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Connected Components of Moduli Stacks of Torsors via Tamagawa Numbers

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Abstract. Let X be a smooth projective geometrically connected curve over a finite field with function field K. Let $\mathcal{G}$ be a connected semisimple group scheme over X. Under certain hypotheses we prove the equality of two numbers associated with $\mathcal{G}$. The first is an arithmetic invariant, its Tamagawa number. The second is a geometric invariant, the number of connected components of the moduli stack of $\mathcal{G}$-torsors on X. Our results are most useful for studying connected components as much is known about Tamagawa numbers.

1 Introduction

We work over a finite ground field k. Let X be a smooth geometrically connected projective curve over k with function field K. Let $\mathcal{G}$ be a semisimple group scheme over X. This means that $\mathcal{G}$ is a smooth group scheme over X, all of whose geometric fibres are (connected) semisimple algebraic groups. We denote the generic fibre of $\mathcal{G}$ by G.

Recall a little bit of terminology: G is *split* if it admits a split maximal torus over K. By the semisimplicity assumption, this implies that G (but not $\mathcal{G}$) is a Chevalley group, *i.e.,* a group scheme which comes from a split semisimple group defined over k by base extension. The fundamental group scheme of $\mathcal{G}$ has as fibres the fundamental groups of the fibres of $\mathcal{G}$. It is a finite Abelian group scheme over X. Thus G, or equivalently $\mathcal{G}$, is simply connected if and only if this fundamental group scheme is trivial.

The roots of this article are a circle of ideas that began to take shape in [Har70, HN75][1] In particular, it was observed that there is a connection between Tamagawa numbers and the trace of the Frobenius endomorphism on the cohomology of certain moduli spaces. This led to the following.

Conjecture 1.1 **(G. Harder)** *If $\mathcal{G}$ is split, the Tamagawa number of G is equal to the number of connected components of the moduli space of $\mathcal{G}$-torsors on X.*

We will study a variation that does not assume that $\mathcal{G}$ is split. We will work with stacks rather than spaces. This has two advantages: first there is a precise relationship between the Lefschetz trace formula on the moduli stack with the Tamagawa number without the need for approximations, see §4. The second advantage is that these stacks always exist without any restriction on the characteristic of the ground field. See for example [Bal04].

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[1] Also, *A reinterpretation of Tamagawa numbers in the function field case.* G. Harder, unpublished note.

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In order to forge a relationship between the stack and the Tamagawa number we will need to assume that G satisfies the Hasse principle, see Section 4. The Hasse principle is known to hold for all split groups, see Corollary 4.2. If $\tilde{G}$ is the universal cover of G, then Weil's conjecture asserts that the Tamagawa number of $\tilde{G}$ is 1. This conjecture is known in the split case by [Har74]. In the number field case the full conjecture is known by [Kot88]. One expects that a variation of the proof would work in the geometric case as well. The main result of this work is that under certain assumptions on the ground field the Tamagawa number of G is in fact the number of connected components of the moduli stack. Furthermore, one can deduce that these components are in fact geometrically connected.

In Section 3 we begin by recalling the definition of the Tamagawa number. The section ends with a precise statement of the main results of this paper.

The purpose of Section 4 is to give the proof of the main results modulo the proof of the trace formula and Ono's formula. Section 4.1 recalls basic facts about the Hasse principle. Section 4.2 use the Siegel formula and the trace formula to give a geometric interpretation of the Tamagawa number. Section 4.3 gives a formula for the canonical open compact subgroup in terms of special values of Artin L-functions. Finally the proof is given in Section 4.4.

Section 5 is devoted to an outline of the proof of the Lefschetz trace formula for the moduli stack of $\mathcal{G}$-torsors. The hardest part of the proof is to show that the trace of the Frobenius converges absolutely on the cohomology of the stack. We only sketch many of the proofs as they are standard (although long) and the details can be found in [Beh90]. Section 5.1, describes the main results on semistabilty for torsors. Section 5.2 introduces the Shatz stratification on the moduli stack of $\mathcal{G}$-torsors. The proof of the trace formula along with some semi-purity results for the weight spaces of the cohomology of the moduli stack of $\mathcal{G}$-torsors are proved in Section 5.3.

The final technical tool needed in Section 4.4 is Ono's formula and some of its consequences. This formula is proved in Section 6.

Let us remark that Ono's formula implies that the Tamagawa number is equal to the number of elements of $\pi_1(G)$ when G is split. Thus, we prove that for a semisimple Chevalley group the stack of $\mathcal{G}$-bundles has $|\pi_1(G)|$ components.

In view of Section 4 below, it is tempting to try to use the results and methods of [AB82] to prove the main assertions of this paper. However one does not have the necessary base change theorems required to transport the results of the cited paper to positive characteristic. The moduli stacks of $\mathcal{G}$-torsors are not proper; indeed they are not even separated. In the case $\mathcal{G}$ is split and the ground field is $\mathbb{C}$, it is known [AB82, Tel98] that the moduli stack has $|\pi_1(G, e)|$ components.

2 Notations and Conventions

- $\mathbb{Q}_\ell$ denotes the ℓ-adic rationals. We fix once and for all an inclusion $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$.
- k, k_n is a ground field, which is assumed to be perfect. Sometimes k will be a finite field. In this case k_n denotes the extension of k of degree n.
- X will be a smooth projective geometrically connected curve over k with structure map $\pi\colon X \to \operatorname{Spec} k$.
- $\mathcal{G}$ is a smooth connected reductive group scheme over X.

- G is the generic fiber of $\mathcal{G}$.
- $\mathrm{Dyn}(\mathcal{G})$ is the scheme of Dynkin diagrams of $\mathcal{G}$. See [DG70, Exp. XXIV].
- $\mathfrak{T}$ is the free Abelian group on the connected components of $\mathrm{Dyn}(\mathcal{G})$.
- $\mathfrak{T}(P)$ is the free Abelian subgroup of $\mathfrak{T}$ generated by the components of the type of P.
- $\mathrm{Bun}_{\mathcal{H},X}$ or $\mathrm{Bun}_{\mathcal{H}}$ is the moduli stack of $\mathcal{H}$-torsors on X where $\mathcal{H}$ is an affine group scheme over X.
- $\mathrm{Bun}_{\mathcal{H}}^{\alpha}$ denotes the moduli stack of $\mathcal{H}$-torsors of degree α.
- $\mathrm{Bun}_{\mathcal{H}}^{\alpha,\leq m}$ denotes the moduli stack of $\mathcal{H}$-torsors of degree of instability at most m and degree α.
- $\mathrm{Bun}_{\mathcal{H}}^{\alpha,m}$ denotes the moduli stack of $\mathcal{H}$-torsors of degree of instability equal to m and degree α.
- $\mathrm{Bun}_{\mathcal{H}}^{\alpha,\mathfrak{o}}$ denotes the moduli stack of $\mathcal{H}$-torsors of type of instability $\mathfrak{o}$ and degree α.
- $\tau(G), \tau_n(G)$ is the Tamagawa number of G or of $G \otimes_k k_n$.

3 The Main Results

Let us begin by recalling the definition of the Tamagawa number of a semisimple algebraic group. To do this, we begin by constructing the Tamagawa measure on $G(\mathbb{A})$.

For any point x of X (place of K), we denote by K_x the completion of K at x. The ring of integers inside K_x is $\widehat{\mathcal{O}_{X,x}}$. The ring of adeles of K, notation $\mathbb{A}$, is the restricted product of all K_x with respect to the $\widehat{\mathcal{O}_{X,x}}$. Throughout we fix an additive Haar measure μ_x on each K_x, normalized so that $\widehat{\mathcal{O}_{X,x}}$ has volume 1.

We fix a section ω of the line bundle $\wedge_{\mathcal{O}_X}^{\dim G} \mathrm{Lie}(\mathcal{G})$. It induces, in a natural way, a Haar measure ω_x on each of the analytic varieties $\mathcal{G}(K_x)$, see [Oes84, Section 2]. The subset $\mathcal{G}(\widehat{\mathcal{O}_{X,x}})$ of $\mathcal{G}(K_x)$ is open and its volume is computed by the formula below, which also characterizes this measure.

Proposition 3.1 *Let n be the order of vanishing of ω at x. Then we have*

$$\mathrm{vol}(\mathcal{G}(\widehat{\mathcal{O}_{X,x}})) = |k(x)|^{-n-d} |\mathcal{G}(k(x))|,$$

where d is the dimension of G and $k(x)$ is the residue field of x.

Proof See [Oes84, 2.5] ∎

For a semisimple group scheme $\mathcal{G}$ on X the vector bundle $\mathrm{Lie}(\mathcal{G})$ is of degree 0 on X.

The *Tamagawa measure* is a measure on $G(\mathbb{A})$ defined by

$$q^{(1-g)\dim G} \prod_{x \in X} \omega_x.$$

It follows from the product formula that this measure does not depend on the choice of ω. The *Tamagawa number* of G, $\tau(G)$, is defined to be the volume of $G(\mathbb{A})/G(K)$ under this measure. It is known to be finite [Har69]. (Also, the Tamagawa number depends only on the generic fibre G of $\mathcal{G}$, even though we used $\mathcal{G}$ in the definition.)

Conjecture 3.2 **(A. Weil)** *If G is simply connected, then $\tau(G) = 1$.*

In the number field case this is a theorem proved by R. Kottwitz [Kot88].

Let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of $\mathcal{G}$-torsors on X. We will show that under various hypotheses, the Tamagawa number computes the number of open and closed substacks $\mathrm{Bun}_{\mathcal{G}}$.

Theorem 3.3 *Assume that there is a splitting field L for G whose constant field is k and that k contains all roots of unity dividing the order of the fundamental group of G. Further assume that Weil's conjecture holds for the universal cover of G and G satisfies the Hasse principle. Then $\mathrm{Bun}_{\mathcal{G}}$ has $\tau(G)$ components and each of these components is geometrically connected.*

Note that given any G, we can always find k_n/k such that by base extending to k_n the first two hypotheses of the theorem are satisfied.

