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SIMULTANEOUS APPROXIMATION OF NUMBERS CONNECTED WITH THE EXPONENTIAL FUNCTION

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Abstract

We give several results concerning the simultaneous approximation of certain complex numbers. For instance, we give lower bounds for $|a-\xi_0|+|e^a-\xi_1|$, where a is any non-zero complex number, and ξ_0, ξ_1 are two algebraic numbers. We also improve the estimate of the so-called Franklin Schneider theorem concerning $|b-\xi_0|+|a-\xi_1|+|a^b-\xi_2|$. We deduce these results from an estimate for linear forms in logarithms.

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1. Introduction

When α is an algebraic number, we denote by $H(\alpha)$ the height (in the usual sense) of α. In the present paper we derive several consequences of the following result.

THEOREM 1.1. *Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers, and $\beta_0,\beta_1,\dots,\beta_n$ be algebraic numbers. For $1\leqslant j\leqslant n$, let $\log\alpha_j$ be any determination of the logarithm of α_j. Let D be a positive integer, and $A_0,A_1,\dots,A_n,B,E_1$ be positive real numbers, satisfying*

$$D\geqslant[\mathbf{Q}(\alpha_1,\dots,\alpha_n,\beta_0,\dots,\beta_n):\mathbf{Q}],$$

$$A_j\geqslant\max\{H(\alpha_j),\exp|\log\alpha_j|,e\}\quad(1\leqslant j\leqslant n),$$

$$B\geqslant\max_{0\leqslant j\leqslant n}H(\beta_j),$$

$$A_n\geqslant A_{n-1}\geqslant\dots\geqslant A_1,\quad A_0=e$$

and

$$1 < E_1 \leqslant \min_{1\leqslant j\leqslant n} \{e(\operatorname{Log} A_j)/|\log \alpha_j|\}.$$

If the number

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$$

does not vanish, then

$$|\Lambda| > \exp\{-C_1(n)\, D^{n+2}.(\operatorname{Log} A_1)\ldots(\operatorname{Log} A_n).(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_n + \operatorname{Log} E_1)$$
$$.(\operatorname{Log}\operatorname{Log} A_{n-1} + \operatorname{Log} E_1)(\operatorname{Log} E_1)^{-n-1}\},$$

where

$$C_1(1) \leqslant 2^{39}, \quad C_1(2) \leqslant 2^{59} \quad \textit{and} \quad C_1(n) \leqslant 2^{10n+53}.n^{2n}.$$

We will deduce this result from Theorem C of [Wa] in Section 2 below. The connection with a previous result of Baker is the following. In [Ba], Baker sets

$$\Omega = \prod_{j=1}^{n} \operatorname{Log}\max\{H(\alpha_j), 4\}$$

and

$$\Omega' = \prod_{j=1}^{n-1} \operatorname{Log}\max\{H(\alpha_j), 4\},$$

and proves that

$$|\Lambda| > \exp\{-(16nD)^{200n}\,\Omega(\operatorname{Log}(B\Omega))\operatorname{Log}\Omega'\},$$

provided the logarithms are principal valued. This last requirement implies

$$|\log \alpha| \leqslant \pi + \operatorname{Log}(H(\alpha)+1).$$

Therefore, by choosing $E_1 = e$ in Theorem 1.1, we can replace $(16nD)^{200n}$ (in Baker's result) by $2^{12n+53}.n^{2n}.D^{n+2}$. However, several of our applications will involve a large value for E_1, which leads to an improved bound for $|\Lambda|$. For instance, when the numbers $|\log \alpha_j|$ are bounded, we obtain the following result (choosing $\operatorname{Log} A_j = R\operatorname{Log} H_j$, $E_1 = \operatorname{Log} H_1$).

COROLLARY 1.2. *With the notations of Theorem* 1.1, *let* $H_0, H_1, \ldots, H_n, R$ *satisfy*

$$R \geqslant 1 + \max_{1\leqslant j\leqslant n} |\log \alpha_j|,$$

$$H_j \geqslant \max\{H(\alpha_j), e^e\} \quad (1 \leqslant j \leqslant n)$$

and

$$H_0 = H_1 \leqslant \ldots \leqslant H_n,$$

then

$$|\Lambda| > \exp\{-C_2(n,R)\, D^{n+2}(\operatorname{Log} H_1)\ldots(\operatorname{Log} H_n)(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} H_n)$$
$$.(\operatorname{Log}\operatorname{Log} H_{n-1})(\operatorname{Log}\operatorname{Log} H_1)^{-n-1}\},$$

with

$$C_2(n,R) \leqslant C_1(n)\,.\,R^n\,.(2+\operatorname{Log} R)^2.$$

In the present paper, we first discuss a problem of K. Mahler on $|e^n - p|$; we then consider the simultaneous approximation of a and e^a; then we deal with Franklin Schneider's theorem [S], and some of its generalizations. Finally, we derive a connection between simultaneous approximations and algebraic independence.

