Note on Numerical Integration.

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1. All the commonly used rules for the approximate quadrature of areas, such as those of Cotes, Simpson, Tchebychef and Gauss, are based on the assumption that y can be expressed as a rational integral function of x with *finite* coefficients. A tacit assumption is thus made that $\frac{dy}{dx}$ is not infinite within the range considered, and it is therefore hardly a matter for surprise that the degree of accuracy obtainable by the use of these rules in the case of a curve which *touches* the end ordinates is very poor.

In such a case, however, the difficulty can be entirely obviated by the use of a rule based on the assumption that the equation of the curve is

$$y = \sqrt{1 - x^2}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots\ldots a_{2n-1} x^{2n-1}) \ldots\ldots (1)$$

where n is the number of ordinates and the limits of the base are taken to be ± 1 as usual.

It is proved in what follows that an expression can be obtained for the exact area of such a curve if the ordinates are measured at the points where

$$x = \cos\left(\frac{k\pi}{n+1}\right) \qquad (k = 1, 2, 3 \ldots . n) \ldots\ldots\ldots\ldots (2)$$

The geometrical interpretation of this is shewn in Fig. 1, which is drawn for $n = 5$. Thus if we divide the circumference of a semicircle into $(n + 1)$ equal parts the ordinates must pass through the n points thus found. The trouble of finding the position of the ordinates, which is the objection to the rules of Tchebychef and Gauss, does not exist in the case of this rule.

The value of the coefficients by which the ordinates have to be multiplied is also easily remembered. Thus, if we write as the expression for the area

$$A = \sum_{k=1}^{n} (p_k\, y_k) \times l \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where l is the perpendicular distance between the end ordinates, we have

$$p_k = \frac{\pi}{2(n+1)} \sin\left(\frac{k\pi}{n+1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

This is easily remembered since $\sin\left(\frac{k\pi}{n+1}\right)$ is the length of the corresponding ordinate of the semicircle in Fig. (1) divided by the radius, so that we have

$$A = \frac{\pi}{n+1} \Sigma \text{(Ordinate of Semicircle) (Ordinate of Curve)} \quad \ldots\ldots (5)$$

2. If now our given curve is of any unsymmetrical form, as in Fig. 3, we can always construct a symmetrical curve as in Fig. 2 having the same length for all the ordinates $a_1 b_1$, $a_2 b_2$, etc., as those in Fig. 3 and the areas of the curves will clearly be equal to one another. Hence the rule will hold good for any curve such as that in Fig. 3.

The accuracy of the rule may be estimated by noting that we have $2n$ values of a_r at our disposal in (1). The area given by (3) is therefore that of a continuous curve which agrees with the given curve in $4n$ points in addition to A and B. Of these points $2n$ are arbitrary, and therefore if these are supposed to move into coincidence with the points $a_1, a_2, a_3 \ldots b_1, b_2$, etc., we may say that (3) *gives the area of a curve which not only passes through all the* $(2n+2)$ *points* $A, a_1, a_2 \ldots B, b_1, b_2 \ldots$ *but has a common tangent with the given curve at every one of these points also.*

3. The unsuitable nature of the rules usually used when applied to curves of the nature considered can be best shewn by considering the case of a circle of radius a.

(1) *Cotes*' Rule with *Five* Ordinates.

$$\text{Area} = \frac{b}{90}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5)$$

$$= \frac{2r}{90}(2 \times 7 \times 0 + 2 \times 32 \times 1 \cdot 4142r + 12 \times 2r)$$

$$= \underline{2 \cdot 6112r^2} \qquad \text{Error} = -17\,\%$$

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(2) *Simpson's* Rule with *Five* Ordinates.

$$\text{Area} = \frac{b}{12}(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5)$$
$$= \frac{2r}{12}(0 + 2 \times 4 \times 1 \cdot 4142 + 2 \times 2)\, r$$
$$= \underline{2 \cdot 5548r^2} \qquad \text{Error} = -\,18 \cdot 7\,\%$$