We will deduce the following from this theorem.

Corollary 3.4 *If the generic fiber of $\mathcal{G}$ is a Chevalley group, then $\mathrm{Bun}_{\mathcal{G}}$ has exactly $\tau(G)$ components each of which are connected.*

Using similar techniques we can also prove the following.

Theorem 3.5 *If Weil's Tamagawa number conjecture is true, then $\mathrm{Bun}_{\mathcal{G}}$ is geometrically connected for every simply connected $\mathcal{G}$*

4 The Proof

4.1 The Hasse Principle

We begin by recalling some theorems of G. Harder on the Hasse principle. Recall that an algebraic group G over K satisfies the *Hasse principle*, if the map of Galois cohomology sets

$$\mathrm{H}^1(K, G) \longrightarrow \prod_{x \in X} \mathrm{H}^1(K_x, G)$$

is injective.

Theorem 4.1 **(G. Harder)** *The Galois cohomology group $\mathrm{H}^1(K, G)$ is trivial, if G is simply connected. In particular, the Hasse principle holds for such G.*

Proof See [Har75]. ■

Corollary 4.2 *The Hasse principle holds when the generic fiber G is a Chevalley group.*

Proof Let G' be the universal cover of G and M the fundamental group scheme of G. As the first Galois cohomology vanishes for G' and all its inner forms, we have an injection $\mathrm{H}^1(K, G) \hookrightarrow \mathrm{H}^2(K, M)$. As G' is also a Chevalley group, there is an exact sequence $0 \to M \to \mathbb{G}_m^r \to \mathbb{G}_m^r \to 0$, where $\mathbb{G}_m^r$ is a maximal torus of G' containing M. The result follows from Hilbert's theorem 90 and the fact that there is an injection of Brauer groups

$$\mathrm{Br}(K) \hookrightarrow \prod_{x \in X} \mathrm{Br}(K_x).$$ ■

Convention 4.3 For the remainder of this section we assume that the generic fiber G satisfies the Hasse principle.

4.2 The Siegel Formula and the Trace Formula

We now begin to give a geometric interpretation of the Tamagawa number of G.

Lemma 4.4 *Let $x \in X$ be a closed point. The étale cohomology set $\mathrm{H}^1(\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}}), \mathcal{G}_x)$ is trivial.*

Proof We need to show that every $\mathcal{G}_x$-torsor over $\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$ has an $\widehat{\mathcal{O}_{X,x}}$-point. By Lang's theorem, such a torsor has a point over the residue field of $\widehat{\mathcal{O}_{X,x}}$ that can be lifted to an $\widehat{\mathcal{O}_{X,x}}$-point by formal smoothness. ■

Proposition 4.5 *Every $\mathcal{G}$-torsor is trivial over the generic point of X.*

Proof We have a diagram of étale cohomology sets

$$\begin{tikzcd} \mathrm{H}^1(X, \mathcal{G}) \arrow[r] \arrow[d] & \mathrm{H}^1(K, G) \arrow[d] \\ \prod_{x \in X} \mathrm{H}^1(\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}}), \mathcal{G}_x) \arrow[r] & \prod_{x \in X} \mathrm{H}^1(K_x, G). \end{tikzcd}$$

The bottom left corner vanishes and the right vertical map is injective by the Hasse principle. So the top map is trivial. ■

Recall that the integral model $\mathcal{G}$ of G defines an open compact subgroup $\mathfrak{K}$ of $\mathcal{G}(\mathbb{A})$ with $\mathfrak{K} = \prod_{x \in X} \mathcal{G}(\widehat{\mathcal{O}_{X,x}})$.

Lemma 4.6 *There is a bijection between elements of $G(\mathbb{A})$ and (isomorphism classes of) triples $(P, \phi, (\rho_x)_{x \in X})$, where P is a $\mathcal{G}$-torsor, ϕ is a trivialization of P over the generic point of P, and ρ_x is a trivialization of P over the formal disc $\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$.*

Proof There is an obvious map from such triples to the elements of $G(\mathbb{A})$. We construct its inverse as follows. Let $\mathbf{a} = (a_x)$ be an adelic point of G. There is an open Zariski subset U of X such that $\mathbf{a}$ is integral over U, that is, $a_x \in \mathcal{G}(\widehat{\mathcal{O}_{X,x}})$ for $x \in U$. Consider the flat cover $U \cup \bigcup_{x \notin U} \mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$ of X. To construct P we need only specify descent data with respect to this cover and apply faithfully flat descent. On the intersection $U \cap \mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$ they are given by a_x. This gives P, together with a generic trivialization and a trivialization at each $x \notin U$. For $x \in U$ the trivialization ρ_x is given by the trivialization over U multiplied by a_x. ■

Proposition 4.7 *There is a bijection between points of the double coset space*

$$G(K) \backslash G(\mathbb{A}) / \mathfrak{K}$$

and the set of isomorphism classes of $\mathcal{G}$-torsors over X.

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Proof Use the above Lemma together Proposition 4.5. ∎

Theorem 4.8 **(Siegel's Formula)** *We have*

$$\tau(G) = \operatorname{vol}(\mathfrak{K}) \sum_{P \in \operatorname{Bun}_{\mathcal{G}}(k)} \frac{1}{|\operatorname{Aut}(P)|}.$$

The sum is over isomorphism classes of $\mathcal{G}$-torsors on X and $|\operatorname{Aut}(P)|$ is the order of the automorphism group of P, which is finite.

Proof We have

$$\tau(G) = \operatorname{vol}(G(\mathbb{A})/G(K)) = \sum_x \operatorname{vol}(\mathfrak{K}xG(K)/G(K))$$

(the sum is over a collection of double coset representatives)

$$= \sum_x \operatorname{vol}(\mathfrak{K}) \frac{1}{|x\mathfrak{K}x^{-1} \cap G(K)|} = \operatorname{vol}(\mathfrak{K}) \sum_{P \in \operatorname{Bun}_{\mathcal{G}}(k)} \frac{1}{\operatorname{Aut}(P)}.$$

The first equality is by the preceding proposition. One checks in the bijection above that the automorphism group of P is identified with $x\mathfrak{K}x^{-1} \cap G(K)$. Note that one can show that $\operatorname{vol}(\mathfrak{K})$ is finite [Kne67]. Furthermore the sum converges [Har69]. ∎

We will prove below a Lefschetz trace formula for the algebraic stack $\operatorname{Bun}_{\mathcal{G}}$. This formula forges a link between the Siegel formula and the cohomology of $\operatorname{Bun}_{\mathcal{G}}$, and we will describe it now.

If $\mathfrak{X}$ is an algebraic stack over the finite field k we define its number of k-rational points by

$$\#\mathfrak{X}(k) = \sum \frac{1}{|\operatorname{Aut}(x)|}$$

where the sum is over isomorphism classes of objects in $\mathfrak{X}(k)$. We denote by Φ acting on the cohomology of $\mathfrak{X}$. We will prove in Section 5 the following version of the trace formula for $\operatorname{Bun}_{\mathcal{G}}$ over k.

Theorem 4.9 *We have*

$$\sum_{x \in \operatorname{Bun}_{\mathcal{G}}(k)} \frac{1}{|\operatorname{Aut}(x)|} = q^{(g-1)(\dim G)} \sum (-1)^i \operatorname{tr} \Phi|_{H^i(\operatorname{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)},$$

and both sides converge absolutely.

Remark 4.10. The trace formula for stacks of finite type is proved in [Beh03]. The stack $\operatorname{Bun}_{\mathcal{G}}$ is not of finite type but it is naturally filtered by stacks of finite type. Our main task in proving the above theorem is to prove the convergence.

Corollary 4.11 *In the current setting we have*

$$\tau(G) = \operatorname{vol}(\mathfrak{K}) q^{(g-1)\dim G} \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} \Phi|_{\mathrm{H}^i(\operatorname{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)}.$$

Proof Combine the above remark with Theorem 4.9. ∎

4.3 Artin L-Functions and the Volume of $\mathfrak{K}$

We begin by recalling Steinberg's formula for the number of points of a semisimple group over a finite field. Let H/k be a semisimple connected linear algebraic group. By Lang's theorem, it is necessarily quasi-split, so let B be a Borel subgroup and $T \subseteq B$ a maximal torus. Let k_n/k be a splitting field for H. We form the Weyl group $W = (N_H(T)/T)(k_n)$. Note that W acts on $\mathrm{X}(T \otimes_k k_n) \otimes \mathbb{Q}$ and hence on its symmetric algebra S. By a theorem of Chevalley, the invariants of this action is the symmetric algebra on finite dimensional graded vector space $V = \bigoplus_{n \geq 2} V_n$.

Theorem 4.12 *Let F be the Frobenius of k_n/k. We have*

$$\frac{|H(k)|}{q^d} = \prod_{n \geq 2} \det((1 - q^{-n}F)|V_n),$$

where d is dimension of H and q is the number of elements of k.

Proof See [Ste68, 11.16]. ∎

We return to our global situation. First we observe the following.