Here, we do not pay a special attention to the degree. Our results will be rather sharp with this respect, but we could improve them by using the refined arguments of [Wa], at the cost of complicating the statements.

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2. Auxiliary results

We first show how to deduce Theorem 1.1 from Theorem C of [Wa]. Let us define

$$V_j = \operatorname{Log} A_j \quad (1 \leqslant j \leqslant n),$$

$$W = \operatorname{Log} B$$

and

$$E = (DE_1)^{\frac{1}{2}}.$$

In view of the inequalities

$$W + \operatorname{Log} V_n + \operatorname{Log} E + \operatorname{Log} D \leqslant \tfrac{4}{3}(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_n + \operatorname{Log} E_1 + \operatorname{Log} D)$$

and

$$\operatorname{Log} V_{n-1} + \operatorname{Log} E + \operatorname{Log} D \leqslant \tfrac{4}{3}(\operatorname{Log}\operatorname{Log} A_{n-1} + \operatorname{Log} E_1 + \operatorname{Log} D)$$

we will obtain $C_1(n) \leqslant 3^{n-1}2^4 C(n) \leqslant 2^{2n+2} C(n)$, (where $C(n)$ is the constant of Theorem C of [Wa]), provided that we prove

$$E \leqslant e^{DV_1}.$$

Since

$$e\,.\,2^D\,.\,D^2 \log A_1 \leqslant A_1^{3D-1},$$

it is sufficient to prove the following lemma.

LEMMA 2.1. *Let* α *be a non-zero algebraic number of degree at most* D *and height at most* A, *and let* $\log\alpha$ *be a non-zero determination of the logarithm of* α. *Then*

$$|\log\alpha| \geqslant 2^{-D}.(AD)^{-1}.$$

PROOF OF LEMMA 2.1. Since $2^D A D \geqslant 2$, we may assume $|\log \alpha| \leqslant \frac{1}{2} < \operatorname{Log} 2$. From the inequality $|e^z - 1| \leqslant |z| e^{|z|}$ for all $z \in \mathbf{C}$ we deduce

$$|\alpha - 1| \leqslant 2 |\log \alpha|.$$

Finally, we have (for example by using Lemma 3 of [M–W])

$$|\alpha - 1| \geqslant 2^{1-D}(AD)^{-1}.$$

This completes the proof of Lemma 2.1.

The following simple lemma is proved in [Wa], Lemma 2.4.

LEMMA 2.2. *Let v and w be two complex numbers satisfying*

$$|w - e^v| \leqslant \tfrac{1}{3} |e^v|.$$

Then there exists a determination of the logarithm of w such that

$$|w - e^v| \geqslant \tfrac{2}{3} |e^v| \, |\log w - v|.$$

We now prove several auxiliary lemmas which will be used in Section 6. We use the notations of [Wa].

LEMMA 2.3. *Let $P \in \mathbf{Z}[X_0, \ldots, X_m]$ be a polynomial of degree at most N_j with respect to X_j $(0 \leqslant j \leqslant m)$, let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers generating a field K of degree at most D, such that the polynomial $P(X_0, \alpha_1, \ldots, \alpha_m) \in \mathbf{C}[X_0]$ does not vanish identically. Let t be a complex number.*

There exist a positive integer k, and an algebraic number γ of degree at most DN_0/k, such that

$$M(\gamma)^k \leqslant L(P)^D . \exp\left\{\sum_{j=1}^{m} N_j h(\alpha_j)\right\}$$

and

$$|\gamma - t|^k \leqslant |P(t, \alpha_1, \ldots, \alpha_m)| \, 2^{4D^2N_0^2} . L(P)^{D^2N_0+D-1} \\ . \max\{1, |t|\}^{N_0(D-1)} . \exp\left\{(1 + DN_0) \sum_{j=1}^{m} N_j h(\alpha_j)\right\}.$$

PROOF OF LEMMA 2.3. For $1 \leqslant j \leqslant m$, let a_j be the leading coefficient of the minimal polynomial of α_j, with, say, $a_j > 0$, and let d_j be the degree of α_j. We denote by $\{\sigma\}$ the set of the embeddings of K into $\mathbf{C}$. The polynomial

$$Q(Y) = \left(\prod_{j=1}^{m} a_j^{N_j D/d_j}\right) \prod_{\{\sigma\}} P(X_0, \alpha_1^\sigma, \ldots, \alpha_m^\sigma)$$

is not identically zero, and has coefficients in $\mathbf{Z}$.

