Proceedings of the Edinburgh Mathematical Society (1995) **38**, 171–177 ©

A SIMPLE CONSTRUCTION OF EXPONENTIAL BASES IN L^2 OF THE UNION OF SEVERAL INTERVALS

by KRISTIAN SEIP

(Received 23rd September 1993)

It is proved that every space L^2 $(I_1 \cup I_2)$, where I_1 and I_2 are finite intervals, has a Riesz basis of complex exponentials $\{e^{i\lambda_k x}\}$, $\{\lambda_k\}$ a sequence of real numbers. A partial result for the corresponding problem for $n \geqq 3$ finite intervals is also obtained.

1991 *Mathematics subject classification.* Primary 30E05, 30E10, 42A10, 46E20; Secondary 94A12.

1. Introduction

Riesz bases of complex exponentials in the space $L^2(I)$, where I is a finite interval of **R**, have been thoroughly explored since the possibility of nonharmonic Fourier expansions was discovered by Paley and Wiener [7]. A complete theory is exposed in the fundamental paper [3], the basic result of which is the necessary and sufficient condition for basicity obtained by Pavlov [8].

From the point of view of communication theory, it is natural to ask for a solution of the corresponding problem of basicity when the one interval is replaced by a union of disjoint intervals $I_1 \cup I_2 \cup \ldots \cup I_n$. The problem is then to find those sequences $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ of real numbers for which the exponential systems

$$\mathscr{E}(\Lambda) = \{e^{i\lambda_k x} \colon \lambda_k \in \Lambda\}$$

form Riesz bases in $L^2(I_1 \cup I_2 \cup \ldots \cup I_n$. Restated in the terms of signal processing, the matter is to find those sampling sets Λ which yield both *stable* and *non-redundant sampling* of corresponding multiband signals; we refer to Landau's (by now) classical paper [5], which also gives a rigorous explanation of the meaning of the "Nyquist rate" in this connection.

Little seems to be known about this problem; in fact, existence of Riesz bases of complex exponentials in $L^2(I_2 \cup I_2 \cup \ldots \cup I_n)$ for $n \geqq 2$ has so far been established only in very special situations. The most general result available appears to be that such Riesz bases exist whenever the intervals $I_1, I_2, \ldots, I_n$ have commensurable lengths. This result can be proved by the methods of the recent paper [2], as pointed out by the authors; it can also be proved by means of another approach, originally used by Kohlenberg in a less general situation [4, 6].

In this paper, we construct Riesz bases of complex exponentials with other constraints on the intervals. Firstly, we confirm that such bases do exist in every space $L^2(I_1 \cup I_2)$. Secondly, we prove existence of bases in $L^2(I_2 \cup I_2 \cup \ldots \cup I_n)$ under certain (nondiscrete) conditions on the gaps between the intervals.

Aside from an application of Avdonin's theorem [**1**], our approach is quite elementary. It is strongly felt that the restrictions we make, as well as those concerning commensurability, reflect our insufficient understanding of the problem rather than genuine obstacles. We can see no reason why exponential bases should not exist in every space $L^2(I_2 \cup I_2 \cup \ldots \cup I_n)$.

2. Construction of exponential bases on two intervals

The solution of the existence problem for $n=2$ is provided by the following theorem.

Theorem. *Let $a_1 < b_1 < a_2 < b_2$, where a_1, b_1, a_2, b_2 are otherwise arbitrary real numbers. Then there exists a subsequence, say $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$, of the sequence $\{2\pi m/(b_2 - a_1)\}_{m=-\infty}^{\infty}$ such that the exponential system $\mathscr{E}(\Lambda)$ is a Riesz basis in $L^2([a_1, b_1] \cup [a_2, b_2])$.*

It is clear that the problem is invariant under simultaneous dilations and translations of the intervals $[a_1, b_1]$ and $[a_2, b_2]$. We may therefore assume (for convenience) that $b_1 = \pi$ and $a_1(b_2 - a_2) = -\pi$. Since we restrict attention only to exponentials which are $(b_2 - a_1)$-periodic and since

$$[-\pi, \pi] = [a_1, b_1] \cup [a_2 - (b_2 - a_1),\, b_2 - (b_2 - a_1)],$$

we may equivalently consider the problem of finding a Riesz basis in the space $L^2(-\pi, \pi)$. For, if $\mathscr{E}(\Lambda)$ is a Riesz basis in $L^2(-\pi, \pi)$ with dual basis $\{g_k\}$, then $\mathscr{E}(\Lambda)$ (restricted to $[a_1, b_1] \cup [a_2, b_2]$) is a Riesz basis in $L^2[a_1, b_1] \cup [a_2, b_2])$ with dual basis $\{h_k\}$, where

$$h_k(x) = \begin{cases} g(x), & a_1 \leqq x \leqq b_1 \\ g(x - (b_2 - a_1)), & a_2 \leqq x \leqq b_2. \end{cases}$$

Thus, our theorem is equivalent to the following lemma.