Proposition 4.13 *Let $\mathfrak{K} = \prod_{x \in X} \mathcal{G}(\widehat{\mathcal{O}_{X,x}})$ be the canonical open compact. Then*

$$\mathrm{vol}(\mathfrak{K}) = q^{(1-g)\dim G} \prod_{x} |k(x)|^{-\dim G} |\mathcal{G}(k(x))|.$$

Proof This is by Proposition 3.1 combined with the fact that the vector bundle $\mathrm{Lie}(\mathcal{G})$ has degree 0. ∎

As there is an integral model $\mathcal{G}$ for G, by Lemma 5.1 we can find an unramified extension L/K that splits G. We may assume that the extension is in fact Galois. Correspondingly, we have a Galois cover $Y \to X$. Again form the Weyl group $W = (N_G(T)/T)(L)$, which acts on the symmetric algebra of $\mathrm{X}(G \bigotimes_K L) \otimes \mathbb{Q}$. Using Chevalley's result again, we obtain a finite dimensional graded vector space $V = \bigoplus_{n \geq 2} V_n$. Each of the V_i's are $\mathrm{Gal}(L/K)$-modules so we can form the associated Artin L-functions

$$L_n(X, s) = L(X, V_n, s) = \prod_{x \in X} \det((1 - q^{-s \deg x} f_x)|_{V_n}).$$

In the above f_x is the Frobenius for the extension of residue fields $k(y)/k(x)$ and y is a point lying over x. Using the above theorem and Proposition 4.13, we have

$$\mathrm{vol}(\mathfrak{K}) = q^{(1-g)\dim G} \prod_{n \geq 2} L_n(X, n)^{-1}.$$

In summary, we have the following.

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Theorem 4.14 *The volume of the open compact is given by*

$$\mathrm{vol}(\mathfrak{K}) = q^{(1-g)\dim G} \prod_{n\geq 2} L_n(X,n)^{-1},$$

where each L_n is an Artin L-function described above. Furthermore, we have

$$L_n(X,s) = Z(X,s)^{\gamma_n} \prod_{i=1}^{m_n} p_{ni}(X,s),$$

where $Z(X,s)$ is the zeta function, $p_{ni}(X,s) = \prod(1-\alpha_{ni}q^{-s})$, $|\alpha_{ni}| = q^{1/2}$, and γ_n is some integer.

Proof The only part that needs justification is the last statement, see [Mil80, p. 126]. ■

4.4 The Proof of the Main Theorems

We are in a position to give the proof of the main theorems modulo some technical results. The proofs of these will be given later.

We denote the Tamagawa number of the base extension by $\tau_n(G) = \tau(G \otimes_k k_n)$. Here k_n is the unique extension of the finite field k of degree n. A key ingredient in the proofs of the main theorems is the fact that the sequence

$$\tau_1(G), \tau_2(G), \ldots,$$

is constant under suitable hypothesis. Using Ono's formula we will show the following.

Proposition 4.15 *Given G with a field of constants k, suppose all roots of unity dividing the order of $|\pi_1(G)|$ are in k. Further assume Weil's conjecture for the universal cover of G. Then we have $\tau_n(G) = \tau(G)$ for every n.*

Proof See Corollary 6.14. ■

We need the following technical definition and lemma to make the proofs of the main results go more smoothly.

Definition 4.16 Let $\sum_{m=1}^{\infty} s_{nm} = t_n$ be a sequence of series of complex numbers. We say that the series *converge uniformly* if for every $\epsilon > 0$ there is an M_0 such that for every $M \geq M_0$ we have

$$\Big| \sum_{m=1}^{M} s_{nm} - t_n \Big| < \epsilon$$

independently of n.

Lemma 4.17 *Let $\sum_{m=1}^{\infty} s_{nm}$ be a sequence of series that all sum to t independently of n. Furthermore, assume that the convergence is uniform and that the series $\sum_{n=1}^{\infty} s_{nm}$ converges absolutely for each m. Then $t = 0$.*

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Proof Let $\epsilon > 0$ and M be as in the definition of uniform convergence. We have

$$\Big| t - \sum_{m=1}^{M} s_{nm} \Big| < \epsilon.$$

However, $\sum_{n=1}^{\infty} \sum_{m=1}^{M} s_{nm}$ converges and hence $\lim_{n\to\infty} \sum_{m=1}^{M} s_{nm} = 0$, and we are done. ■

Recall that the zeta function of X can be written in the form

$$Z(X,s) = \frac{\prod_{i=1}^{2g}(1-\alpha_i q^{-s})}{(1-q^{-s})(1-q^{-s+1})},$$

where α_i are the eigenvalues of the Frobenius on $\mathrm{H}^1(X,\mathbb{Q}_\ell)$. It will be important below to note that the α_i have absolute value $q^{1/2}$.

Finally, before giving the proof, we will need two facts about the cohomology of $\mathrm{Bun}_{\mathcal{G}}$. First we will need to know that the vector spaces $\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)$ are finite dimensional. Secondly, we will need that the eigenvalues of Φ have absolute value at most $q^{-i/2}$ on $\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)$. Both these facts are proved in §5.

Proof of Theorem 3.3 We have

$$\begin{aligned}
\tau(G) &= \mathrm{vol}(\mathfrak{K}) q^{(g-1)\dim G}\Big(\sum_{i=0}^{\infty} \mathrm{tr}\,\Phi|_{\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)}\Big) && \text{(Theorem 4.8)}\\
&= \prod_{n\geq 2} L_n(X,s)^{-1}(\mathrm{tr}\,\Phi|_{\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)}) && \text{(Theorem 4.14).}
\end{aligned}$$

Let $\{\beta_j\}$ be the eigenvalues of Φ on $\bigoplus_{i>0}\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)$ and ϵ_j their signs in the above formula. So we have

$$\begin{aligned}
\tau(G) - \mathrm{tr}\,\Phi|_{\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)} = \Big(\prod_{n\geq 2} L_n(X,n)^{-1} - 1\Big)&(\mathrm{tr}\,\Phi|_{\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)})\\
&+ \prod_{n\geq 2} L_n(X,n)^{-1}\Big(\sum_j \beta_j\epsilon_j\Big).
\end{aligned}$$

We remind the reader that ℓ-adic cohomology of a stack $\mathfrak{X}$ over a finite field is defined by first passing to the algebraic closure, *i.e.*, it is really defined on $\mathfrak{X}\otimes_k \bar{k}$. With this in mind, the action of the Frobenius on the cohomology of the base extension $\mathrm{Bun}_{\mathcal{G}_m}$ is just given by Φ^m.

$$\begin{aligned}
\tau_m(G) - \mathrm{tr}\,\Phi^m|_{\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)} = \Big(\prod_{n\geq 2} L_n(X_m,n)^{-1} - 1\Big)&(\mathrm{tr}\,\Phi^m|_{\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}},\mathbb{Q}_\ell)})\\
&+ \prod_{n\geq 2} L_n(X_m,n)^{-1}\Big(\sum_j \beta_j^m\epsilon_j\Big).
\end{aligned}$$

For future use, we denote the series on the right-hand side of the above equation by A_m.

Note that

$$L_n(X_m, s) = L_n(X, s) = Z(X_m, s)^{\gamma_n} \prod_{i=1}^{m_n} p_{ni}(X_m, s),$$

where the γ_n is the same as that in Theorem 4.14. We have

$$Z(X_m, s) = \frac{\prod_{i=1}^{2g}(1 - \alpha_i^m q^{-ms})}{(1 - q^{-ms})(1 - q^{m(1-s)})}$$

and

$$p_{ni}(X_m, s) = \prod (1 - \alpha_{ni}^m q^{-ms}).$$

In the above formulas the α_i and α_{ni} are the same as those in the formulas for X. Now $\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)$ is finite dimensional by the results of Section 5. So there is a k_m/k such that the connected components of $\mathrm{Bun}_{\mathcal{G}_m}$ are geometrically connected and have a rational point. It follows that Φ^{lm} is the identity on $\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)$ for all $l > 1$. So using Proposition 4.15, the series A_{lm} satisfy the first of the hypothesis of Lemma 4.17. Now using the fact that $|\beta_j| \leq q^{-1/2}$, by Corollary 5.22 the remaining hypotheses are easily checked. It follows that $A_{lm} = 0$. It follows that there are exactly $\tau(G)$ components.

Now consider $1 < r < m$. A similar analysis shows that each of the series A_{lm+r} is zero. So $\operatorname{tr} \Phi^r|_{\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)} = \tau(G)$ also. It follows that Φ must be the identity and we are done. ■

Proof of Theorem 3.4 Note that Weil's conjecture is true for Chevalley groups by [Har74]. The result is now obtained by combining the above with Corollary 6.8. ■

Proof of Theorem 3.5 Argue as in the proof of Theorem 3.3. ■

5 The Lefschetz Trace Formula for $\mathrm{Bun}_{\mathcal{G}}$

5.1 Semistabilty for $\mathcal{G}$-Torsors

The purpose of this subsection is to recall the main results and constructions of [Beh95]. The main point of that paper is to extend notions such as (semi)stability and Harder–Narasimhan filtration to torsors over a reductive group scheme.

For concepts such as root systems with complementary convex solids, special facets and semistabilty of root systems, the reader is referred to the first three sections of [Beh95]. The relationship of these concepts with what is to follow can be found in Section 6 of that paper.

The following construction will be used throughout this work.

Lemma 5.1 *There is a finite étale cover $f\colon Y \to X$ such that $f^*\mathcal{G}$ is an inner form.*

Proof We make use of the notations of [DG70]. Let $\mathcal{G}_0$ be the constant reductive group scheme over X having the same type as $\mathcal{G}$. Being an inner form means that the scheme $\mathrm{Isomext}(\mathcal{G}, \mathcal{G}_0)$ has a section over X. By [DG70, XXIV, theorem 1.3] and

[DG70, XXII, corollary 2.3], G is quasi-isotrivial and hence so is Isomext$(\mathcal{G}, \mathcal{G}_0)$. This implies by [DG70, X, corollary 5.4] that Isomext$(\mathcal{G}, \mathcal{G}_0)$ is étale and finite over X. So we take Y to be one of these components and the section is the tautological section. ■

Note that such an inner form is generically split by the Hasse principle. See Corollary 4.2 and Proposition 4.5 above.

Definition 5.2 Let $\mathcal{H}$ be a smooth affine group scheme over X with connected fibers. We define the *degree* of $\mathcal{H}$ to be $\deg \mathcal{H} = \deg \mathrm{Lie}(\mathcal{H})$, where $\mathrm{Lie}(\mathcal{H})$ is the Lie algebra of $\mathcal{H}$ viewed as a vector bundle on X.

