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THE DIRICHLET PROBLEM FOR THE TWO-DIMENSIONAL LAPLACE EQUATION IN A MULTIPLY CONNECTED DOMAIN WITH CUTS

P. A. KRUTITSKII

Department of Mathematics, Faculty of Physics, Moscow State University, Moscow 119899, Russia

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Abstract The Dirichlet problem for the Laplace equation in a connected-plane region with cuts is studied. The existence of a classical solution is proved by potential theory. The problem is reduced to a Fredholm equation of the second kind, which is uniquely solvable.

Keywords: Laplace equation; Dirichlet problem; cracked domain

AMS 1991 *Mathematics subject classification:* Primary 31A05; 31A25; 35J05; 45E05; 30E25

1. Introduction

The theory of boundary-value problems for two-dimensional PDEs mostly deals with connected domains bounded by closed curves. A small number of investigations are devoted to the problems outside cuts in the plane. There are almost no results concerning well-posedness of classical problems in domains bounded by closed curves and containing cuts. It seems that the difficulty in analysing these problems comes from the different technique of the proof of the solvability theorems for domains bounded by closed curves and for plane with cuts. It is very likely that there is no great difference between these problems in nature. In the present paper, we try to overcome technical difficulties for the Laplace equation in the internal domain with cuts, and, therefore, to suggest an approach to the analysis of similar problems.

The two-dimensional Dirichlet boundary-value problem for the Laplace equation in a multiply connected domain bounded by closed curves is considered, for instance, in [**2**, **9**]. The Dirichlet problem for this equation in the exterior of cuts is studied in [**9**]. The present note is an attempt to join these problems together and to consider domains containing cuts. Similar domains have great significance, because cuts model cracks, screens or wings in physical problems. Domains without cuts are a particular case of our problem. Our approach is different from [**2**, **9**] even in this case.

The approach proposed in the present paper can be applied to other elliptic problems in domains with closed and open boundaries. The Dirichlet and Neumann problems for

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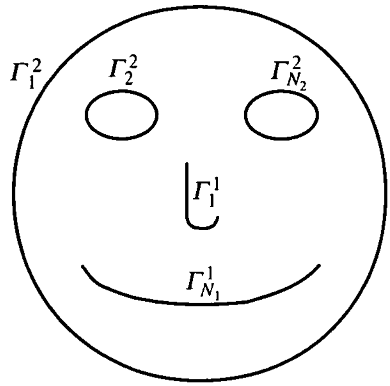

Figure 1. An internal domain.

the Helmholtz equation in a plane domain with cuts have recently been investigated in [**5**, **7**, **8**]. Some nonlinear problems on fluid flow over several obstacles, including wings, were treated in [**6**].

The uniqueness theorem in the Dirichlet problem for the Laplace equation follows from the maximum principle, unlike the Dirichlet problem for the Helmholtz equation [**5**, **8**], where the energy equalities are used. This enables one to study the problem in the present paper under weakened smoothness conditions in comparison with [**5**, **8**].

In addition, the Dirichlet problem for the Laplace equation is more complicated than the relative problem for the dissipative Helmholtz equation, even in classical multiply connected domains without cuts, because, generally, the solution for the Laplace equation cannot be expressed in the form of pure double layer potential. In view of these reasons, different modified approaches were suggested, for instance, in [**9**], but they are not appropriate in domains with cuts.

Let us note the basic difficulties in the analysis of the Dirichlet problem in a plane domain with cuts by potential theory. Problems in a domain bounded by closed curves can be reduced to a Fredholm equation of the second kind. Problems in the exterior of cuts can be reduced to the integral equation of the first kind with a weak or strong singularity in the kernel. If we consider a domain with cuts, then, on the whole boundary, we obtain a non-classical integral equation, for which the analysis is quite complicated. Our approach enables one to reduce the Dirichlet problem in a domain with cuts to a uniquely solvable Fredholm equation of the second kind in the appropriate Banach space.

2. Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [**9**].

In the plane $x = (x_1, x_2) \in R^2$ we consider the multiply connected domain bounded by simple open curves $\Gamma_1^1, \dots, \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, \dots, \Gamma_{N_2}^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, so that the curves do not have points in common and the curve Γ_1^2 encloses all others. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \qquad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \qquad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by Γ^2 will be called $\mathcal{D}$. We assume that each curve Γ_n^k is parametrized by the arc length s:

$$\Gamma_n^k = \{x : x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n^k, b_n^k]\}, \quad n = 1, \ldots, N_k, \quad k = 1, 2,$$

so that $a_1^1 < b_1^1 < \cdots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \cdots < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is on the right when the parameter s increases on Γ_n^2. Therefore, points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence, except a_n^2, b_n^2, which correspond to the same point x for $n = 1, \ldots, N_2$. Below the sets of the intervals on the Os axis,

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \qquad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \qquad \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k],$$

will be denoted by the same symbols as corresponding sets of curves, that is, by Γ^1, Γ^2 and Γ, respectively.

We put $C^0(\Gamma_n^2) = \{\mathcal{F}(s) : \mathcal{F}(s) \in C^0[a_n^2, b_n^2],\ \mathcal{F}(a_n^2) = \mathcal{F}(b_n^2)\}$, and

$$C^0(\Gamma^2) = \bigcap_{n=1}^{N_2} C^0(\Gamma_n^2).$$

By $\mathcal{D}_n$ we denote the internal domain bounded by the curve Γ_n^2, if $n = 2, \ldots, N_2$. The external domain bounded by Γ_1^2 will be called $\mathcal{D}_1$.

The tangent vector to Γ at the point $x(s)$ we denote by $\boldsymbol{\tau}_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\boldsymbol{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\boldsymbol{n}_x$ is chosen such that it will coincide with the direction of $\boldsymbol{\tau}_x$ if $\boldsymbol{n}_x$ is rotated anticlockwise through an angle of $\pi/2$.