(3) *Cotes*' Rule with *Seven* Ordinates.

$$\text{Area} = \frac{b}{840}(41y_1 + 216y_2 + 27y_3 + 272y_4 + 27y_5 + 216y_6 + 41y_7)$$
$$= \frac{2r}{840}(2 \times 216 \times 1 \cdot 49071 + 2 \times 27 \times 1 \cdot 88562 + 272 \times 2)\, r$$
$$= \underline{3 \cdot 7097r^2} \qquad \text{Error} = -\,2 \cdot 25\,\%$$

(4) *Weddle's* Rule with *Seven* Ordinates.

$$\text{Area} = \frac{b}{20}(y_1 + 5y_2 + y_3 + 6y_4 + y_5 + 5y_6 + y_7)$$
$$= \frac{2r}{20}(2 \times 5 \times 1 \cdot 49071 + 2 \times 1 \cdot 88562 + 6 \times 2)\, r$$
$$= \underline{3 \cdot 06784r^2} \qquad \text{Error} = -\,2 \cdot 34\,\%$$

(5) *Simpson's* Rule with *Seven* Ordinates.

$$\text{Area} = \frac{b}{18}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + y_7)$$
$$= \frac{2r}{18}(2 \times 4 \times 1 \cdot 49071 + 2 \times 2 \times 1 \cdot 88562 + 4 \times 2)\, r$$
$$= \underline{3 \cdot 05202r^2} \qquad \text{Error} = -\,2 \cdot 85\,\%$$

(6) *Tchebychef's* Rule with *Five* ordinates.

Here the ordinates are measured along the straight lines

$$x = 0,\ \pm \cdot 37454r \text{ and } \pm \cdot 83250r$$

and the area is given by the base multiplied by the arithmetical mean of the ordinates.

In the case of the circle the lengths of the ordinates are

$$1 \cdot 10804r,\ 1 \cdot 85442r,\ 2r,\ 1 \cdot 85442r,\ 1 \cdot 10804r$$

and the area is found to be

$$\tfrac{1}{5} \times 7 \cdot 92492r \times 2r = \underline{3 \cdot 16997r^2} \qquad \text{Error} = +\,0 \cdot 90\%$$

(7) *Gauss's* Rule with *Five* ordinates.

Here the ordinates are measured along the straight lines

$$x = 0,\ \pm \cdot 53847r,\ \pm \cdot 90618r$$

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and the corresponding coefficients or weights are

$$\cdot 28444,\ \cdot 23931,\ \cdot 11846.$$

The lengths of the measured ordinates in the case of a circle are therefore

$$2r,\ 1\cdot 6876r,\ \cdot 8458r.$$

The area is thus found to be

$$2r\,(\cdot 28444\times 2 + 2\times\cdot 23931\times 1\cdot 6876 + 2\times\cdot 11846\times\cdot 8458)r$$
$$= \underline{3\cdot 15399r^2} \qquad\qquad \text{Error} = +\,0\cdot 39\%$$

The last two rules clearly give much better results than the first five, but owing to the trouble of setting off the ordinates and also to the fact that the values of the coefficients necessitate reference to books, they are hardly ever used in practice. The rule given in this paper gives exact results for a circle, even with only one ordinate, and no difficulty is met with however many ordinates are used, and there is therefore no difficulty in obtaining any desired degree of accuracy with any curve between parallel tangents.

4. As an example of a case in which the rule does not give mathematically exact results we may take the area of a *cycloid* between successive cusps. Here the equations are

$$\left.\begin{aligned} x &= a\,(\theta + \sin\theta) \\ y &= a\,(1 + \cos\theta) \end{aligned}\right\}$$

where a is the radius of the generating circle. The length of the base is $2\pi a$ and the area is $3\pi a^2$.