By Lemma 5.1, a reductive group scheme has degree 0.

Definition 5.3
- We say that $\mathcal{G}$ is *semistable* if for every parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ we have $\deg \mathcal{P} \leq 0$.
- We say that $\mathcal{G}$ is *stable* if for every parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ we have $\deg \mathcal{P} < 0$.
- The largest integer d such that there exists a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ of degree d is called the *degree of instability of* $\mathcal{G}$ and is denoted $\deg_i(\mathcal{G})$.

By [Beh95, Lemma 4.3], the integer $\deg_i(\mathcal{G})$ is finite.

Let $\mathrm{Dyn}(\mathcal{G})$ be the scheme of Dynkin diagrams of $\mathcal{G}$, see [DG70, XXIV]. The power scheme of $\mathrm{Dyn}(\mathcal{G})$, denoted $P(\mathrm{Dyn}(\mathcal{G}))$ is the scheme that represents the functor

$$\begin{aligned} \textbf{schemes}/X &\to \textbf{sets} \\ T &\mapsto \mathcal{P}(\mathrm{Dyn}(\mathcal{G}_T)). \end{aligned}$$

Here $\mathcal{P}$ means the set of open and closed subschemes. For a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ recall the definition of the type of $\mathcal{P}$, denoted $t(\mathcal{P})$, from [Beh95, p. 294]. The *type* $t(\mathcal{P})$ is a section of $P(\mathrm{Dyn}(\mathcal{G})) \to X$. In a nutshell, the type of $\mathcal{P}$ can be thought of in the following way: think of $\mathcal{G}$ as a family of reductive groups over X and then $\mathrm{Dyn}(\mathcal{G})$ is their Dynkin diagrams glued together in the appropriate way. Over a point $x \in X$ choose a Borel subgroup contained inside $\mathcal{P}_x$. This Borel gives a choice of simple roots which correspond to the vertices of the Dynkin diagram over x. We consider $\mathrm{Lie}(\mathcal{P})_x \subseteq \mathrm{Lie}(\mathcal{G})_x$, and let R be the subset of the simple roots that consists of those roots α such that the weight space for $-\alpha$ is in $\mathrm{Lie}(P)$ with respect to the above inclusion. The value of $t(\mathcal{P})$ over x is the complement of R.

Let $\mathfrak{T}$ be the free Abelian group on $\pi_0(\mathrm{Dyn}(\mathcal{G}))$. By definition of power scheme, the section $t(\mathcal{P})$ chooses some connected components of $\mathrm{Dyn}(\mathcal{G})$; let $\mathfrak{T}(\mathcal{P})$ be the free Abelian group on these components.

Let $\mathfrak{o}$ be a positive element of $\mathfrak{T}(\mathcal{P})$, that is, an element of the form $\sum n_i \mathfrak{o}_i$ with the n_i positive. Given such an $\mathfrak{o}$, one can construct a vector bundle $W(\mathcal{P}, \mathfrak{o})$. We refer the reader to [Beh95, p. 293] for the construction and basic properties.

Definition 5.4 Let $\mathcal{P}$ be a parabolic subgroup of $\mathcal{G}$ and let $\mathfrak{o}$ be a component of its type. We define the *numerical invariants* of $\mathcal{P}$ to be all $\deg W(P, \mathfrak{o})$ as $\mathfrak{o}$ ranges over the components of the type of $\mathcal{P}$.

Definition 5.5 A parabolic subgroup $\mathcal{P} \subseteq \mathcal{G}$ is called *canonical* if

- the numerical invariants of $\mathcal{P}$ are all positive,
- the Levi component $\mathcal{P}/R_u(\mathcal{P})$ of $\mathcal{P}$ is semistable.

The main results of [Beh95] can be summarized in the following theorem.

Theorem 5.6 *There is a unique canonical parabolic subgroup of $\mathcal{G}$. It is maximal among parabolic subgroups of maximal degree. It commutes with pullback under separable covers.*

The above constructions and definitions apply to a $\mathcal{G}$-torsor E as follows. One forms the inner form ${}^E\mathcal{G} = E \times_{\mathcal{G},\mathrm{Ad}} \mathcal{G}$. Then ${}^E\mathcal{G}$ is a reductive group scheme over X and we define the degree of E, etc. to be that of ${}^E\mathcal{G}$.

5.2 The Shatz Stratification on $\mathrm{Bun}_\mathcal{G}$

We will describe in this subsection the Shatz stratification on $\mathrm{Bun}_\mathcal{G}$ and state its elementary properties. The proofs are often just generalizations of facts about the usual Shatz stratification for vector bundles. When new ideas are involved, we sketch these. The interested reader is referred to [Beh90] for complete proofs.

Let $\mathrm{Bun}_\mathcal{G}$ be the moduli stack of $\mathcal{G}$-torsors. Let $\mathrm{X}(\mathcal{G})$ be the group of characters of $\mathcal{G}$. Each $\mathcal{G}$-torsor E defines a map

$$\begin{aligned} \deg E\colon \mathrm{X}(\mathcal{G}) &\to \mathbb{Z} \\ \phi &\mapsto \deg(E \times_\phi \mathbb{G}_m). \end{aligned}$$

For $\alpha \in \mathrm{X}(\mathcal{G})^\vee$ we denote by $\mathrm{Bun}_\mathcal{G}^\alpha$ the open and closed substack of $\mathrm{Bun}_\mathcal{G}$ of torsors of degree α. For m an integer we denote by $\mathrm{Bun}_\mathcal{G}^{\alpha,\le m}$ the substack of torsors of degree of instability at most m. It is an open substack of $\mathrm{Bun}_\mathcal{G}^\alpha$ that is in fact of finite type. To show this last fact we proceed in several steps.

By a vector group over X we mean the underlying additive group of a vector bundle over X.

Proposition 5.7 *Let V be a vector group on X. Then the natural map* $\mathrm{Bun}_V \to \mathrm{H}^1(X,V)$ *makes* Bun_V *into an affine gerbe over the vector space* $\mathrm{H}^1(X,V)$*. This gerbe is trivial,* i.e., *isomorphic to* $B\mathrm{H}^0(X,V) \times \mathrm{H}^1(X,V)$*. It follows that* Bun_V *is a smooth stack of finite type of dimension $r(g-1)-d$, where r is the rank of V and d its degree.*

Proof A torsor for V defines, via cocycles, a cohomology class, and this defines the canonical map. It is easy to see it is a gerbe. Let $t\colon T \to \mathrm{H}^1(X,V)$ be a an affine morphism. The T-points of $\mathrm{H}^1(X,V)$ are in bijection with $\mathrm{H}^1(X_T,V_T)$, thinking of T as a k-scheme by composition of structure maps. Hence t defines a cohomology class $\xi \in \mathrm{H}^1(X_T,V_T)$ which corresponds to a V_T torsor E. This gives a map $\tilde{t}\colon T \to \mathrm{Bun}_V$ that lifts t. Hence the triviality result. ∎

Proposition 5.8 *Let $\mathcal{P}$ be a parabolic subgroup of $\mathcal{G}$ and let $\mathcal{H} = \mathcal{P}/R_u(\mathcal{P})$. The natural map* $\mathrm{Bun}_\mathcal{P} \to \mathrm{Bun}_\mathcal{H}$ *is a smooth epimorphism of stacks that is of finite type and relative dimension* $\dim_X R_u(\mathcal{P})(g-1) - \deg({}^E\mathcal{P})$*, where E is the universal $\mathcal{G}$-torsor. It induces an isomorphism on cohomology.*

Proof The unipotent radical is filtered by subgroups all of whose quotients are vector bundles [DG70, XXVI, 2.1]. One then proceeds by induction. ■

Proposition 5.9 *Let B be a Borel subgroup of $\mathcal{G}$ and assume that $\mathcal{G}$ is split over the generic point of X. Then for each $\beta \in \mathsf{X}(B)^\vee$ the stack Bun_B^β is of finite type.*

Proof The quotient $B/R_u(B)$ is a split torus. It is well known that the components of $\mathrm{Bun}_{\mathbb{G}_m}$ are of finite type and the result follows from the above proposition. ■

Proposition 5.10 *Let Z be a projective scheme over k and let $f\colon Z' \to Z$ be a projective flat cover. Let $\mathcal{H}$ be a smooth affine group scheme over Z. Then the natural pullback map $\mathrm{Bun}_{\mathcal{H}} \to \mathrm{Bun}_{f^*\mathcal{H}}$ is affine and of finite presentation.*

Proof Straightforward. See [Beh90, 4.4.3]. ■

Before we get to the proof of the fact that $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}$ is of finite type, we need some constructions. Let $\mathcal{P}$ be a parabolic subgroup of $\mathcal{G}$. The type of $\mathcal{P}$ is an open and closed subscheme of $\mathrm{Dyn}(\mathcal{G})$. Its connected components $\mathfrak{o}_1, \mathfrak{o}_2, \dots, \mathfrak{o}_s$ generate a subgroup $\mathfrak{T}(\mathcal{P})$ of $\mathfrak{T}$. There is an action of $\mathcal{P}$ on $W(P, \mathfrak{o}_i)$. Taking the determinant of the action produces a character χ_i of $\mathcal{P}$.

Definition 5.11 We say that an element α of $\mathsf{X}(\mathcal{P})^\vee$ is positive if $\alpha(\chi_i) > 0$ for $i = 1, \dots, s$. We denote by $\mathsf{X}(\mathcal{P})^\vee_+$ the collection of all such positive elements.

