambda}q(t)(a+\varepsilon)^{\lambda-1}\frac{|u_2(t)-u_1(t)|}{t} = \frac{k(t)}{t}|u_2(t)-u_1(t)|
\end{aligned}$$

for all $u_{1,2} \in F_{a,b,c}$ and $t \geq t_0$. ∎

***Theorem 2.9*** *Fix $\lambda \geq 1$ and $c \geq 0$, $d > 0$ such that*

$$\max\Big\{ \lambda(c+d)^{\lambda-1}, \frac{(c+d)^{\lambda}}{d} \Big\} \cdot \int_{t_0}^{+\infty} t^{\lambda} q(t)dt < 1.$$

*Then equation* (1.1) *possesses the solution $x(t)$ defined in $[t_0, +\infty)$ with the property that*

$$c - d \leq x'(t) < \frac{x(t)}{t} \leq c + d, \qquad t > t_0.$$

*As a plus, the solution has the asymptotic profile $x(t) = ct + o(t)$ when $t \to +\infty$.*

**Proof** Consider $S = (D, d)$ the metric space given by the formulas

$$D = \Big\{ u \in C\big([t_0, +\infty), \mathbb{R}\big) : ct \leq u(t) \leq (c+d)t \text{ for all } t \geq t_0 \Big\}$$

and

$$d(u_1, u_2) = \sup_{t \geq t_0} \Big\{ \frac{|u_1(t) - u_2(t)|}{t} \Big\}, \qquad u_{1,2} \in D.$$

For the operator $T\colon D \to C([t_0, +\infty), \mathbb{R})$ defined by

$$T(u)(t) = t\Big\{ c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau q(\tau)[u(\tau)]^{\lambda} d\tau ds \Big\}, \qquad u \in D, t \geq t_0,$$

we have the estimates:

$$\begin{aligned}
c \leq \frac{T(u)(t)}{t} &= c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau^{\lambda+1} q(\tau) \Big[ \frac{u(\tau)}{\tau} \Big]^{\lambda} d\tau ds \\
&\leq c + (c+d)^{\lambda} \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau^{\lambda+1} q(\tau) d\tau ds \\
&\leq c + (c+d)^{\lambda} \Big[ \frac{1}{t} \int_{t_0}^{t} \tau^{\lambda+1} q(\tau) d\tau + \int_t^{+\infty} \tau^{\lambda} q(\tau) d\tau \Big] \\
&\leq c + (c+d)^{\lambda} \int_{t_0}^{+\infty} \tau^{\lambda} q(\tau) d\tau < c + d
\end{aligned}$$

and

$$\begin{aligned}
&\frac{|T(u_2)(t) - T(u_1)(t)|}{t} \\
&\leq \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau^{\lambda+1} q(\tau) \Big| \Big( \frac{u_2(\tau)}{\tau} \Big)^{\lambda} - \Big( \frac{u_1(\tau)}{\tau} \Big)^{\lambda} \Big| d\tau ds \\
&\leq \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau^{\lambda+1} q(\tau) \big[ \lambda(c+d)^{\lambda-1} \big] \frac{|u_2(\tau) - u_1(\tau)|}{\tau} d\tau ds \\
&\leq \lambda(c+d)^{\lambda-1} \Big[ \frac{1}{t} \int_{t_0}^{t} \tau^{\lambda+1} q(\tau) d\tau + \int_t^{+\infty} \tau^{\lambda} q(\tau) d\tau \Big] d(u_1, u_2) \\
&\leq \lambda(c+d)^{\lambda-1} \int_{t_0}^{+\infty} \tau^{\lambda} q(\tau) d\tau \cdot d(u_1, u_2) = \vartheta \cdot d(u_1, u_2).
\end{aligned}$$

These imply that $T(D) \subseteq D$, respectively $T\colon S \to S$ is a contraction of coefficient $\vartheta$.