We consider the curves Γ^1 as a set of cuts. The side of Γ^1 that is on the left when the parameter s increases will be denoted by $(\Gamma^1)^+$, and the opposite side will be denoted by $(\Gamma^1)^-$.

Let us formulate the Dirichlet problem for the Laplace equation in the domain $\mathcal{D}\backslash\Gamma^1$.

Problem U. To find a function $u(x) \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$, which satisfies the Laplace equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0, \quad x \in \mathcal{D}\backslash\Gamma^1, \tag{2.1 a}$$

and the boundary conditions

$$u(x(s))|_{(\Gamma^1)^+} = F^+(s), \qquad u(x(s))|_{(\Gamma^1)^-} = F^-(s), \qquad u(x(s))|_{\Gamma^2} = F(s). \tag{2.1 b}$$

All conditions of Problem U must be satisfied in the classical sense.

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Remark 2.1. By $C^0(\overline{\mathcal{D}\backslash\Gamma^1})$ we denote functions which are continuously extended on cuts Γ^1 from the left and right, but their values on Γ^1 from the left and right can be different, so that the functions may have a jump on Γ^1.

If $N_1 = 0$ and cuts Γ^1 are absent, then Problem U transforms to the classical Dirichlet problem in a domain $\mathcal{D}$ without cuts.

On the basis of the maximum principle for harmonic functions [**3, 10**], we can readily prove the following assertion.

Theorem 2.2. *The Problem U has at most one solution.*

3. Integral equations at the boundary

Below, we assume that

$$F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1), \quad F(s) \in C^0(\Gamma^2), \quad \lambda \in (0,1], \tag{3.1 a}$$

$$F^+(a_n^1) = F^-(a_n^1), \quad F^+(b_n^1) = F^-(b_n^1), \quad n = 1, \dots, N_1. \tag{3.1 b}$$

If $\mathcal{B}_1(\Gamma^1), \mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then, for functions given on Γ, we introduce the Banach space $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ with the norm

$$\|\cdot\|_{\mathcal{B}_1(\Gamma^1)\cap\mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}.$$

An example of such a Banach space is $C^0(\Gamma) = C^0(\Gamma^1) \cap C^0(\Gamma^2)$.

We say that the function $u(x)$ belongs to the smoothness class $\boldsymbol{K}$ if

(1) $u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$;

(2) $\nabla u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}\backslash\Gamma^2\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1,

$$X = \bigcup_{n=1}^{N_1} (x(a_n^1) \cup x(b_n^1));$$

(3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\epsilon > -1$, the inequality

$$|\nabla u| \leqslant \mathcal{C}|x - x(d)|^{\epsilon}, \tag{3.2}$$

holds, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \dots, N_1$.

We shall construct the solution of Problem U from the smoothness class $\boldsymbol{K}$ with the help of potential theory for the Laplace equation (2.1 a).

By $\int_{\Gamma^k} \dots \,\mathrm{d}\sigma$ we mean

$$\sum_{n=1}^{N_k} \int_{a_n^k}^{b_n^k} \cdots \,\mathrm{d}\sigma.$$

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We consider an angular potential [**1**] for the equation (2.1 a):

$$v_1[\nu](x) = -\frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma) V(x, y(\sigma))\,\mathrm{d}\sigma. \tag{3.3}$$

The kernel $V(x, y(\sigma))$ is defined (up to indeterminacy $2\pi m$, $m = \pm 1, \pm 2, \dots$) by the formulae

$$\cos V(x, y(\sigma)) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \qquad \sin V(x, y(\sigma)) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where

$$y = y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma^1, \qquad |x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see that $V(x, y(\sigma))$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1-axis. More precisely, $V(x, y(\sigma))$ is a many-valued harmonic function of x connected with $\ln|x - y(\sigma)|$ by the Cauchy–Riemann relations.

Below, by $V(x, y(\sigma))$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ_n^1 ($n = 1, \dots, N_1$) for given fixed $x \notin \Gamma^1$.

Under this definition of $V(x, y(\sigma))$, the potential $v_1[\nu](x)$ is a many-valued function. In order that the potential $v_1[\nu](x)$ be single-valued, it is necessary to impose the following additional conditions

$$\int_{a_n^1}^{b_n^1} \nu(\sigma)\,\mathrm{d}\sigma = 0, \quad n = 1, \dots, N_1. \tag{3.4}$$

Below, we suppose that the density $\nu(\sigma)$ belongs to $C^{0,\lambda}(\Gamma^1)$ and satisfies conditions (3.4). As shown in [**1**, **4**], for such $\nu(\sigma)$, the angular potential $v_1[\nu](x)$ belongs to the class $\boldsymbol{K}$. In particular, the condition (3.2) is satisfied for any $\epsilon \in (0, 1)$. Moreover, integrating $v_1[\nu](x)$ by parts and using (3.4), we express the angular potential in terms of a double layer potential

$$v_1[\nu](x) = \frac{1}{2\pi}\int_{\Gamma^1} \rho(\sigma)\frac{\partial}{\partial n_y}\ln|x - y(\sigma)|\,\mathrm{d}\sigma, \tag{3.5}$$

with the density

$$\rho(\sigma) = \int_{a_n^1}^{\sigma} \nu(\xi)\,\mathrm{d}\xi, \quad \sigma \in [a_n^1, b_n^1], \quad n = 1, \dots, N_1. \tag{3.6}$$

Consequently, $v_1[\nu](x)$ satisfies equation (2.1 a) outside Γ^1.