We have a homomorphism $\mathfrak{T}(\mathcal{P}) \to \mathsf{X}(\mathcal{P})$ and taking duals and identifying the dual of $\mathfrak{T}(\mathcal{P})$ with itself via the basis $\mathfrak{o}_1, \mathfrak{o}_2, \dots, \mathfrak{o}_s$, we obtain $\sigma\colon \mathsf{X}(\mathcal{P})^\vee \to \mathfrak{T}(\mathcal{P})$. As P acts on $R_u(\mathcal{P})$ and this group is filtered by subgroups with vector bundle quotients, we may take determinants to obtain a character χ_0. Evaluation at χ_0 gives a map $m\colon \mathsf{X}(\mathcal{P})^\vee \to \mathbb{Z}$. Finally the inclusion $\mathcal{P} \subseteq \mathcal{G}$ gives a map $\delta\colon \mathsf{X}(\mathcal{P})^\vee \to \mathsf{X}(\mathcal{G})^\vee$.

Proposition 5.12 *The map $\delta \times \sigma\colon \mathsf{X}(\mathcal{P})^\vee \to \mathsf{X}(\mathcal{G})^\vee \times \mathfrak{T}(\mathcal{P})$ is injective with finite cokernel.*

Proof The details can be found in [Beh90, 7.3.11], but the idea is as follows. Using Lemma 5.1, one can assume that $\mathcal{G}$ is generically split. To see the reduction observe that the cover in Lemma 5.1 may be taken to be Galois with group Γ. The character groups of the original groups are just the groups of Γ invariants.

In the split case one uses the correspondences set up in [Beh95, §6] to reduce the problem to questions about root systems with complementary solids. ■

Theorem 5.13 *The stack $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}$ is of finite type.*

Proof See [Beh90, 8.2.6] for full details. Again, by passing to Galois covers, we may assume that $\mathcal{G}$ is generically split as the natural map

$$\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m} \to \mathrm{Bun}_{f^*\mathcal{G}}^{\mathrm{tr}^{\check{}}(\alpha),\leq m}$$

is of finite type by Proposition 5.10. Choose a Borel $B \subseteq \mathcal{G}$ and let $\chi_1, \chi_2, \dots, \chi_s$ be the associated characters inducing a map $\coprod_\beta \mathrm{Bun}_B^\beta \to \mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}$, where the disjoint union is over all characters such that

$$d(\beta) = \alpha, \quad m(\delta) \leq m, \quad \beta(\chi_i) \geq -2g.$$

The above proposition shows this disjoint union is finite. A calculation shows that the morphism exists, *i.e.*, the degree of instability of the torsors in the image is at most m. Furthermore, the morphism is surjective. By Proposition 5.9, we are done. ■

Every parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ determines an element

$$\sum_{i=1}^{s} n(P, \mathfrak{o}_i)\mathfrak{o}_i, \quad \text{in } \mathfrak{T},$$

where $\mathfrak{o}_i$ are the connected components of the type of $\mathcal{P}$. For a reductive group scheme $\mathcal{G}$ on a family of curves $\mathfrak{X} \to S$, we define a function $n\colon S \to \mathfrak{T}$ as follows. For a point $s \in S$ choose an algebraic closure $\overline{k(s)}$ of the residue field at s. Define $n(s)$ to be $n(\mathcal{P}_s)$ where $\mathcal{P}_s$ is the canonical parabolic subgroup of $\mathcal{G}_s$.

Proposition 5.14 *Let S_d be the locally closed subscheme of S where the degree of instability of $\mathcal{G}$ is d. Then n is a continuous function on S_d.*

Proof See [Beh90] (7.2.9) ■

Denote by $\mathrm{Bun}_{\mathcal{G}}^{\alpha,m}$ the locally closed substack of torsors of degree α and degree of instability m. Denote by $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\mathfrak{o}}$ the locally closed substack of torsors of degree α and type of instability $\mathfrak{o}$.

If E is a $\mathcal{P}$ torsor of degree α, then the torsor $E \times_{\mathcal{P}} \mathcal{G}$ has degree $\delta(\alpha)$. If $\sigma(\alpha) = \sum_{i=1}^{s} n_i \mathfrak{o}_i$, then $n_i = n(E \times_{P,\mathrm{Ad}} P)$, where we think of $E \times_{P,\mathrm{Ad}} P$ as a parabolic subgroup of $E \times_{P,\mathrm{Ad}} G$. Furthermore, $\deg_i(E \times_{P,\mathrm{Ad}} P)) = m(\alpha)$.

Theorem 5.15 *Denote by $\bar{\mathcal{G}}$ the reductive group scheme $\mathcal{G} \times_k \bar{k}$ over the curve $X \times_k \bar{k}$. Let $\mathcal{P}$ be a parabolic subgroup of $\bar{\mathcal{G}}$ and let $\alpha \in \mathrm{X}(\mathcal{P})_+^\vee$. The natural map*

$$\mathrm{Bun}_{\mathcal{P}}^{\alpha,0} \to \mathrm{Bun}_{\bar{\mathcal{G}}}^{\delta(\alpha),m(\alpha)}$$

is finite radical and surjective.

Proof Recall that a morphism is radical if it induces a bijection on L-points for every field L. Representability of this morphism is easy to show. The fact that it is radical and surjective amounts to the existence and uniqueness of the canonical parabolic. ■

5.3 The Lefschetz Trace Formula for $\mathrm{Bun}_{\mathcal{G}}$

Proposition 5.16 *Let $\mathcal{P}$ be a parabolic subgroup of $\bar{\mathcal{G}}$. Let $\alpha \in \mathrm{X}(\mathcal{P})_+^\vee$. Let $\mathcal{H} = \mathcal{P}/R_u(\mathcal{P})$. Then there is a natural isomorphism*

$$\mathrm{H}^i(\mathrm{Bun}_{\bar{\mathcal{G}}}^{d(\alpha),\sigma(\alpha)}, \mathbb{Q}_\ell) \to \mathrm{H}^i(\mathrm{Bun}_{\mathcal{H}}^{\alpha,0}).$$

Proof Use Theorem 5.15 and Proposition 5.8. Note that a finite radical and surjective morphism induces an isomorphism on cohomology [AGV72, Expose VII]. ■

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Lemma 5.17 *There is a function $r\colon \mathfrak{T} \to \mathbb{Z}$ such that if E is a $\bar{\mathcal{G}}$-torsor of type of instability $\mathfrak{o}$, then $r(\mathfrak{o}) = \dim_X R_u(\mathcal{P})$ where $\mathcal{P}$ is the canonical parabolic of E.*

Proof This is just because two parabolic subgroups having the same type are twisted forms of each other. ■

Proposition 5.18 *The closed immersion $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\mathfrak{o}} \to \mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m(\mathfrak{o})}$ is of codimension $c(\mathfrak{o}) = r(\mathfrak{o})(g-1) + m(\mathfrak{o})$.*

Proof This is a standard dimension calculation. ■

Define $\gamma(i)$ to be the smallest integer such that

$$\gamma(i) \geq \begin{cases} 1 + i/2 & \text{if } g > 0, \\ 1 + i/2 + |\Phi| & \text{if } g = 0, \end{cases}$$

where $|\Phi|$ is the number of roots of $\mathcal{G}$.

Proposition 5.19 *Let $i \geq 0$ be such that $m \geq \gamma(i)$. Then the canonical map $\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}, \mathbb{Q}_\ell) \to \mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}}^{\alpha}, \mathbb{Q}_\ell)$ is an isomorphism.*

Proof Let c be the codimension of $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\mathfrak{o}}$ in $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}$ where $m(\mathfrak{o}) = m$. Using the above, one shows that for $m \geq \gamma(i)$ we have $i \leq 2c - 2$. A Gysin sequence yields the result. ■

For a divisor D on X we denote by $\mathrm{Bun}_{\mathcal{G}}(D)$ the moduli stack of $\mathcal{G}$-torsors with level structure over D. By *level structure* over D for a torsor E we mean a section of i_D^*E where $i_D\colon D \hookrightarrow X$ is the natural inclusion.

Proposition 5.20 *Let D be a divisor on X. Let $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}(D)$ be the moduli stack of G-torsors on X with level structure at D. Let E be the universal torsor on $\mathrm{Bun}_{\mathcal{G}}^{\alpha}$. If*

$$p_*{}^E \operatorname{Lie}(\mathcal{G})(-D)|_{\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}} = 0,$$

then $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}(D)$ is a Deligne–Mumford stack.

Proof We need to show that the diagonal morphism of $\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}(D)$ is unramified. Let K be a field and consider a K-point $\operatorname{Spec} K \to \mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}(D)$. It corresponds to a pair (E, s) where E is a a torsor on X_K and s is a section. Denote by $\operatorname{Aut}(E, s)$ the automorphism group of E compatible with s. If $\pi\colon X \to \operatorname{Spec} k$ is the structure morphism, then we need to show that $\pi_K(\operatorname{Aut}(E, s))$ is unramified over $\operatorname{Spec} K$. A calculation shows that one may identify the Zariski tangent spaces of this group with $\mathrm{H}^0(X_K, \operatorname{Lie}(\mathcal{G})(-D))$ which vanishes. ■

Theorem 5.21 *The eigenvalues of the arithmetic Frobenius acting on*

$$\mathrm{H}^i(\mathrm{Bun}_{\mathcal{G}}^{\alpha,\leq m}, \mathbb{Q}_\ell)$$

have absolute value at most $q^{-i/2}$.

Proof First, for a smooth Deligne–Mumford stack the analogous statement is true by comparison to its coarse moduli space [Beh93]. Now for a quotient stack $[\mathfrak{X}/\Gamma]$, where $\mathfrak{X}$ is a smooth Deligne–Mumford stack and Γ any algebraic group, we proceed as follows. First we may assume that $\Gamma = \mathrm{GL}_n$ by choosing a faithful representation $\Gamma \hookrightarrow \mathrm{GL}_n$ and replace $\mathfrak{X}$ by $\mathfrak{X} \times_\Gamma \mathrm{GL}_n$. The Leray spectral sequence for

$$[\mathfrak{X}/\,\mathrm{GL}_n] \to B\,\mathrm{GL}_n$$

has E_2 term $\mathrm{H}^i(B\,\mathrm{GL}_n, \mathbb{Q}_\ell) \otimes \mathrm{H}^j(\mathfrak{X}, \mathbb{Q}_\ell)$. The result follows for the quotient as it is known for $\mathfrak{X}$ and $B\,\mathrm{GL}_n$. ∎

Corollary 5.22 *The eigenvalues of the arithmetic Frobenius acting on* $\mathrm{H}^i(\mathrm{Bun}^\alpha_{\mathcal{G}})$ *have absolute value at most* $q^{-i/2}$.