If we take *five* ordinates we shall have

$$x_1 = \tfrac{1}{2}\pi a \text{ and therefore } (\theta_1 + \sin\theta_1) = \frac{\pi}{2} = 1\cdot 570796.$$

By trial we find

$$\theta_1 = \cdot 83171 \text{ and } \sin\theta_1 = \cdot 73909$$
$$\therefore\quad y_1 = a\,(1 + \cos\theta_1) = \underline{1\cdot 67360a}$$

similarly $\qquad x_2 = \dfrac{\sqrt{3}}{2}\pi a$ and therefore $(\theta_2 + \sin\theta_2) = \dfrac{\sqrt{3}}{2}\pi = 2\cdot 72070$

$$\therefore\quad \theta_2 = 1\cdot 733985 \text{ and } \sin\theta_2 = \cdot 986715$$
$$\therefore\quad y_2 = a\,(1 + \cos\theta_2) = \underline{\cdot 837535a.}$$

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Hence the area given by the rule

$$= \frac{\pi}{12}\left\{\left(\frac{\sqrt{3}}{2} \times 1{\cdot}67360 + \tfrac{1}{2} \times {\cdot}837535\right)2 + 2\right\} \times 2\pi a^2$$

$$= \underline{3{\cdot}0035\,\pi a^2} \qquad \text{Error} = 0{\cdot}12\%.$$

The area given by *Cotes'* Rule with *five* ordinates

$$= \frac{2\pi a^2}{90}(2 \times 32 \times 1{\cdot}6736 + 12 \times 2)$$

$$= \underline{2{\cdot}9136\pi a^2} \qquad \text{Error} = -\,2{\cdot}88\%,$$

and by *Simpson's* Rule with *five* ordinates

$$= \frac{2\pi a^2}{12}(2 \times 4 \times 1{\cdot}6736 + 2 \times 2)$$

$$= \underline{2{\cdot}8981\pi a^2} \qquad \text{Error} = -\,3{\cdot}40\%.$$

Proof of Formula.

(1) From the Exponential Expression for $\cos\theta$ we have $2^{2r}\cos^{2r}\theta = (e^{i\theta} + e^{-i\theta})^{2r}$

$$= e^{2ir\theta} + 2re^{2i(r-1)\theta} + \ldots\ldots + \frac{(2r)!}{r!\,r!} + \ldots\ldots + 2re^{-2i(r-1)\theta} + e^{-2ir\theta}.$$

Hence if $\theta = \dfrac{k\pi}{n+1}$, then $2^{2r}\cos^{2r}\left(\dfrac{k\pi}{n+1}\right) =$

$$e^{\frac{2irk\pi}{n+1}} + 2re^{\frac{2i(r-1)k\pi}{n+1}} + \ldots + \frac{(2r)!}{r!\,r!} + \ldots + 2re^{\frac{-2i(r-1)k\pi}{n+1}} + e^{\frac{-2irk\pi}{n+1}}$$

Now if we form a series of equations by putting $k = 0, 1, 2 \ldots n$ in turn, we get on adding

$$2^{2r}\sum_{k=0}^{n}\cos^{2r}\left(\frac{k\pi}{n+1}\right) = \sum_{k=0}^{n} e^{\frac{2irk\pi}{n+1}} + 2r\sum_{k=0}^{n} e^{\frac{2i(r-1)k\pi}{n+1}} + \ldots + (n+1)\frac{(2r)!}{r!\,r!}$$
$$+ \ldots + 2r\sum_{k=0}^{n} e^{\frac{-2i(r-1)k\pi}{n+1}} + \sum_{k=0}^{n} e^{\frac{-2irk\pi}{n+1}}$$

Now each of the terms on the right, with the exception of the middle term, represents a Geometrical Progression with $(n+1)$ terms. Thus

$$\sum_{k=0}^{n} e^{\frac{2irk\pi}{n+1}} = \frac{e^{2ir\pi} - 1}{e^{\frac{2ir\pi}{n+1}} - 1} = 0 \text{ if } r < (n+1)$$

since the numerator vanishes for all integral values of r, and the denominator does not vanish unless r is a multiple of $(n+1)$. Thus