Let us construct a solution of Problem U. We seek a solution of the problem in the following form

$$u[\nu, \mu](x) = v_1[\nu](x) + w[\mu](x), \tag{3.7}$$

where $v_1[\nu](x)$ is given by (3.3), (3.5) and

$$\left.\begin{aligned} w[\mu](x) &= w_1[\mu](x) + w_2[\mu](x),\\ w_1[\mu](x) &= -\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\ln|x-y(\sigma)|\,\mathrm{d}\sigma,\\ w_2[\mu](x) &= -\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial \boldsymbol{n}_y}\ln|x-y(\sigma)|\,\mathrm{d}\sigma + h[\mu](x). \end{aligned}\right\} \tag{3.8}$$

By $h[\mu](x)$ we denote the sum of point sources placed at the fixed points Y_k lying inside Γ_k^2 $(k = 2,\dots,N_2)$:

$$h[\mu](x) = -\frac{1}{2\pi}\sum_{k=2}^{N_2}\int_{\Gamma_k^2}\mu(\sigma)\,\mathrm{d}\sigma\ln|x-Y_k|, \quad Y_k \in \mathcal{D}_k, \quad k = 2,\dots,N_2.$$

Clearly, $h[\mu](x)$ obeys equation (2.1 a) and belongs to

$$C^\infty\biggl(R^2\setminus\bigcup_{k=2}^{N_2} Y_k\biggr);$$

besides, if $x(s) \in \Gamma$, then $h[\mu](x(s)) \in C^{1,\lambda}(\Gamma)$ in s.

As noted above, we will look for the density $\nu(\sigma)$ satisfying conditions (3.4) and belonging to $C^{0,\lambda}(\Gamma^1)$.

We will seek $\mu(s)$ from the Banach space $C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, with the norm

$$\|\cdot\|_{C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)} = \|\cdot\|_{C_q^\omega(\Gamma^1)} + \|\cdot\|_{C^0(\Gamma^2)}.$$

We say that $\mu(s) \in C_q^\omega(\Gamma^1)$ if

$$\mu(s)\prod_{n=1}^{N_1}|s-a_n^1|^q|s-b_n^1|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with the index ω and

$$\|\mu(s)\|_{C_q^\omega(\Gamma^1)} = \biggl\|\mu(s)\prod_{n=1}^{N_1}|s-a_n^1|^q|s-b_n^1|^q\biggr\|_{C^{0,\omega}(\Gamma^1)}.$$

It can be checked directly, with the help of [**4, 9**], that for such $\mu(s)$, the function $w_1[\mu](x)$ obeys equation (2.1 a) and belongs to the class $\boldsymbol{K}$. In particular, inequality (3.2) holds with $\epsilon = -q$ if $q \in (0,1)$. The potential $w_2[\mu](x)$ satisfies equation (2.1 a) and belongs to $C^0(\overline{\mathcal{D}})\cap C^2(\mathcal{D})$. Consequently, $w_2[\mu](x)$ belongs to the class $\boldsymbol{K}$, and so $u[\nu,\mu](x) \in \boldsymbol{K}$.

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To satisfy the boundary conditions, we put (3.7) in (2.1 b) and arrive at the system of the integral equations for the densities $\mu(s), \nu(s)$:

$$\pm \tfrac{1}{2}\rho(s) - \frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma)V(x(s),y(\sigma))\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma$$
$$-\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial n_y}\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + h[\mu](x(s)) = F^{\pm}(s), \quad s\in\Gamma^1, \tag{3.9 a}$$

$$-\frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma)V(x(s),y(\sigma))\,\mathrm{d}\sigma - \frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + \tfrac{1}{2}\mu(s)$$
$$-\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial n_y}\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + h[\mu](x(s)) = F(s), \qquad s\in\Gamma^2, \tag{3.9 b}$$

where $\rho(s)$ is defined in terms of $\nu(s)$ in (3.6). The kernels of the second integral term in (3.9 a) and the third integral term in (3.9 b) have a weak singularity as $s=\sigma$.

To derive limit formulae for the angular potential, we used its expression in the form of a double layer potential (3.5).

Equation (3.9 a) is obtained as $x\to x(s)\in(\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$; the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have the conditions (3.4).

Subtracting the integral equation (3.9 a) and using (3.6), we find

$$\left.\begin{aligned}\rho(s) &= (F^+(s)-F^-(s))\in C^{1,\lambda}(\Gamma^1),\\ \nu(s) &= (F'^+(s)-F'^-(s))\in C^{0,\lambda}(\Gamma^1), \quad F'^{\pm}(s)=\frac{\mathrm{d}}{\mathrm{d}s}F^{\pm}(s).\end{aligned}\right\} \tag{3.10}$$

We note that $\nu(s)$ is found completely and satisfies all required conditions, and in particular (3.4). Hence, the angular potential (3.3), (3.5) is found completely as well.

We introduce the function $f(s)$ on Γ by the formula

$$f(s) = F(s) + \frac{1}{2\pi}\int_{\Gamma^1}(F'^+(\sigma)-F'^-(\sigma))V(x(s),y(\sigma))\,\mathrm{d}\sigma, \quad s\in\Gamma, \tag{3.11}$$

where $F(s)$ is a function defined on Γ, so that $F(s)$ on Γ^2 is specified in (2.1 b), while $F(s)$ on Γ^1 is specified by the relationship

$$F(s) = \tfrac{1}{2}(F^+(s)+F^-(s)), \quad s\in\Gamma^1.$$

As shown in [4], if $s\in\Gamma^1$, then $f(s)\in C^{1,\lambda}(\Gamma^1)$. Consequently,

$$f(s)\in C^{1,\lambda}(\Gamma^1)\cap C^0(\Gamma^2).$$

Adding the integral equations (3.9 a) and taking into account (3.9 b), we obtain the integral equation for $\mu(s)$ on Γ,

$$\begin{aligned} w[\mu](x(s))|_{\Gamma} = &-\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + \tfrac{1}{2}\delta(\Gamma^2,s)\mu(s) \\ &-\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial \boldsymbol{n}_y}\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + h[\mu](x(s)) \\ = &\, f(s), \quad s\in\Gamma, \end{aligned} \tag{3.12}$$

where $f(s)$ is given in (3.11), and the limit values of the function (3.8) as $x \to x(s) \in \Gamma$ $(x \in \mathcal{D})$ are denoted by $w[\mu](x(s))|_{\Gamma}$. Furthermore,

$$\delta(\Gamma^2,s) = \begin{cases} 0, & \text{if } s \notin \Gamma^2, \\ 1, & \text{if } s \in \Gamma^2. \end{cases}$$

Thus, if $\mu(s)$ is a solution of equation (3.12) from the space $C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q\in[0,1)$, then the potential (3.7) with $\nu(s)$ from (3.10) satisfies all conditions of Problem U and belongs to the class $\boldsymbol{K}$.