Lemma 5.23 *For the group* $\bar{\mathcal{G}}$ *there is a function* $m\colon \mathfrak{T} \to \mathbb{Z}$ *such that if* E *is a* $\bar{\mathcal{G}}$*-torsor with type of instability* $\mathfrak{o} \in \mathfrak{T}$*, then* $m(\mathfrak{o})$ *is the degree of instability of* ${}^E\bar{\mathcal{G}}$. *If* $\mathcal{P}$ *is a parabolic subgroup of* $\bar{G}$*, then the diagram*

$$\begin{tikzcd} \mathsf{X}(\mathcal{P})^\vee \arrow[r, "\mathfrak{o}"] \arrow[dr] & \mathfrak{T} \arrow[d, "m"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes.

Proof If E and F are torsors with the same type of instability $\mathfrak{o}$, there is a parabolic $\mathcal{P}$ of type η where η is the support of $\mathfrak{o}$ and E and F have reductions E' and F' to $\mathcal{P}$. Then $\mathfrak{o}(\deg E') = \mathfrak{o} = \mathfrak{o}(\deg F')$, and our lemma follows from the fact that m factors through $\mathfrak{T}(\mathcal{P})$. ∎

Let η be a closed and open subscheme of $\mathrm{Dyn}(\bar{\mathcal{G}})$. Let

$$C(\eta, mu) = \Bigl|\{\mathfrak{o} \in \mathfrak{T}(\eta)^+ \mid m(\mathfrak{o}) = \mu\}\Bigr|.$$

Here $\mathfrak{T}(\eta)^+$ denotes the set of linear combinations in the support of η all of whose coefficients are positive.

Lemma 5.24 *We have* $C(\eta, \mu) = O(\mu^s)$ *where* s *is the number of components of the type of* $\mathcal{P}$.

Proof Recall the definition of the characters χ_i and χ_0 from the discussion after Proposition 5.10. The result follows from the fact that there are positive rational numbers such that $\chi_0 = \sum_{i=1}^s y_i \chi_i$. ∎

Lemma 5.25 *Let* R *be the radical of* $\bar{\mathcal{G}}$. *There exist finitely many* $d_1, \dots, d_n \in \mathsf{X}(\bar{\mathcal{G}})^\vee$ *such that for every* $d \in \mathsf{X}(\bar{\mathcal{G}})^\vee$ *there is an* R*-torsor of degree* $d_i - d$.

Proof Let $M = \{\delta \in \mathsf{X}(R)^\vee \mid \mathrm{Bun}_R^\delta \neq \varnothing\}$. By looking at dual objects one constructs an exact sequence of tori over X, $1 \to S \to R \to \mathbb{G}_m^r \to 1$, where the last map has a quasi-section. We can identify $\mathsf{X}(R)^\vee$ with $\mathsf{X}(\mathbb{G}_m^r)^\vee$ and using this identification and the quasi-section, we observe that M has finite index in $\mathsf{X}(R)^\vee$. It follows that M has finite index in $\mathsf{X}(\bar{\mathfrak{G}})^\vee$ and we take d_i to be a set of coset representatives for $\mathsf{X}(\bar{\mathfrak{G}})^\vee/M$. ■

Let A be the set of open and closed subschemes $\eta \subseteq \mathrm{Dyn}\,\bar{\mathfrak{G}}$ such that there is a torsor E whose canonical parabolic has type η. For each $\eta \in A$, fix a torsor E_η. Let $P_\eta \subset {}^{E_\eta}\bar{\mathfrak{G}}$ be the canonical parabolic and let $H_\eta = P_\eta/R_u(P_\eta)$ be its Levi factor. We choose for each η degrees $d(\eta,1), d(\eta,2),\dots,d(\eta,n) \in \mathsf{X}(H_\eta)^\vee$ according to the above lemma. We get a finite family of stacks $\mathrm{Bun}_{H_\eta}^{d(\eta,j),0}$ parameterized by $A \times \{1,\dots,n\}$. Set $b_i(\eta,j) = \dim_{\mathbb{Q}_\ell} \mathrm{H}^i(\mathrm{Bun}_{H_\eta}^{d(\eta,j),0}, \mathbb{Q}_\ell)$. Choose $B(i) = \sup b_i(\eta,j)$.

Lemma 5.26 *There is an integer N so that $B(i) = O(i^N)$.*

Proof The stacks in question are quotients of smooth Deligne–Mumford stacks by Proposition 5.20. The result follows from the usual spectral sequence. ■

Recall the definitions of $\gamma(i)$ after Proposition 5.18 and $r(\mathfrak{o})$ in Lemma 5.17. We set

$$D(i) = \sum_{eta\in A} \sum_{\mu=0}^{\gamma(i)} C(\eta,\mu) B(i - 2\mu - 2r(\eta)(g-1)).$$

Lemma 5.27 *The following sum converges:*

$$\sum_{\eta\in A}\sum_{m} q^{-m+(1-g)R_uP} \sum_{i} \dim H^i(\mathrm{Bun}_{\mathfrak{G}}^{H_\eta,C}\, d(m), 0, \mathbb{Q}_\ell) q^{-i/2}.$$

Proof Use the above estimate. ■

The convergence now follows from some general observations about cohomology of stacks that we outline below. Denote by $W^iH^j(X,\mathbb{Q}_\ell)$ the i-th weight space of the j-th cohomology group.

Recall that a morphism $Z \to \tilde{Z}$ is called a universal homeomorphism if it is finite radical and surjective. Such a morphism induces an equivalence of étale sites by [GR71, Expose IX 4.10].

Theorem 5.28 *Let $\mathfrak{X}$ be a smooth stack with a countable stratification by locally closed stacks $\widetilde{\mathfrak{Z}}_i$. We assume that the union $\mathfrak{X}_n = \bigcup_{i=0}^n \widetilde{\mathfrak{Z}}_i$ is open. Suppose that there are universal homeomorphisms $\mathfrak{Z}_i \to \widetilde{\mathfrak{Z}}_i$ with each of the $\mathfrak{Z}_i$'s smooth and the following sum converges:*

$$\sum_{n=0}^{\infty} q^{-\mathrm{codim}(\mathfrak{Z}_n,\mathfrak{X})} \sum_{i,j} \dim Gr_i^W H^j(\overline{\mathfrak{Z}}_n, \mathbb{Q}_\ell) q^{-i/2} < \infty.$$

Then the trace of the Frobenius converges absolutely on $\mathfrak{X}$.

Proof Let $Z \to X$ be a morphism of finite type smooth schemes which factors as $Z \to \widetilde{Z} \to X$, where $\pi\colon Z \to \widetilde{Z}$ is a universal homeomorphism and $i\colon \widetilde{Z} \to X$ a closed immersion with complement U. We have a long exact sequence

$$\cdots \to H^*(\widetilde{Z}, i^!\mathbb{Q}_\ell) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \cdots .$$

Let $c = \dim X - \dim Z$. We have $H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) = H^*(Z, \pi^! i^! \mathbb{Q}_\ell)$ because Z and X are smooth. Now pulling back via π induces an isomorphism of étale sites [GR71, Expose IX,4.10]. As π_* is the right adjoint of π^*, it is the inverse of π^* and hence also a left adjoint of π^*. Since π is proper, we conclude that $\pi^! = \pi^*$. Thus, we have $H^*(Z, \pi^! i^! \mathbb{Q}_\ell) = H^*(Z, \pi^* i^! \mathbb{Q}_\ell) = H^*(\widetilde{Z}, i^! \mathbb{Q}_\ell)$, Thus we have a natural long exact sequence

$$\cdots \to H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \cdots .$$

This result extends to stacks and filtrations of schemes and stacks consisting of more than two pieces. Assembling these long exact sequences we get the required result and some simple analysis gives the required result. ∎

Theorem 5.29 *We have*

$$\sum_{x \in \mathrm{Bun}^\alpha_{\mathcal{G}}(k)} \frac{1}{\mathrm{Aut}(x)} = q^{(g-1)(\dim G)} \sum (-1)^i \operatorname{tr} \Phi|_{H^i(\mathrm{Bun}_{\mathcal{G}}, \mathbb{Q}_\ell)},$$

and both sides converge absolutely.

Proof The convergence of the trace is by the above proposition and lemma, noting that the natural maps $\mathrm{Bun}_P^{m,0} \to \mathrm{Bun}_{\mathcal{G}}$ are finite radical and surjective onto their image by the uniqueness of the canonical parabolic. The stratification being used is Shatz stratification induced by reduction to the canonical parabolic. The open substacks of bounded degree of instability are of finite type by Theorem 5.13. As the trace formula holds for these, the result follows. ∎

6 Ono's Formula and Applications

Let M be the fundamental group scheme of G. So M is an Abelian group scheme over K and there is an exact sequence $1 \to M \to \tilde{G} \to G \to 1$, where $\tilde{G}$ is the universal cover of G. For a continuous $\mathrm{Gal}(\overline{K}^s/K)$-module N, let

$$\text{Ш}^1(K, N) = \ker\bigl(\mathrm{H}^1(K, N) \to \prod_{x \in X} \mathrm{H}^1(K_x), N)\bigr).$$

Here K_x is the completion of the global field K at x.