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all the terms except the middle term vanish, and we therefore get on dividing out by $2^{2r}(n+1)$

$$\frac{1}{n+1}\sum_{k=0}^{n}\cos^{2r}\left(\frac{k\pi}{n+1}\right) = \frac{1}{2^{2r}}\ \frac{(2r)!}{r!\ r!}\ \text{ if } r<(n+1)$$

and similarly

$$\frac{1}{n+1}\sum_{k=0}^{n}\cos^{2r+2}\left(\frac{k\pi}{n+1}\right) = \frac{1}{2^{2r+2}}\ \frac{(2r+2)!}{(r+1)!\ (r+1)!}\text{ if } r<n$$

hence we find on subtraction

$$\frac{1}{n+1}\sum_{k=0}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)\cos^{2r}\left(\frac{k\pi}{n+1}\right) = \frac{1}{2^{2r}}\ \frac{(2r)!}{r!\ r!}\left(1-\tfrac{1}{4}\ \frac{(2r+2)(2r+1)}{(r+1)^2}\right)$$

$$= \frac{1}{2^{2r+1}}\ \frac{(2r)!}{r!\ (r+1)!}\ \ \dots\dots\dots\dots(1)$$

Now when $k=0$, the corresponding term on the left hand side of this equation vanishes (since $\sin^2\left(\frac{k\pi}{n+1}\right)=0$ when $k=0$). Hence we must have

$$\frac{1}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)\cos^{2r}\left(\frac{k\pi}{n+1}\right) = \frac{1}{2^{2r+1}}\ \frac{(2r)!}{r!\ (r+1)!}\ \ \dots\dots\dots(2)$$

(2) Again, we get on putting $x=\cos\theta$

$$\tfrac{1}{2}\int_{-1}^{1}x^{2r}\sqrt{1-x^2}\,dx = \tfrac{1}{2}\int_0^{\pi}\sin^2\theta\cos^{2r}\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi}\cos^{2r}\theta\,d\theta - \tfrac{1}{2}\int_0^{\pi}\cos^{2r+2}\theta\,d\theta$$

$$= \frac{(2r-1)(2r-3)\dots 1}{2r(2r-2)\dots\dots 2}\ \frac{\pi}{2} - \frac{(2r+1)(2r-1)\dots 1}{(2r+2)(2r)\dots 2}\ \frac{\pi}{2}$$

$$= \frac{(2r-1)(2r-3)\dots 1}{(2r+2)(2r)\dots 4}\ \frac{\pi}{4}.$$

Now $(2r-1)(2r-3)\dots\dots 1 = \dfrac{(2r)!}{2^r r!}$ and $(2r+2)2r\dots 4 = 2^r(r+1)!$

and we therefore find on substituting

$$\tfrac{1}{2}\int_{-1}^{1}x^{2r}\sqrt{1-x^2}\,dx = \frac{\pi}{2r+2}\ \frac{(2r)!}{r!\ (r+1)!}\ \ \dots\dots\dots\dots(3)$$

Combining equations (1) and (3) we get finally

$$\sum_{k=1}^{n}\frac{\pi}{2(n+1)}\sin^2\left(\frac{k\pi}{n+1}\right)\cos^{2r}\left(\frac{k\pi}{n+1}\right) = \tfrac{1}{2}\int_{-1}^{1}x^{2r}\sqrt{1-x^2}\,dx\ \ (4)$$

where r and n are any integers $(r<n)$.

(3) If now we consider the expression

$$\sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2\left(\frac{k\pi}{n+1}\right) \cos^{2r+1}\left(\frac{k\pi}{n+1}\right)$$

we see that $\cos^{2r+1}\left(\frac{\pi}{n+1}\right) = -\cos^{2r+1}\left(\frac{n\pi}{n+1}\right)$

$$\cos^{2r+1}\left(\frac{2\pi}{n+1}\right) = -\cos^{2r+1}\left(\frac{(n-1)\pi}{n+1}\right)$$

etc.,

so that the terms cancel out in pairs. Also, if n is an odd number the middle term will be $\cos^{2r+1}\frac{\pi}{2} = 0$. Hence we find

$$\sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2\left(\frac{k\pi}{n+1}\right) \cos^{2r+1}\left(\frac{k\pi}{n+1}\right) = 0 \quad \ldots\ldots(5)$$