The following theorem holds.

Theorem 3.1. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and the conditions in (3.1) hold. If equation (3.12) has a solution $\mu(s)$ from the Banach space $C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)$ for some $\omega\in(0,1]$ and $q\in[0,1)$, then a solution of Problem U exists, belongs to the class $\boldsymbol{K}$ and is given by (3.7), where $\nu(s)$ is defined in (3.10).*

If $s \in \Gamma^2$, then (3.12) is an equation of the second kind. If $s\in\Gamma^1$, then (3.12) is an equation of the first kind, and its kernel has the logarithmic singularity.

Our further treatment will be aimed at the proof of the solvability of (3.12) in the Banach space $C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)$. Moreover, we reduce (3.12) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

By differentiating (3.12) on Γ^1, we reduce it to the following Cauchy singular integral equation on Γ^1,

$$\begin{aligned} \frac{\partial}{\partial s}w[\mu](x(s)) = &\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}\,\mathrm{d}\sigma \\ &-\frac{1}{2\pi}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial s}\frac{\partial}{\partial \boldsymbol{n}_y}\ln|x(s)-y(\sigma)|\,\mathrm{d}\sigma + \frac{\partial}{\partial s}h[\mu](x(s)) \\ = &\, f'(s), \quad s\in\Gamma^1, \end{aligned} \tag{3.13}$$

where $\varphi_0(x,y)$ is the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\boldsymbol{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\boldsymbol{n}_x$ and negative if it is measured clockwise from $\boldsymbol{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$

if $x \neq y$. Note that for $x(s), y \in \Gamma$ and $x \neq y$, we have

$$\frac{\partial}{\partial s}\ln|x(s)-y| = \frac{\partial}{\partial \tau_x}\ln|x-y| = -\frac{\partial}{\partial \boldsymbol{n}_x}V(x,y) = -\frac{\sin\varphi_0(x,y)}{|x-y|}$$
$$= \frac{\cos(V(x(s),y)-\alpha(s))}{|x(s)-y|},$$

$$\frac{\partial}{\partial s}V(x(s),y) = \frac{\partial}{\partial \tau_x}V(x,y) = \frac{\partial}{\partial \boldsymbol{n}_x}\ln|x-y| = -\frac{\cos\varphi_0(x,y)}{|x-y|}$$
$$= -\frac{\sin(V(x(s),y)-\alpha(s))}{|x(s)-y|},$$

where $\alpha(s)$ is the inclination of the tangent τ_x to the $\mathrm{O}x_1$-axis and $V(x,y)$ is the kernel of the angular potential from (3.3).

Equation (3.12) on Γ^2 we rewrite in the form

$$\mu(s) + \int_\Gamma \mu(\sigma)A_2(s,\sigma)\,\mathrm{d}\sigma = 2f(s), \quad s \in \Gamma^2, \tag{3.14}$$

where

$$A_2(s,\sigma) = \Big\{-\frac{1}{\pi}(1-\delta(\Gamma^2,\sigma))\ln|x(s)-y(\sigma)| - \frac{1}{\pi}\delta(\Gamma^2,\sigma)\frac{\partial}{\partial \boldsymbol{n}_y}\ln|x(s)-y(\sigma)|$$
$$-\frac{1}{\pi}\sum_{k=2}^{N_2}\delta(\Gamma_k^2,\sigma)\ln|x(s)-Y_k|\Big\}.$$

The function $\delta(\Gamma^2,\sigma)$ was introduced in (3.12), and

$$\delta(\Gamma_k^2,\sigma) = \begin{cases} 0, & \text{if } \sigma \notin \Gamma_k^2, \\ 1, & \text{if } \sigma \in \Gamma_k^2, \end{cases} \qquad (k = 2,\dots,N_2).$$

The kernel $A_2(s,\sigma)$ has a weak singularity if $s = \sigma \in \Gamma^2$. Consequently, the integral operator from (3.14) is a compact operator mapping $C^0(\Gamma)$ into $C^0(\Gamma^2)$.

Remark 3.2. Evidently,

$$f(a_n^2) = f(b_n^2) \text{ and } A_2(a_n^2,\sigma) = A_2(b_n^2,\sigma), \quad \text{for } \sigma \in \Gamma,\ \sigma \neq a_n^2, b_n^2\ (n = 1,\dots,N_2).$$

Hence, if $\mu(s)$ is a solution of equation (3.14) from $C^0(\bigcup_{n=1}^{N_2}[a_n^2,b_n^2])$, then, according to the equality (3.14), $\mu(s)$ automatically satisfies matching conditions $\mu(a_n^2) = \mu(b_n^2)$ for $n = 1,\dots,N_2$, and, therefore, belongs to $C^0(\Gamma^2)$. This observation is true for equation (3.12) also, and can be helpful in finding numerical solutions, since we may refuse from matching conditions $\mu(a_n^2) = \mu(b_n^2)$, $(n = 1,\dots,N_2)$, which are fulfilled automatically.

We note that equation (3.13) is equivalent to (3.12) on Γ^1 if and only if (3.13) is accompanied by the following additional conditions

$$w[\mu](x(a_n^1)) = f(a_n^1), \quad n = 1,\dots,N_1. \tag{3.15}$$

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The system (3.13)–(3.15) is equivalent to equation (3.12).