By denoting with $x$, where $x \in D$, the fixed point of operator $T$, we notice that

$$x'(t) = [T(x)]'(t) = \frac{x(t)}{t} - \frac{1}{t}\int_{t_0}^{t} \tau q(\tau)[x(\tau)]^{\lambda} d\tau < \frac{x(t)}{t}$$

and

$$x'(t) \geq c - \frac{1}{t}\int_{t_0}^{t} \tau q(\tau)[x(\tau)]^{\lambda} d\tau \geq c - (c+d)^{\lambda}\int_{t_0}^{t} \tau^{\lambda} q(\tau) d\tau$$

for all $t > t_0$. ∎

## 3 Non-Vanishing Solution to Equation (1.4)

To give an application of Theorem 2.9, let us assume that, in accordance with the analysis from [6, 9], the functions $f\colon \overline{G}_A \times \mathbb{R} \to \mathbb{R}$ and $g\colon [A, +\infty) \to [0, +\infty)$ are locally Hölder continuous. Moreover,

$$0 \leq f(x,u) \leq a(|x|)u, \quad x \in \overline{G}_A,\ u \in [0, \varepsilon],$$

for a certain $\varepsilon > 0$. Here, the function $a\colon [A, +\infty) \to [0, +\infty)$ is continuous such that

$$\int^{+\infty} t a(t) dt < +\infty.$$

Following the presentations in [2, 10, 16], if $u_2(x)$ is a positive radially symmetric solution of the linear elliptic equation

$$\Delta u + a(|x|)u = 0, \qquad |x| > A, \tag{3.1}$$

such that $x \cdot \nabla u_2(x) \leq 0$ in $G_A$, and $u_1(x)$ is a nonnegative radially symmetric solution of the linear elliptic equation

$$\Delta u + g(|x|)x \cdot \nabla u = 0, \qquad |x| > A, \tag{3.2}$$

that satisfies the inequality $u_1(x) \leq u_2(x)$ throughout $G_A$, then equation (1.4) will possess a solution $u(x)$, not necessarily with radial symmetry, such that

$$u_1(x) \leq u(x) \leq u_2(x), \qquad |x| > A.$$

We introduce the quantities $u_{1,2}(x) = \frac{h_{1,2}(s)}{s}$, where

$$|x| = \left(\frac{s}{n-2}\right)^{\frac{1}{n-2}} = \beta(s).$$

Now, the existence of solution $u_2$ to equation (3.1) is implied by the (eventual) existence of a solution $h_2(s)$ of the equation

$$h'' + \frac{\beta(s)\beta'(s)}{(n-2)s} a(\beta(s))h = 0, \qquad s \geq s_0 \geq 1, \quad (\text{here, } \beta(s_0) > A)$$

such that, in $[s_0, +\infty)$,

$$\frac{1}{s_0} < \rho C \le h'(s) < \frac{h(s)}{s} \le C \qquad \text{for given } C \in (0, \varepsilon), \rho \in (0, 1). \tag{3.3}$$

Since

$$\int_{s_0}^{+\infty} s\Big[\frac{\beta(s)\beta'(s)}{(n-2)s} a\big(\beta(s)\big)\Big]\,ds = \frac{1}{n-2}\int_{\beta(s_0)}^{+\infty} \tau a(\tau)d\tau < +\infty,$$

the hypotheses of Theorem 2.9 are verified. So, there exists the supersolution $u_2(x)$ of equation (1.4).

Further, the problem of existence for the subsolution $u_1(x)$ that satisfies equation (3.2) reduces to the existence of a nonnegative solution $h_1(s)$ to the equation

$$h'' + k(s)\Big(h' - \frac{h}{s}\Big) = 0, \qquad s \ge s_0,$$

where $k(s) \equiv \beta(s)\beta'(s)g(\beta(s))$ is a continuous nonnegative-valued function.

By fixing $h_0 \in (1, s_0\rho C)$ (see (3.3)), we have

$$h_1(s) = s\Big(\frac{h_0}{s_0} + \int_{s_0}^{s} \frac{H(\tau)}{\tau^2} d\tau\Big), \quad H(\tau) = -\exp\Big(-\int_{s_0}^{\tau} k(\xi)d\xi\Big)$$

for all $s \ge \tau \ge s_0$. In this way,

$$\frac{h_0 - 1}{s_0} \le \frac{h_1(s)}{s} \le \frac{h_0}{s_0}, \qquad s \ge s_0.$$

In conclusion, we have demonstrated that equation (1.4) admits a bounded solution $u$ estimated by

$$0 < \frac{h_0 - 1}{s_0} \le u(x) \le C, \qquad x \in G_{\beta(s_0)}.$$

The result improves and clarifies the inferences of [16, Section 3].

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## References

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