Further, putting $x = \cos\theta$ we find

$$\tfrac{1}{2}\int_{-1}^{1} x^{2r+1}\sqrt{1-x^2}\,dx = \tfrac{1}{2}\int_0^\pi \cos^{2r+1}\theta\,d\theta - \tfrac{1}{2}\int_0^\pi \cos^{2r+3}\theta\,d\theta = 0 \ldots.(6)$$

Thus it follows that if $m < 2n$

$$\sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2\left(\frac{k\pi}{n+1}\right) \cos^m\left(\frac{k\pi}{n+1}\right) = \tfrac{1}{2}\int_{-1}^{1} x^m\sqrt{1-x^2}\,dx \ldots.(7)$$

for all values of m and n, and therefore if

$$x_k = \cos\left(\frac{k\pi}{n+1}\right) \text{ and } p_k = \frac{\pi}{2(n+1)} \sin\left(\frac{k\pi}{n+1}\right) \ldots\ldots\ldots(8)$$

we have

$$\sum_{k=1}^{n} p_k \sqrt{1-x_k^2}.\,x_k{}^m = \tfrac{1}{2}\int_{-1}^{1} x^m\sqrt{1-x^2}\,dx \quad (m < 2n) \quad \ldots\ldots(9)$$

From this it follows that if

$$y = \sqrt{1-x^2}\,(a_0 + a_1x + a_2x^2 + \ldots.\, a_{2n-1}\,x^{2n-1})$$

and if we write y_k for the value of y corresponding to $x = x_k$

$$\sum_{k=1}^{n} p_k\, y_k = \tfrac{1}{2}\int_{-1}^{1} y\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

provided that p_k and x_k have the values given in (8). This follows since equation (7) is satisfied for each term in the expression for y separately.

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Note.—The vital part of this proof consists in shewing that

$$\frac{1}{n+1} \sum_{k=0}^{n} \sin^2\left(\frac{k\pi}{n+1}\right) \cos^m\left(\frac{k\pi}{n+1}\right) \quad \ldots\ldots\ldots\ldots (1)$$

is independent of n if $m < 2n$.

For it clearly follows from this that

$$\sum_{k=0}^{n+1} \frac{\pi}{n+1} \sin^2\left(\frac{k\pi}{n+1}\right) \cos^m\left(\frac{k\pi}{n+1}\right) \quad \ldots\ldots\ldots\ldots (2)$$

is also independent of n if $n > \frac{m}{2}$. If now n is made to increase indefinitely, and we write θ for $\frac{k\pi}{n+1}$, $\frac{\pi}{n+1}$ becomes $d\theta$ and (2) becomes

$$\int_0^{\pi} \sin^2\theta \cos^m\theta \, d\theta = \int_{-1}^{1} \sqrt{1-x^2}\, x^m dx \quad \ldots\ldots\ldots\ldots (3)$$

if $\cos\theta$ is put equal to x.

Extension for Moments and Moments of Inertia of Section.

(1) The Moment of the Area bounded by the curve in Fig. 3 about the central ordinate $a_3 b_3$ is given by

$$M = \frac{l^2}{4} \int_{-1}^{1} yx dx$$

and if we wish to find this in the form

$$M = \sum_{k=1}^{n} (p'_k y_k) \times l^2$$

we must clearly make

$$p'_k = \frac{\pi}{2(n+1)} \sin\left(\frac{k\pi}{n+1}\right) \times \tfrac{1}{2} \cos\left(\frac{k\pi}{n+1}\right) = \frac{\pi}{8(n+1)} \sin\left(\frac{2k\pi}{n+1}\right) \quad ..(1)$$