It can be easily proved that

$$\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}-\frac{1}{\sigma-s}\in C^{0,\lambda}(\Gamma^1\times\Gamma^1)$$

(see [**4, 9**] for details). Therefore, we can rewrite (3.13) in the form

$$\begin{aligned}2\frac{\partial}{\partial s}w[\mu](x(s))&=\frac{1}{\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\mathrm{d}\sigma}{\sigma-s}+\int_{\Gamma}\mu(\sigma)M(s,\sigma)\,\mathrm{d}\sigma\\&=2f'(s),\quad s\in\Gamma^1,\end{aligned}\tag{3.16}$$

where $f'(s)=(\mathrm{d}/\mathrm{d}s)f(s)\in C^{0,\lambda}(\Gamma^1)$ and

$$\begin{aligned}M(s,\sigma)=\frac{1}{\pi}\biggl\{&(1-\delta(\Gamma^2,\sigma))\biggl[\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}-\frac{1}{\sigma-s}\biggr]\\&-\delta(\Gamma^2,\sigma)\frac{\partial}{\partial s}\biggl[\frac{\partial}{\partial n_y}\ln|x(s)-y(\sigma)|+\sum_{k=2}^{N_2}\delta(\Gamma_k^2,\sigma)\ln|x(s)-Y_k|\biggr]\biggr\}\\&\in C^{0,\lambda}(\Gamma^1\times\Gamma).\end{aligned}$$

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (3.16), we arrive at the following integral equation of the second kind [**9**],

$$\mu(s)+\frac{1}{Q_1(s)}\int_{\Gamma}\mu(\sigma)A_1(s,\sigma)\,\mathrm{d}\sigma+\frac{1}{Q_1(s)}\sum_{n=0}^{N_1-1}G_ns^n=\frac{1}{Q_1(s)}\Phi_1(s),\quad s\in\Gamma^1,\tag{4.1}$$

where

$$\begin{aligned}A_1(s,\sigma)&=-\frac{1}{\pi}\int_{\Gamma^1}\frac{M(\xi,\sigma)}{\xi-s}Q_1(\xi)\,\mathrm{d}\xi,\\Q_1(s)&=\prod_{n=1}^{N_1}|\sqrt{s-a_n^1}\sqrt{b_n^1-s}|\operatorname{sgn}(s-a_n^1),\\\Phi_1(s)&=-\frac{1}{\pi}\int_{\Gamma^1}\frac{2Q_1(\sigma)f'(\sigma)}{\sigma-s}\,\mathrm{d}\sigma,\end{aligned}$$

and $G_0,\dots,G_{N_1-1}$ are arbitrary constants.

It can be shown, using the properties of singular integrals [**9**], that $\Phi_1(s)$, $A_1(s,\sigma)$ are Hölder functions if $s\in\Gamma^1$, $\sigma\in\Gamma$. Consequently, any solution of (4.1) belongs to $C^{\omega}_{1/2}(\Gamma^1)$, and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s)=(1-\delta(\Gamma^2,s))Q_1(s)+\delta(\Gamma^2,s),\quad s\in\Gamma.$$

Instead of $\mu(s) \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, we introduce the new unknown function $\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, and rewrite equations (4.1) and (3.14) in the form of one equation,

$$\mu_*(s) + \int_\Gamma \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma + (1-\delta(\Gamma^2,s))\sum_{n=0}^{N_1-1} G_n s^n = \Phi(s), \quad s \in \Gamma, \tag{4.2}$$

where

$$A(s,\sigma) = (1-\delta(\Gamma^2,s))A_1(s,\sigma) + \delta(\Gamma^2,s)A_2(s,\sigma),$$
$$\Phi(s) = (1-\delta(\Gamma^2,s))\Phi_1(s) + 2\delta(\Gamma^2,s)f(s).$$

To derive equations for $G_0,\ldots,G_{N_1-1}$, we substitute $\mu(s)$ from (4.1), (3.14) in the conditions (3.15); then, in terms of $\mu_*(s)$, we obtain

$$\int_\Gamma Q^{-1}(\xi)\mu_*(\xi)l_n(\xi)\,d\xi + \sum_{m=0}^{N_1-1} B_{nm}G_m = H_n, \quad n = 1,\ldots,N_1, \tag{4.3}$$

where

$$\left.\begin{aligned} l_n(\xi) = -w[Q^{-1}(\cdot)A(\cdot,\xi)](a_n^1), \qquad H_n = -w[Q^{-1}(\cdot)\Phi(\cdot)](a_n^1) + f(a_n^1),\\ B_{nm} = -w[Q^{-1}(\cdot)(1-\delta(\Gamma^2,\cdot))(\cdot)^m](a_n^1).\end{aligned}\right\} \tag{4.4}$$

By $\cdot$ we denote the variable of integration in the potential (3.8).

Thus, the system of equations (3.13)–(3.15) for $\mu(s)$ has been reduced to the system (4.2), (4.3) for the function $\mu_*(s)$ and constants $G_0,\ldots,G_{N_1-1}$. It is clear from our considerations that any solution of system (4.2), (4.3) gives a solution of system (3.13)–(3.15).

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [9]), $\Phi_1(s) \in C^{0,p}(\Gamma^1)$, $p = \min\{\frac{1}{2},\lambda\}$, and $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$. We arrive at the following assertion.

Lemma 4.1. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$ and $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, where $p = \min\{\lambda, \frac{1}{2}\}$. If $\mu_*(s)$ from $C^0(\Gamma)$ satisfies equation (4.2), then $\mu_*(s)$ belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.*

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if the conditions in (3.1) hold.

Hence, below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

It was noted above that the integral operator from (3.14) with the kernel $A_2(s,\sigma)$ is compact from $C^0(\Gamma)$ into $C^0(\Gamma^2)$.

Since $A_1(s,\sigma) \in C^0(\Gamma^1 \times \Gamma)$, the integral operator from (4.2),

$$\boldsymbol{A}\mu_* = \int_\Gamma \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)\,d\sigma,$$

is a compact operator mapping $C^0(\Gamma)$ into itself.