***Theorem 6.1* (Ono's Formula)** *Assume that Weil's conjecture holds true for the universal cover $\tilde{G}$ of G, that is, $\tau(\tilde{G}) = 1$. Then we have*

$$\tau(G) = \frac{|\mathrm{H}^0(K, \widehat{M})|}{|\text{Ш}^1(\widehat{M})|},$$

where $\widehat{M} = \mathrm{Hom}(M, \mathbb{G}_m)$ is the dual Galois module.

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A notational clarification is in order here. The object $\widehat{M}$ is to be viewed as functor on field extensions of K. We have $\widehat{M}(L) = \mathrm{X}(M) \otimes_K L$. We will often write $\mathrm{H}^i(L, \widehat{M})$ when we really mean $\mathrm{H}^i(L, \widehat{M}(\bar{L}^{\mathrm{sep}}))$.

The above theorem is the main result of [Ono65]. It was originally only proved in the number field case and some modifications are needed in the function field case. We will detail these below.

To prove this result we need to generalize the theory of Tamagawa measures to reductive groups. We refer the reader to [Oes84, 1.4] for the definition of the Tamagawa measure $d\tau_H$ for a reductive group H.

In [Ono65], the theorem is proved by reducing to the case of an isogeny of tori. This was treated in [Ono63]. However this last paper contains a small error in the function field case that was corrected in [Oes84, p.23, Ch. IV].

We need some background results before giving the proof of the above theorem. Let $\Lambda_1 \subseteq \Lambda$ be an inclusion of free Abelian groups of the same rank r. Let

$$\mathbf{x} = \{x_1 = 0, x_1, \dots, x_t\}$$

be a set of coset representatives for Λ/Λ_1. A function $f\colon \Lambda \to \mathbb{R}$ is said to be $\mathbf{x}$-*compatible* if

- f has finite support;
- $f(\alpha) = f(\alpha + x_i)$ for all $\alpha \in \Lambda_1$ and all i.

Such a function is said to be Λ_1-*compatible* if it is $\mathbf{x}$-compatible for some choice of coset representatives containing 0.

Lemma 6.2 *Suppose we have three lattices $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3$ of the same rank r. Let $f\colon \Lambda_3 \to \mathbb{R}$ be Λ_1 compatible. Then*

$$\sum_{y\in\Lambda_2} f(y) = \Big(\sum_{y\in\Lambda_3} f(Y)\Big)\frac{1}{[\Lambda_3 : \Lambda_2]}.$$

Proof This is elementary. ■

We denote by D_q the set $\{q^i | i \in \mathbb{Z}\}$. Choose a basis $\{\chi_1, \dots, \chi_r\} = \chi$ for the group $\mathrm{X}(H)$ of rational characters of H. We define

$$\psi_H^\chi = \psi_H \colon H(\mathbb{A}) \to D_q^r$$

by sending $x \mapsto (\|\chi_1(x)\|, \|\chi_2(x)\|, \dots, \|\chi_r(x)\|)$. In the above χ_i is really the adelization of χ_i. The image of ψ is of finite index in D_q^r. Denote by $H(\mathbb{A})^1$ the kernel of ψ. We denote by $d\tau_H^1$ the measure on $H(\mathbb{A})^1$ that is the quotient of $d\tau_H$ by $[D_q^r\colon \mathrm{Im}(\psi)](\log q)^r$. The Tamagawa number of H is

$$\tau(H) = \int_{H(\mathbb{A})^1/H(K)} d\tau_H^1.$$

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Proposition 6.3 *Let $\Lambda \subseteq \operatorname{Im}(\psi) \subseteq D_q^r$ be a sublattice of maximal rank. If F is Λ-compatible, then*

$$\int_{H(\mathbb{A})/H(K)} F(\psi_H(x))\, d\tau_H = \tau(H)\Big(\sum_{x\in D_q^r} F(x)\Big)(\log q)^r.$$

Proof We have, noting the previous lemma,

$$\begin{aligned}\int_{H(\mathbb{A})/H(K)} F(\psi_H(x)) d\tau_H &= \sum_{y\in \operatorname{Im}(\psi)} F(y) \int_{H(\mathbb{A})^1/H(K)} d\tau \\ &= \sum_{y\in D_q^r} F(y) \int_{H(\mathbb{A})^1/H(K)} \frac{d\tau}{[D_q^r : \operatorname{Im}(\psi)]} \\ &= \sum_{y\in D_q^r} F(y) \int_{H(\mathbb{A})^1/H(K)} d\tau_H^1 (\log q)^r. \qquad \blacksquare\end{aligned}$$

Proposition 6.4 *Consider an exact sequence $1 \to H' \xrightarrow{i} H \to H'' \to 1$. Suppose H' is a torus, H'' is semisimple, and the sequence is generically split. Then*

$$\tau(H')\tau(H'') = \tau(H)|\operatorname{cok}\widehat{i}\,|.$$

In the above formula $\widehat{i}$ is the dual map on character groups.

Proof See [Ono65, Proposition(1.2.2)]. The fact that the sequence is generically split implies that the induced map on adelic and K-points is exact. Furthermore $\mathsf{X}(H)$ is a subgroup of $\mathsf{X}(H')$ of finite index $|\operatorname{cok}\widehat{i}|$. By the elementary divisors theorem, we may choose a basis $\chi_1 \cdots \chi_r$ of $\mathsf{X}(H')$ such that $m_1\chi_1 \cdots m_r\chi_r$ is a basis of $\mathsf{X}(H)$. We have a diagram

$$\begin{tikzcd} D_q^r \arrow[r, "\mathbf{m}"] & D_q^r \\ \operatorname{Im}(\psi_{H'}) \arrow[u] \arrow[r] & \operatorname{Im}(\psi_H) \arrow[u] \\ H'(\mathbb{A}) \arrow[u, "\psi_{H'}"'] \arrow[r] & H(\mathbb{A}). \arrow[u, "\psi_H"] \end{tikzcd}$$

We have a sequence of inclusions $\operatorname{Im}(\psi_{H'}) \hookrightarrow D_q^r \hookrightarrow D_q^r$. In what follows we write the group operation on D_q^r additively. By the elementary divisors theorem, there are bases of the two outside lattices of the form $\{e_1, \dots e_r\}$ and $\{d_1e_1, \dots d_re_r\}$. Define a function on D_q^r by

$$f(e_1^{\alpha_1} + \cdots + e_r^{\alpha_r}) = \begin{cases} 1 & 0 \le \alpha_i < d_i, \\ 0 & \text{otherwise.} \end{cases}$$

This function has the property that

$$\int_{H'(\mathbb{A})/H'(K)} f(\psi_H(x')+t)\, d\tau_{H'} = \int_{H'(\mathbb{A})/H'(K)} f(\psi_H(x'))\, d\tau_{H'},$$

which just follows from the definitions. Using the compatibility properties of f and Lemma 6.2 one shows that

$$\int_{H'(\mathbb{A})/H'(K)} f(\psi_H(x'))\, d\tau_{H'} = \frac{1}{|\operatorname{cok}\widehat{i}|}\int_{H'(\mathbb{A})/H'(K)} f(\psi_{H'}(x')).$$

This follows from the definition of f and the fact that $d\tau_{H'}$ is a Haar measure. We have

$$\begin{aligned}
\tau(H)(\sum_{y\in D_q^r} f(y))(\log q)^r &= \int_{H(\mathbb{A})/H(K)} f(\psi_H(x))\, d\tau_H \\
&= \int_{H''(\mathbb{A})/H''(K)} d\tau_{H''} \int_{H'(\mathbb{A})/H'(K)} f(\psi_H(x')\psi(x))\, d\tau_{H'} \\
&= \int \tau(H'') \int_{H'(\mathbb{A})/H'(K)} f(\psi_H(x'))\, d\tau_{H'} \\
&= \frac{\tau(H')\tau(H'')}{|\operatorname{cok}\widehat{i}|}\Big(\sum_{y\in D_q^r} f(y)\Big)(\log q)^r,
\end{aligned}$$

which finishes the proof. ■

Lemma 6.5 *Let* $1 \to H' \to H \xrightarrow{\kappa} H'' \to 1$ *be an exact sequence of linear algebraic groups over* K*. Assume that* H' *is semisimple simply connected,* H'' *is a torus, and* H *is reductive. Then*

(i) $\kappa(H_{\mathbb{A}}) \cap H''_K = \kappa(H_k)$;
(ii) $H''_{\mathbb{A}} = \kappa_{\mathbb{A}}(H_A)$.

Proof Let $x \in \kappa(H_{\mathbb{A}}) \cap H''_K$. Then $\kappa^{-1}(x)$ is a torsor for H'. By [Har75], this torsor is trivial which yields the result.

(ii) For each $x \in X$ the map κ_x, obtained by base change to K_x, is surjective. This is because the Galois cohomology $\mathrm{H}^1(K_x, H')$ vanishes since H' is simply connected. ■

Proposition 6.6 *Let* $1 \to H' \to H \xrightarrow{\kappa} H'' \to 1$ *be an exact sequence of connected reductive groups with* H' *semisimple simply connected and* H'' *a torus. Then*

$$\tau(H')\tau(H'') = \tau(H).$$

Proof Compare with [Ono65, Prop 1.2.3]. Let F be compatible with $\operatorname{Im}\psi_{H''}$. Consider the integral

$$J = \int_{H(\mathbb{A})/H(K)} F(\psi_{H''}(\kappa(x)))\, d\tau_H = \tau(H)\Big(\sum_{x\in D_q^r} F(x)\Big)(\log q)^r.$$

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To see this, note that $\widehat{\kappa}$ induces an isomorphism on character groups, as H' is semi-simple. Then apply the above lemma along with Proposition 6.3. Again by the lemma we may apply [Wei82, Theorem 2.4.4] to this integral and obtain

$$J = \tau(H') \int_{H(\mathbb{A})/H(K)} F(\psi_{H''}(y))\, d\tau_{H''} = \tau(H')\tau(H'')\Big(\sum_{x \in D_q^r} F(x)\Big) (\log q)^r .$$

Again we have made use of the above lemma. ∎

We now recall Ono's construction of crossed diagrams. Let $\tilde{G}$ be the universal cover of G, so that we have an exact sequence $1 \to M \to \tilde{G} \to G \to 1$. Note that M is of multiplicative type [DG70, IX] over the field K. Recall the following result from [DG70, X].