(2) In a similar manner we see that the Moment of Inertia about the same axis is given by

$$I = \frac{l^3}{8} \int_{-1}^{1} yx^2 dx$$

and this will be given by

$$I = \sum_{k=1}^{n} (p''_k y_k) \times l^3$$

if

$$p''_k = \frac{\pi}{2(n+1)} \sin\left(\frac{k\pi}{n+1}\right) \times \tfrac{1}{4} \cos^2\left(\frac{k\pi}{n+1}\right)$$

$$= \frac{\pi}{32(n+1)} \left(\sin\left(\frac{k\pi}{n+1}\right) + \sin\left(\frac{3k\pi}{n+1}\right)\right) \quad \ldots\ldots\ldots\ldots (2)$$

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(3) From these formulae the Moment of Inertia about an axis through the Centre of Gravity parallel to the ordinates is found by use of the formula

$$I_{C.G.} = I - \frac{M^2}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where I, M and A are found by means of the formulae already given.

Note.—In the case of the formula for the Moments *half the coefficients will have a negative sign.*

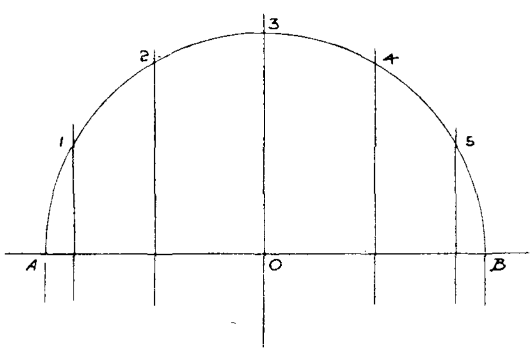

Fig. 1

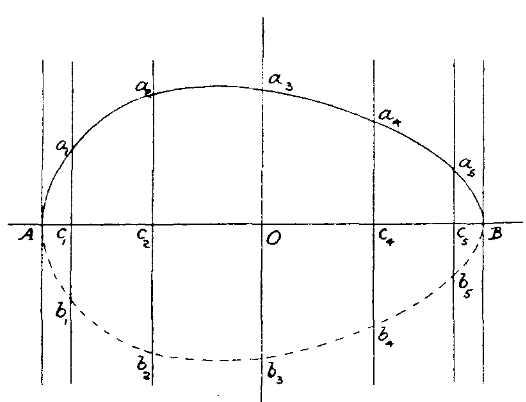

Fig. 2

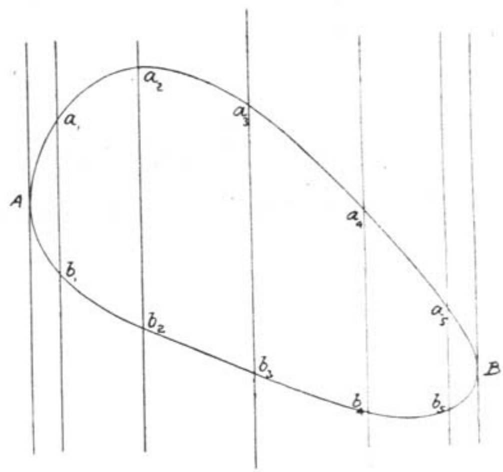

Fig 3.

Note on Discussion of the Paper.

In the course of the discussion it was suggested (I think by Prof. Whittaker) that a simpler method of dealing with curves of the form considered in the paper would be to apply Simpson's Rule to the area enclosed between the given curve and a semicircle. This method is still open to the original objection *unless the radius of curvature at the two extremities of the base is equal to half the length of the base.*

Thus, taking the equation as being

$$y = \sqrt{1 - x^2}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots .)$$

the subtraction of the ordinates of the semicircle will leave

$$y' = \sqrt{1 - x^2}\,((a_0 - 1) + a_1 x + a_2 x^2 + \dots .)$$

which is of the same form as before, though the error will be reduced in the ratio $(A_2 - A_1)/A_2$, where A_2 is the area of the given curve and A_2 that of the semicircle. Further, I am doubtful if any saving in time would be effected.

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