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We rewrite (4.2) in the operator form

$$(I + \boldsymbol{A})\mu_* + PG = \Phi, \tag{4.5}$$

where P is the operator multiplying the row

$$P = (1 - \delta(\Gamma^2, s))(s^0, \dots, s^{N_1-1})$$

by the column $G = (G_0, \dots, G_{N_1-1})^{\mathrm{T}}$. The operator P is finite-dimensional from E_{N_1} into $C^0(\Gamma)$ and, therefore, compact.

Now we rewrite equation (4.3) in the form

$$I_{N_1}G + L\mu_* + (B - I_{N_1})G = H, \tag{4.6}$$

where $H = (H_1, \dots, H_{N_1})^{\mathrm{T}}$ is a column of N_1 elements, I_{N_1} is an identity operator in E_{N_1} and B is an $N_1 \times N_1$ matrix consisting of the elements B_{nm} from (4.4). The operator L acts from $C^0(\Gamma)$ into E_{N_1}, so that $L\mu_* = (L_1\mu_*, \dots, L_{N_1}\mu_*)^{\mathrm{T}}$, where

$$L_n\mu_* = \int_\Gamma Q^{-1}(\xi)\mu_*(\xi)l_n(\xi)\,\mathrm{d}\xi.$$

The operators $(B - I_{N_1})$, L are finite-dimensional and, therefore, compact.

We consider the columns

$$\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix}, \qquad \tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix}$$

in the Banach space $C^0(\Gamma) \times E_{N_1}$ with the norm $\|\tilde{\mu}\|_{C^0(\Gamma)\times E_{N_1}} = \|\mu_*\|_{C^0(\Gamma)} + \|G\|_{E_{N_1}}$.

We write system (4.5), (4.6) in the form of one equation,

$$(\boldsymbol{I} + \boldsymbol{R})\tilde{\mu} = \tilde{\Phi}, \qquad \boldsymbol{R} = \begin{pmatrix} \boldsymbol{A} & P \\ L & B - I_{N_1} \end{pmatrix}, \tag{4.7}$$

where $\boldsymbol{I}$ is an identity operator in the space $C^0(\Gamma) \times E_{N_1}$. It is clear that $\boldsymbol{R}$ is a compact operator mapping $C^0(\Gamma) \times E_{N_1}$ into itself. Therefore, (4.7) is a Fredholm equation in this space.

Let us show that the homogeneous equation (4.7) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (4.7) has a unique solution for any right-hand side. We will prove this by a contradiction. Let

$$\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$$

be a non-trivial solution of the homogeneous equation (4.7). According to the lemma,

$$\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}, \quad p = \min\{\lambda, \tfrac{1}{2}\}.$$

Therefore, the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ and the column G^0 converts the homogeneous equations (4.1), (3.14) and (4.3) into identities. For instance, (3.14) takes the form

$$\lim_{x \to x(s) \in \Gamma^2} w[\mu^0](x) = 0, \quad x \in \mathcal{D}. \tag{4.8 a}$$

Using the homogeneous identities (4.1), (3.14), we check that the homogeneous identities (4.3) are equivalent to

$$w[\mu^0](a_n^1) = 0, \quad n = 1, \ldots, N_1. \tag{4.8 b}$$

Besides, acting on the homogeneous identity (4.1) with a singular operator with the kernel $(s-t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (3.16):

$$\frac{\partial}{\partial s} w[\mu^0](x(s))\bigg|_{\Gamma^1} = 0. \tag{4.8 c}$$

It follows from (4.8) that $\mu^0(s)$ satisfies the homogeneous equation (3.12). On the basis of Theorem 3.1, $u[0, \mu^0](x) \equiv w[\mu^0](x)$ is a solution of the homogeneous Problem U. According to Theorem 2.2, $w[\mu^0](x) \equiv 0$, $x \in \mathcal{D} \backslash \Gamma^1$. Using the limit formulae for normal derivatives of a single-layer potential on Γ^1, we have

$$\lim_{x \to x(s) \in (\Gamma^1)^+} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) - \lim_{x \to x(s) \in (\Gamma^1)^-} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) = \mu^0(s) \equiv 0, \quad s \in \Gamma^1.$$

Hence $w[\mu^0](x) = w_2[\mu^0](x) \equiv 0$, $x \in \mathcal{D}$, and $\mu^0(s)$ satisfies (4.8 a), which can be written as

$$\tfrac{1}{2}\mu^0(s) - \frac{1}{2\pi} \int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x(s) - y(\sigma)|\, \mathrm{d}\sigma + h[\mu^0](x(s)) = 0, \quad s \in \Gamma^2. \tag{4.9}$$

The Fredholm equation (4.9) arises when solving the Dirichlet problem for the Laplace equation (2.1 a) in the domain $\mathcal{D}$ by the double-layer potential with the sum of point sources placed inside the curves $\Gamma_2^2, \ldots, \Gamma_{N_2}^2$. Equation (4.9) has only the trivial solution $\mu^0(s) \equiv 0$ in $C^0(\Gamma^2)$. This will be shown in Appendix A.

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu_*^0(s) = \mu^0(s)Q^{-1}(s) \equiv 0$, and it follows from the homogeneous identity (4.1) for $\mu^0(s)$ and $G_0^0, \ldots, G_{N_1-1}^0$ that

$$G^0 = (G_0^0, \ldots, G_{N_1-1}^0)^{\mathrm{T}} \equiv 0.$$

Hence $\tilde{\mu}^0 \equiv 0$, and we arrive at the contradiction to the assumption that $\tilde{\mu}^0$ is a non-trivial solution of the homogeneous equation (4.7). Thus, the homogeneous Fredholm equation (4.7) has only a trivial solution in $C^0(\Gamma) \times E_{N_1}$.

We have proved the following assertion.

Theorem 4.2. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, then (4.7) is a Fredholm equation of the second kind in the space $C^0(\Gamma) \times E_{N_1}$. Moreover, equation (4.7) has a unique*

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solution

$$\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix} \in C^0(\Gamma) \times E_{N_1},$$

for any

$$\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}.$$

As a consequence of Theorem 4.2 and Lemma 4.1, we obtain the corollary.