Proposition 6.7 *The category of groups of multiplicative type over K is antiequivalent to the category of* $\mathrm{Gal}(\bar{K}^s/K)$*-modules that are finitely generated as Abelian groups.*

The above duality is induced by $\mathrm{Hom}(-, \mathbb{G}_m)$, *i.e.*, by taking character modules. Now observe that we can find an exact sequence $0 \to M \to T' \to T \to 0$, with $\mathrm{X}(T' \otimes_K K')$ a projective $\mathrm{Gal}(K'/K)$-module for some splitting field K' of T'. To see this set $G_M = \{g \in \mathrm{Gal}(\bar{K}^s/K) \mid g \text{ fixes } \widehat{M}\}$. Let $\Gamma = \mathbb{Z}[G_M]$ and we can find an exact sequence $0 \to \text{kernel} \to \Gamma + \Gamma + \cdots + \Gamma \to M \to 0$. Set $G^* = (\tilde{G} \times T')/M$ and we have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T' \arrow[d, "i"] \arrow[dr] & & \\
0 \arrow[r] & \tilde{G} \arrow[r] \arrow[dr] & G^* \arrow[r] \arrow[d] & T \arrow[r] & 0 \\
 & & G \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Proof of Theorem 6.1 We use the above notations. As in [Ono65, Lemma 2.1.1] the vertical column above has a generic section. Assuming Weil's conjecture that $\tau(\tilde{G}) = 1$, we obtain

$$\tau(G) = \frac{\tau(T)|\operatorname{cok}(\widehat{i})|}{\tau(T')}$$

using Proposition 6.4 and Proposition 6.6. Theorem 6.1 follows now from arithmetic duality theorems and the arguments in [Ono65, p.99–102]. Also note [Oes84, Corollary 3.3]. ∎

Corollary 6.8 *Suppose that the group G is split. Then the sequence*

$$\tau_1(G), \tau_2(G), \ldots$$

is constant. Here $\tau_n(G)$ is the Tamagawa number of the base change $G \times_k k_n$.

Proof See the cited work of Ono, in particular [Ono65, Theorem 2.1.1] and [Ono63, Proposition 4.5.1]. Essentially, under the stated hypothesis, the Tamagawa number is the cardinality of the fundamental group which is stable under base change. ∎

The remainder of this section will be devoted to studying how the Tamagawa number changes under base extensions of the form k_n/k under various hypotheses. We begin by recalling the explicit construction of the localization maps for $\widehat{M}$ in our particular setting.

We view the Galois module $\widehat{M}$ as a functor on field extensions of K in the usual way. Given a diagram of fields

$$\begin{tikzcd} K_1' \arrow[r] & K_2' \\ K_1 \arrow[u] \arrow[r] & K_2 \arrow[u] \end{tikzcd}$$

with the vertical maps being Galois extensions, we obtain morphisms

$$\mathrm{Gal}(K_2'/K_2) \to \mathrm{Gal}(K_1'/K_1) \quad \text{and} \quad \widehat{M}(K_1') \to \widehat{M}(K_2').$$

This gives maps $\mathrm{H}^i(K_1'/K_1, \widehat{M}) \to \mathrm{H}^i(K_2'/K_2, \widehat{M})$. In particular we have diagrams

$$\begin{tikzcd} \bar{K}^s \arrow[r] & \overline{K_x}^s \\ K \arrow[u] \arrow[r] & K_x \arrow[u] \end{tikzcd}$$

for every $x \in X$. This yields the map $\mathrm{H}^1(K, \widehat{M}) \to \prod_{x \in X} \mathrm{H}^1(K_x, \widehat{M})$, whose kernel is $Ш^1(K, \widehat{M})$.

We record the following.

Lemma 6.9 *Consider the projection $\pi\colon X_n \to X$.*

(i) *The map π is étale.*

(ii) *Let $x \in X$. Then $\pi^{-1}(x)$ consists of* $\gcd(n, \deg x)$ *points.*
(iii) *Suppose $\pi(y) = x$. Then $K_{n,y}/K_x$ is a cyclic Galois extension of degree $\frac{n}{\gcd(n,\deg x)}$. Its Galois group is generated by the Frobenius.*

Proof This is well known. See for example, [Ros00]. ■

We denote by K_n the function field of X_n. We assume from now on that there is a splitting field L for G that has field of constants k and further k contains all roots of unity of order dividing the fundamental group of G. The field L_n has its obvious meaning.

Lemma 6.10 *Let $y \in X_n$ and denote by π the projection $X_n \to X$. Under the above hypothesis we have that $\widehat{M}(K_n)$ (resp. $\widehat{M}(K_{n,y})$) is a trivial* $\mathrm{Gal}(K_n/K)$*-module (resp.* $\mathrm{Gal}(K_{n,y}/K_{\pi(y)})$*-module).*

Proof Note that $\widehat{M}(K_n) = \widehat{M}(\bar{K}^s)^{\mathrm{Gal}(\bar{K}^s/K_n)}$. The group $\mathrm{Gal}(K_n/K)$ is cyclic and generated by the Frobenius. Let F be a lift of the Frobenius to $\mathrm{Gal}(\bar{K}^s/K)$. As the field of constants of L and K are the same, we may assume F fixes L. Now by the assumption on the roots of unity we have that F acts on $\widehat{M}(\bar{K}^s)$ trivially. The result follows. The other case is similar. ■

Proposition 6.11 *Suppose that G has a splitting field with field of constants k and if k contains all roots of unity dividing the order of M, then $|\mathrm{H}^0(K_n, \widehat{M})|$ does not depend on n.*

Proof Follows from the above lemma. ■

Lemma 6.12 *The natural map*

$$\mathrm{H}^i(K_n/K, \widehat{M}) \to \prod_{y \in X_n} \mathrm{H}^i(K_{n,y}/K_{\pi(y)}, \widehat{M})$$

is injective.

Proof By the Riemann hypothesis for function fields, we can find a point $x \in X$ with $\deg x$ coprime to n. If y lifts this point, we have that $\mathrm{Gal}(K_n/K) \cong \mathrm{Gal}(K_{n,y}/K_x)$. The result follows from Lemma 6.10. ■

Theorem 6.13 *Suppose that G has a splitting field with field of constants k and k contains all roots of unity dividing the order of M. Then there is a natural isomorphism*

$$\text{Ш}^1(K, \widehat{M}) \cong \text{Ш}^1(K_n, \widehat{M}).$$

Proof We have an inflation-restriction sequence inducing the diagram

$$\begin{tikzcd}
0 \arrow[d] & \\
\mathrm{H}^1(K_n/K, \widehat{M}) \arrow[d] & \\
\mathrm{H}^1(K, \widehat{M}) \arrow[r, "l"] \arrow[d] & \prod_{x\in X} \mathrm{H}^1(K_y, \widehat{M}) \arrow[d] \\
\mathrm{H}^1(K_n, \widehat{M}) \arrow[r, "l_n"] \arrow[d] & \prod_{y\in X_n} \mathrm{H}^1(K_y, \widehat{M}) \\
\mathrm{H}^2(K_n/K, \widehat{M}) &
\end{tikzcd}$$

The previous lemma shows that we have an injection $\text{Ш}^1(K, \widehat{M}) \hookrightarrow \text{Ш}^1(K_n, \widehat{M})$. Let $\alpha \in \text{Ш}^1(K_n, \widehat{M})$. Also by the lemma we can lift α to $\tilde{\alpha} \in \mathrm{H}^1(K, \widehat{M})$. We need to show that $\tilde{\alpha}$ is in the subgroup $\text{Ш}^1(K, \widehat{M})$ modulo the image of $\mathrm{H}^1(K_n/K, \widehat{M})$. Let $l(\tilde{\alpha}) = (\beta_x)_{x\in X}$. Choose for each $x \in X$ a lift $\tilde{x} \in X_n$. Since $\alpha \in \text{Ш}^1(K_n, \widehat{M})$ we have that $\beta_x \in \mathrm{H}^1(K_{\tilde{x}}/K_x, \widehat{M})$ for every x. The hypotheses imply that $\widehat{M}(\widehat{k(\tilde{x})})$ is trivial as a $\mathrm{Gal}(\widehat{k(\tilde{x})}/\widehat{k(x)})$-module. So we may think of each β_x as a homomorphism

$$\beta_x : \mathrm{Gal}(K_{\tilde{x}}/K_x) \to \widehat{M}(K_{\tilde{x}}).$$

We have an inclusion of Abelian groups $\widehat{M}(K_n) = \widehat{M}(\bar{K}^s)^{\mathrm{Gal}(\bar{K}^s/K_n)} \hookrightarrow \widehat{M}(K_{\tilde{x}})$. As each β_x comes from $\tilde{\alpha}$, the above homomorphisms factor through $\widehat{M}(K_n)$. Define a homomorphism $\alpha_0\colon\ \mathrm{Gal}(K_n/K) \to \widehat{M}(K_n)$ by $F \mapsto \tilde{\alpha}(F)$, where F is the Frobenius and we are choosing a representing cocycle for $\tilde{\alpha}$ in the above expression. Now an easy diagram chase shows that $\tilde{\alpha} - \alpha_0$ is in $\text{Ш}^1(K, \widehat{M})$. ■

Corollary 6.14 *Under the hypothesis of the theorem and assuming Weil's conjecture for the universal cover of G, we have $\tau_n(G) = \tau(G)$. for every n.*

Proof Combine the theorem with Theorem 6.1 and Proposition 6.11. ■

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