Lemma. *For every $a \in (0, 1)$, we can find a subsequence, say $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$, of the sequence $\{am\}_{m=-\infty}^{\infty}$ such that the exponential system $\mathscr{E}(\Lambda)$ is a Riesz basis in the space $L^2(-\pi, \pi)$.*

The lemma is an immediate consequence of Kadec's 1/4-theorem [**9**, p. 42] if $0 < a < 1/2$, since then we can find a sequence Λ satisfying

$$|\lambda_k - k| \leqq a/2 < 1/4$$

for every integer k.

The case of rational a is also easy. Then we may use the fact that the union Λ of p mutually disjoint translates of the sequence $\{pk\}_{k=-\infty}^{\infty}$ is a sequence of the desired type (see [3]). With $a=p/q$ and p and q relatively prime integers $(0<p<q)$, we may clearly extract such a subsequence Λ from $\{am\}_{m=-\infty}^{\infty}$.

The problem is more delicate for irrational a in the range $1/2<a<1$, and here it seems necessary to use more advanced tools. We shall base our argument on an important theorem due to Avdonin (see [1] and [3, p. 251]). The following statement is a special version of Avdonin's theorem: Let $\lambda_k=k+\delta_k$ and suppose the sequence $\{\lambda_k\}$ is separated, i.e., $\inf_{j\neq k}|\lambda_j-\lambda_k|>0$. If there exist a positive integer N and a positive number $d<1/4$ such that

$$\left|\sum_{k=mN+1}^{(m+1)N}\delta_k\right|\leqq dN$$

for all integers m, then the system $\mathscr{E}(\Lambda)$ is a Riesz basis in $L^2(-\pi,\pi)$.

Let us now see how to pick from $\{am\}$ the points $\lambda_{mN+1},\lambda_{mN+2},\dots,\lambda_{(m+1)N}$, with N chosen so that Avdonin's theorem applies. We require these points to lie in the interval $[mN+1/2,\,(m+1)N+1/2)$. Letting $S_{\min}(m)$ and $S_{\max}(m)$ denote respectively the smallest and largest possible value of the sum

$$\sum_{k=mN+1}^{(m+1)N}\delta_k,$$

we have

$$S_{\min}(m)\leqq\sum_{k=1}^{N}(ak+1/2-k)=\frac{N-(1-a)N(N+1)}{2}$$

and

$$S_{\max}(m)\geqq\sum_{k=1}^{N}(N+1/2-ak-k)=\frac{(1-a)N(N+1)-N}{2}.$$

Therefore, if choosing $N\geqq(1-a)^{-1}$, we have

$$S_{\min}(m)<0<S_{\max}(m),$$

and it follows that the λ_k may be chosen so that

$$\left|\sum_{k=mN+1}^{(m+1)N}\delta_k\right|\leqq a/2=\frac{a}{2N}N.$$

With these choices of N and the λ_k we are done, in view of Avdonin's theorem.

3. Construction of exponential bases on $n \geqq 3$ intervals

The argument above which allowed us to reduce the original problem to a problem concerning one interval, can be generalized to yield some information about the corresponding problem for n intervals. We obtain rather severe restrictions on the lengths of the intermediate gaps; however, since we avoid discrete conditions, we find our observations noteworthy.

We shall need the following elementary fact about Riesz bases (see [9, p. 32]): A sequence of vectors $\{f_n\}_{n=1}^{\infty}$ in a separable Hilbert space $\mathscr{H}$ is a Riesz basis in $\mathscr{H}$ if and only if the sequence $\{f_n\}$ is complete in $\mathscr{H}$ and there exist positive constants A and B such that for every finite sequence of scalars $c_1, c_2, \ldots, c_N$ we have

$$A \sum_{n=1}^{N} |c_n|^2 \leqq \left\| \sum_{n=1}^{N} c_n f_n \right\|^2 \leqq B \sum_{n=1}^{N} |c_n|^2;$$

we refer to the largest possible A and the smallest possible B as the *bounds* of the Riesz basis $\{f_n\}$.

We introduce the auxiliary function

$$\eta(x) = 1 - \cos\frac{\pi}{2}x + \sin\frac{\pi}{2}x$$

and prove the following proposition.