Corollary 4.3. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, then equation (4.7) has a unique solution*

$$\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}$$

for any

$$\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1},$$

where $p = \min\{\lambda, \frac{1}{2}\}$.

We recall that $\tilde{\Phi}$ belongs to the class of smoothness required in the corollary if the conditions in (3.1) hold. Besides, equation (4.7) is equivalent to the system (4.2), (4.3). As mentioned above, if $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ and $G_0, \ldots, G_{N_1-1}$ is a solution of system (4.2), (4.3), then $\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ is a solution of system (3.13)–(3.15), and, therefore, $\mu(s)$ satisfies equation (3.12). We obtain the following statement.

Theorem 4.4. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and the conditions in (3.1) hold, then equation (3.12) has a solution $\mu(s)$ from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\frac{1}{2}, \lambda\}$. This solution is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_*(s)$ from $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ is found by solving the Fredholm equation (4.7), which is uniquely solvable.*

Remark 4.5. The solution of equation (3.12), ensured by Theorem 4.4, is unique in the space $C^{p_o}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ for any $p_o \in (0,p]$. The proof can be given by a contradiction to the assumption that the homogeneous equation (3.12) has a non-trivial solution in this space. The proof almost coincides with the proof of Theorem 4.2. Consequently, the numerical solution of equation (3.12) can be obtained by the direct numerical inversion of the integral operator from (3.12). In doing so, Hölder functions can be approximated by continuous piecewise linear functions, which also obey the Hölder inequality. The simplification for numerically solving equation (3.12) is suggested in Remark 3.2.

On the basis of Theorems 3.1 and 4.4 we arrive at the final result.

Theorem 4.6. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and the conditions in (3.1) hold, then the solution of Problem U exists, belongs to the class $\boldsymbol{K}$ and is given by (3.7), where $\nu(s)$ is defined in (3.10) and $\mu(s)$ is a solution of equation (3.12) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\frac{1}{2}, \lambda\}$, ensured by Theorem 4.4.*

It can be checked directly that the solution of Problem U constructed in Theorem 4.6 satisfies condition (3.2) with $\epsilon = -\frac{1}{2}$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can easily be obtained with the help of formulae presented in [**4**].

Theorem 4.6 ensures existence of a classical solution of Problem U when $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ and the conditions in (3.1) hold. The uniqueness of the classical solution follows from Theorem 2.2. It appears that in our assumptions, the classical solution of Problem U belongs to the smoothness class $\boldsymbol{K}$. On the basis of our considerations, we suggest the following scheme for solving Problem U. First, we find the unique solution of the Fredholm equation (4.7) from $C^0(\Gamma) \times E_{N_1}$. This solution automatically belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}$, $p = \min\{\lambda, \frac{1}{2}\}$. Second, we construct the solution of equation (3.12) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. Finally, putting $\nu(s)$ from (3.10) and $\mu(s)$ in (3.7), we obtain the solution of Problem U from the class $\boldsymbol{K}$.

Appendix A.

Here, we prove the following assertion.

Proposition A 1. *If $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, then there is only the trivial solution of the homogeneous Fredholm equation (4.9) in $C^0(\Gamma^2)$.*

We shall give a proof by a contradiction. Let $\mu^0(s) \in C^0(\Gamma^2)$ be a non-trivial solution of the homogeneous equation (4.9). The kernel of the integral term in (4.9) has a weak singularity. It can be shown with the help of [**9**, § 51] that the integral term in (4.9) belongs to $C^{0,\lambda/2}(\Gamma^2)$ in s, therefore $\mu^0(s) \in C^{0,\lambda/2}(\Gamma^2)$. Now we consider the function $w_2[\mu^0](x)$ introduced in (3.8). This function belongs to $C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ and satisfies the following homogeneous Dirichlet problem for the Laplace equation:

$$\Delta w_2 = 0 \text{ in } \mathcal{D}, \quad w_2|_{\Gamma^2} = 0. \tag{A 1}$$

Indeed, putting $w_2[\mu^0](x)$ in the boundary condition, we get the identity (4.9). According to the uniqueness theorem for the Dirichlet problem (A 1), we obtain

$$w_2[\mu^0](x) \equiv 0, \quad x \in \overline{\mathcal{D}}. \tag{A 2}$$

We consider the function

$$w_2^*[\mu^0](x) = -\frac{1}{2\pi}\left[-\int_{\Gamma^2} \mu^0(\sigma)\frac{\partial}{\partial\sigma}\ln|x-y(\sigma)|\,\mathrm{d}\sigma + \sum_{k=2}^{N_2}\int_{\Gamma_k^2}\mu^0(\sigma)\,\mathrm{d}\sigma V(x, Y_k)\right], \tag{A 3}$$

where $V(x,y)$ is the kernel of the angular potential from (3.3). The function $w_2^*[\mu^0](x)$ is connected with $w_2[\mu^0](x)$ by the Cauchy–Riemann relations $\partial_{x_1}w_2 = \partial_{x_2}w_2^*$, $\partial_{x_2}w_2 =$

$-\partial_{x_1} w_2^*$, and, consequently, $w_2^*[\mu^0](x) \equiv C = \text{const.}$ in $\mathcal{D}$. It is clear from (A 3) that $w_2^*[\mu^0](x)$ is a many-valued function, because the $V(x, Y_k)$ are many-valued functions ($k = 2, \dots, N_2$). Indeed, when passing round the point Y_k, the value of the function $V(x, Y_k)$ changes by 2π. Evidently, $w_2^*[\mu^0](x)$ can be constant in $\mathcal{D}$ only if $w_2^*[\mu^0](x)$ is single-valued. In order for $w_2^*[\mu^0](x)$ to be single-valued, the following $(N_2 - 1)$ conditions must hold

$$\int_{\Gamma_k^2} \mu^0(\sigma)\, \mathrm{d}\sigma = 0, \quad k = 2, \dots, N_2. \tag{A 4}$$

Under these conditions, $w_2^*[\mu^0](x)$ takes the form of the modified single-layer potential [**9**]

$$w_2^*[\mu^0](x) = \frac{1}{2\pi} \int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \sigma} \ln |x - y(\sigma)|\, \mathrm{d}\sigma, \tag{A 5}$$

and $w_2[\mu^0](x)$ transforms to the ordinary double-layer potential

$$w_2[\mu^0](x) = -\frac{1}{2\pi} \int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \boldsymbol{n}_y} \ln |x - y(\sigma)|\, \mathrm{d}\sigma \in C^0(\overline{R^2 \backslash \Gamma^2}) \cap C^2(R^2 \backslash \Gamma^2). \tag{A 6}$$

Potentials (A 5) and (A 6) are connected by the Cauchy–Riemann relations in $R^2 \backslash \Gamma^2$. Because of $\mu^0(s) \in C^{0,\lambda/2}(\Gamma^2)$, the potential (A 5) is a harmonic function that belongs to $C^0(R^2) \cap C^2(R^2 \backslash \Gamma^2)$ (see [**9**] for details). Note that (A 5) is continuous when passing through Γ^2 and is represented on Γ^2 by a singular integral (for this, we stress that $\mu^0(s)$ is a Hölder function).

As stated above, $w_2^*[\mu^0](x) \equiv C$ in $\overline{\mathcal{D}}$.

We consider the internal domain $\mathcal{D}_k$ bounded by Γ_k^2 ($k = 2, \dots, N_2$). In this domain, the potential (A 5) satisfies the following Dirichlet problem

$$\Delta w_2^* = 0 \text{ in } \mathcal{D}_k, \quad w_2^*|_{\Gamma_k^2} = C,$$

which has unique solution

$$w_2^*[\mu^0](x) \equiv C, \quad x \in \overline{\mathcal{D}_k} \quad (k = 2, \dots N_2).$$

It follows from the Cauchy–Riemann relations and the smoothness of the double-layer potential that

$$w_2[\mu^0](x) \equiv c_k, \quad x \in \overline{\mathcal{D}_k}, \quad k = 2, \dots N_2,$$

where $c_2, \dots, c_{N_2}$ are constants. Using (A 2) and the jump relation for the double-layer potential $w_2[\mu^0](x)$ on Γ^2, we get

$$\mu^0(s)|_{\Gamma_k^2} \equiv -c_k, \quad k = 2, \dots N_2.$$

According to (A 4), $c_k = 0$, $k = 2, \dots, N_2$, and, therefore,

$$\mu^0(s)|_{\Gamma_k^2} \equiv 0, \quad k = 2, \dots N_2. \tag{A 7}$$

We consider the external domain $\mathcal{D}_1$ bounded by Γ_1^2. In this domain, the potential (A 5) satisfies the following external Dirichlet problem

$$\Delta w_2^* = 0 \text{ in } \mathcal{D}_1, \quad w_2^*|_{\Gamma_1^2} = C, \quad |w_2^*| < \text{const. in } \mathcal{D}_1,$$

which has a unique solution $w_2^*[\mu^0](x) \equiv C$. It follows from the Cauchy-Riemann relations and the smoothness of the double-layer potential, that

$$w_2[\mu^0](x) \equiv c_1, \quad x \in \overline{\mathcal{D}_1},$$

where c_1 is a constant. Because (A 6) tends to zero at infinity, we have $c_1 = 0$ and $w_2[\mu^0](x) \equiv 0$ in $\overline{\mathcal{D}_1}$. Using (A 2) and the theorem on the jump for the double-layer potential [**10**], we obtain $\mu^0(s)|_{\Gamma_1^2} \equiv 0$. Taking into account (A 7), we have

$$\mu^0(s) \equiv 0 \text{ on } \Gamma^2,$$

and we arrive at the contradiction to the assumption, that $\mu^0(s)$ is a non-trivial solution of the homogeneous equation (4.9). Hence, the homogeneous equation (4.9) has only a trivial solution.

Because (4.9) is a Fredholm equation of the second kind, the following corollary holds.

Corollary A 2. *If $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, then the inhomogeneous Fredholm equation (4.9) is uniquely solvable in $C^0(\Gamma^2)$ for any right-hand side from $C^0(\Gamma^2)$.*

The inhomogeneous equation (4.9) is a particular case of (3.12) if the domain $\mathcal{D}$ does not contain cuts.

References

1. S. A. Gabov, An angular potential and its applications. *Math. USSR Sbornik* **32** (1977), 423–436.
2. F. D. Gakhov, *Boundary value problems* (Pergamon, Oxford; Addison-Wesley, Reading, MA, 1966).
3. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer, Berlin, 1983).
4. P. A. Krutitskii, Dirichlet problem for the Helmholtz equation outside cuts in a plane. *Comp. Math. Math. Phys.* **34** (1994), 1073–1090.
5. P. A. Krutitskii, Wave propagation in a 2-D external domain with cuts. *Applicable Analysis* **62** (1996), 297–309.
6. P. A. Krutitskii, Fast stratified flow over several obstacles, including wings. *IMA J. Appl. Math.* **57** (1996), 243–256.
7. P. A. Krutitskii, The Neumann problem for the 2-D Helmholtz equation in a multiply connected domain with cuts. *Zeitschr. Analysis Anwendungen* **16** (1997), 349–361.
8. P. A. Krutitskii, The Dirichlet problem for the 2-D Helmholtz equation in a multiply connected domain with cuts. *ZAMM* **77** (1997), 883–890.
9. N. I. Muskhelishvili, *Singular integral equations* (Noordhoff, Groningen, 1972) (3rd Russian edn: Nauka, Moscow, 1968).
10. V. S. Vladimirov, *Equations of mathematical physics* (Marcel Dekker, New York, 1971).