Proposition. *Let E be a compact subset of R with left end-point a_0 and right end-point b_0, and suppose that the exponential system $\mathscr{E}(\Lambda)$ $(\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty})$ is a Riesz basis in $L^2(E \cup [a_0 - (b-a), a_0])$ with bounds A and B. Then if $b_0 < a < b$ and*

$$\mu = \sqrt{\frac{B}{A}}\,\eta\left(1 - \frac{a-b_0}{b-a_0}\right) < 1$$

there exists a subsequence, say $\Lambda' = \{\lambda'_k\}_{k=-\infty}^{\infty}$, of the sequence $\{m2\pi/(b-a_0)\}$ such that the exponential system $\mathscr{E}(\Lambda')$ is a Riesz basis in $L^2(E \cup [a,b])$ with bounds A' and B' satisfying

$$A' \geqq (1-\mu)^2 A \quad \textit{and} \quad B' \leqq (1+\mu)^2 B.$$

We may argue that our problem is translation and dilation invariant in such a way that we may assume (for convenience) that $b_0 = \pi$ and $a_0 - (b-a) = -\pi$. Since we consider only exponentials which are $(b-a_0)$-periodic, we may equivalently consider the problem of obtaining a Riesz basis in the space $L^2(E')$, where $E' = E \cup [-\pi, a_0]$. For each integer k, we pick a $\lambda'_k \in \{m2\pi/(b-a_0)\}$ such that

$$|\lambda_k - \lambda'_k| \leqq d = \frac{\pi}{b-a_0} = \frac{1}{2}\left(1 - \frac{a-b_0}{b-a_0}\right).$$

We now apply the Paley–Wiener stability criterion [9, p. 38], considering $\mathscr{E}(\Lambda')$ $(\Lambda'=\{\lambda'_k\})$ as a perturbation of the given basis $\mathscr{E}(\Lambda)$ in $L^2(E')$. In other words, we need to prove that for some $\mu<1$ and all finite sequences of scalars $\{c_k\}$, we have

$$\left\|\sum_k c_k(e^{i\lambda_k x}-e^{i\lambda'_k x})\right\|_{L^2(E')}\leqq\mu\left\|\sum_k c_k e^{i\lambda_k x}\right\|_{L^2(E')}. \tag{1}$$

Put $\delta_k=\lambda_k-\lambda'_k$ and write

$$e^{i\lambda^k x}-e^{i\lambda_k x}(1-e^{i\delta_k x}).$$

We expand each of the functions $1-e^{i\delta_k x}$ in the same ingenious way as done by Kadec in his proof of the 1/4-theorem and follow his estimates step by step (see [9, pp. 42–44]). This gives us (1) with μ as in the Proposition; the procedure is verbatimly as in [9, pp. 42–44] (to which we refer for details), except that we have to take into account the non-orthogonality of the given basis by using the estimate

$$\left\|\sum_k a_k c_k e^{i\lambda_k x}\right\|_{L^2(E')}\leqq\sqrt{B/A}\,\sup_j|a_j|\left\|\sum_k c_k e^{i\lambda_k x}\right\|_{L^2(E')}$$

at a certain point in the proof.

Two applications of the triangle inequality to (1) give us the estimates for A' and B', and this completes the proof of the Proposition.

Let us now apply the proposition to obtain a result for $n\geqq 3$ intervals.

Corollary. *Let $n\geqq 3$, suppose we have $a_1<b_1<a_2<b_2<\dots<a_n<b_n$, and put*

$$\xi=\max_{2\leqq k\leqq n}\frac{a_k-b_{k-1}}{b_k-a_1+\sum_{j=k+1}^n(b_j-a_j)}.$$

If

$$\eta(1-\xi)\leqq(\sqrt{2}-1)^{n-2},$$

then there exists a subsequence, say $\Lambda'=\{\lambda'_k\}_{k=-\infty}^{\infty}$, of the sequence $\{m2\pi/(b_n-a_1)\}$ such that the exponential system $\mathscr{E}(\Lambda')$ is a Riesz basis in the space $L^2(\bigcup_{j=1}^n[a_j,b_j])$.

To prove this assertion, we argue as follows. Define

$$E_j=\left[a_1-\sum_{i=l+1}^n(b_l-a_l),b_1\right]\cup\left(\bigcup_{i=2}^j[a_i,b_i]\right)$$

for $j=1,2,\dots,n$ so that, in particular, we have

https://doi.org/10.1017/S0013091500006295 Published online by Cambridge University Press

$$E_1=\left[a_1-\sum_{l=2}^{n}(b_l-a_l),b_1\right] \quad\text{and}\quad E_n=\bigcup_{i=1}^{n}[a_i,b_i].$$

Observe that for each $j=2,3,\dots,n$, the Proposition applies to $L^2(E_j)$ with $a=a_j$, $b=b_j$, $E=E_j$, $E\cup[a_0-(b-a),a_0]=E_{j-1}$, and

$$\frac{a-b_0}{b-a_0}=\frac{a_j-b_{j-1}}{b_j-a_1+\sum_{i=j+1}^{n}(b_i-a_i)}.$$

It follows that we may start with an orthonormal exponential basis for the space $L^2(E_1)$, apply our Proposition recursively, and thereby obtain a basis for $L^2(E_n)$, *provided* the numbers

$$\xi_j=\frac{a_j-b_{j-1}}{b_j-a_1+\sum_{i=j+1}^{n}(b_i-a_i)}$$

are sufficiently small. More precisely, define

$$h(x)=\frac{1+x}{1-x}x,$$

let h^j denote h composed with itself j times, and check, by a straightforward argument based on repeated use of our Proposition, that it is sufficient that

$$h^j((\eta(1-\xi))<1$$

for all $j=0,1,\dots,n-2$. Since h is invertible on the interval $(0,1)$ and the inverse function h^{-1} satisfies

$$h^{-1}(x)<(\sqrt{2}-1)x$$

for $x\in(0,1)$, we obtain from this condition the conclusion of the Corollary.

4. Two remarks

If we observe that

$$b_k-a_1+\sum_{j=k+1}^{n}(b_j-a_j)=\sum_{j=1}^{n}(b_j-a_j)+\sum_{j=2}^{k}(a_j-b_{j-1}),$$

it becomes apparent that the bases constructed in the proofs of both the Theorem and the Corollary depend only on the total measure of the set $\bigcup_{j=1}^{n}[a_j,b_j]$ and on the lengths of the gaps between the intervals. In other words, the individual lengths of the intervals have no influence on the construction. This is a curious fact when comparing with the requirement, mentioned in the introduction, about intervals of commensurable lengths.

Let us finally observe that our approach may be adapted to the special case of symmetrical "multiband regions", which is a reasonable situation to consider from a practical point of view. As an example, let

$$E=[-b_2,-a_2]\cup[-a_1,a_1]\cup[a_2,b_2]$$

with $0<a_1<a_2<b_2$. Then repeating the arguments of the proof of the Theorem and assuming that $b_2-a_2<a_2-a_1$, we obtain that for some subsequence, say Λ, of the sequence $\{2\pi m/(b_2+a_1)\}$, the system $\mathscr{E}(\Lambda)$ is a Riesz basis in $L^2(E)$. A similar modification can be made in the Proposition and in the Corollary.

REFERENCES

1. S. A. Avdonin, On the question of Riesz bases of complex exponential functions in L^2, *Vestnik Leningrad Univ. Ser. Mat.* **13** (1974); English transl. in *Vestnik Leningrad Univ. Ser. Math.* **7** (1979), 5–12.

2. L. Bezuglaya and V. Katsnelson, The sampling theorem for functions with limited multi-band spectrum. I, *Z. Anal. Anwendungen* **12** (1993), 511–534.

3. S. V. Khrushchev, N. K. Nikol'skii and B. S. Pavlov, Unconditional bases of complex exponentials and of reproducing kernels, in *Complex Analysis and Spectral Theory* (Seminar, Leningrad 1979/80, ed. V. P. Havin and N. K. Nikol'skii, Lecture Notes in Math. **864**, Springer-Verlag, Berlin Heidelberg, 1981), 214–336.

4. A. Kohlenberg, Exact interpolation of band-limited functions, *J. Appl. Phys.* **24** (1953), 1432–1436.

5. H. J. Landau, Sampling, data transmission, and the Nyquist rate, *Proc. IEEE* **55** (1967), 1701–1706.

6. Yu. I. Lyubarskii, Private Communication, 1993.

7. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (Amer. Math. Soc. Colloq. Publ., **19**, Amer. Math. Soc., New York, 1934).

8. B. Pavlov, Basicity of an exponential system and Muckenhoupt's condition, *Soviet Math. Dokl.* **20** (1979), 655–659.

9. R. M. Young, *An Introduction to Nonharmonic Fourier Series* (Academic Press, New York, 1980).

Department of Mathematical Sciences
Norwegian Institute of Technology
N-7034 Trondheim
Norway

E-mail: seip@imf.unit.no