Further,

$$M(Q) = \left(\prod_{j=1}^{m} a_j^{N_j D/d_j}\right) \prod_{\{\sigma\}} \exp\left(\int_0^1 \operatorname{Log}\left|P(e^{2i\pi u}, \alpha_1^\sigma, \ldots, \alpha_m^\sigma)\right| du\right)$$

$$\leqslant \left(\prod_{j=1}^{m} a_j^{N_j D/d_j}\right) \prod_{\{\sigma\}} \left(L(P) \prod_{j=1}^{m} \max(1, |\alpha_j^\sigma|)^{N_j}\right)$$

$$\leqslant L(P)^D . \prod_{j=1}^{m} M(\alpha_j)^{N_j D/d_j}.$$

Furthermore,

$$|Q(t)| \leqslant |P(t, \alpha_1, \ldots, \alpha_m)| \left(\prod_{j=1}^{m} a_j^{N_j D/d_j}\right) \prod_{\substack{\{\sigma\}\\ \sigma \neq 1}} |P(t, \alpha_1^\sigma, \ldots, \alpha_m^\sigma)|$$

$$\leqslant |P(t, \alpha_1, \ldots, \alpha_m)| . L(P)^{D-1} . \max\{1, |t|\}^{N_0(D-1)}$$

$$. \exp\left\{\sum_{j=1}^{m} N_j h(\alpha_j)\right\}.$$

Let γ be a root of Q which is at minimal distance from t, and let k be its multiplicity. Then (see, for example, the proof of Lemma 2.3 of [Wa], or [M–W] Lemma 9)

$$M(\gamma)^k \leqslant M(Q),$$

$$[\mathbf{Q}(\gamma) : \mathbf{Q}] \leqslant DN_0/k$$

and

$$|\gamma - t|^k \leqslant 4^{(DN_0)^2} (2DN_0 H(Q))^{DN_0} |Q(t)|.$$

Since, for n integer $\geqslant 1$,

$$4^{n^2}(2n)^n . 2^{n^2} \leqslant 2^{4n^2},$$

and since

$$H(Q) \leqslant 2^{DN_0} M(Q),$$

the desired result follows.

We will use only a weaker form of Lemma 2.3.

COROLLARY 2.4. *With the notations of Lemma* 2.3, *we have*

$$[\mathbf{Q}(\gamma) : \mathbf{Q}] \leqslant DN_0,$$

$$H(\gamma) \leqslant (H_0 H_1 \ldots H_m)^{C_3}$$

and

$$|\gamma - t| \leqslant |P((t, \alpha_1, \ldots, \alpha_m)| . (H_0 H_1 \ldots H_m)^{C_4},$$

where

$$H_0 = \max\{e, H(P)\}, \quad H_j = \max\{e, H(\alpha_j)\} \quad (1 \leqslant j \leqslant m),$$

and C_3, C_4 *depend only on* $D, N_0, \ldots, N_m$.

LEMMA 2.5. *Let $P \in \mathbf{C}[X_1, \ldots, X_m]$ be a polynomial of total degree at most N, and let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be complex numbers. Then*

$$|P(x_1, \ldots, x_m) - P(y_1, \ldots, y_m)| \leqslant NL(P)\, R^{N-1} \sum_{j=1}^{m} |x_j - y_j|,$$

with

$$R = \max\{1, |x_1|, \ldots, |x_m|, |y_1|, \ldots, |y_m|\}.$$

PROOF. Straightforward, using the identity

$$x_1^{h_1} \ldots x_m^{h_m} - y_1^{h_1} \ldots y_m^{h_m} = \sum_{j=1}^{m} (x_j - y_j)\, x_1^{h_1} \ldots x_{j-1}^{h_{j-1}} y_{j+1}^{h_{j+1}} \ldots y_m^{h_m} \sum_{k=0}^{h_j - 1} x_j^k y_j^{h_j - k - 1}.$$

LEMMA 2.6. *Let $\psi: \mathbf{R}_+ \to \mathbf{R}_+$ be a positive function defined over the set $\mathbf{R}_+$ of positive real numbers, such that*

$$\lim_{x \to +\infty} \psi(x)/x = +\infty.$$

Let $\theta_0, \ldots, \theta_m$ be complex numbers, and N a positive integer. There exist two easily computable numbers C_5 and H_0, depending only on m, ψ, N and $\max_{0 \leqslant j \leqslant m} |\theta_j|$, *with the following property.*

Let H be an integer with $H \geqslant H_0$, let $\xi_1, \ldots, \xi_m$ be algebraic numbers of degree at most N and height at most H, and let $P \in \mathbf{Z}[X_0, \ldots, X_m]$ be a polynomial of degree at most N and height at most H, such that the polynomial $P(X_0, \xi_1, \ldots, \xi_m) \in \mathbf{C}[X_0]$ is not identically zero. Assume

$$|\theta_1 - \xi_1| + \ldots + |\theta_m - \xi_m| + |P(\theta_0, \theta_1, \ldots, \theta_m)| < \exp\{-\psi(\operatorname{Log} H)\}.$$

Then there exists an algebraic number ξ_0 of degree at most N^{m+1} and height at most H^{C_5}, such that

$$|\theta_0 - \xi_0| \leqslant \exp\{-\tfrac{1}{2}\psi(\operatorname{Log} H)\}.$$

PROOF OF LEMMA 2.6. By Lemma 2.5, we have

$$\begin{aligned} &|P(\theta_0, \xi_1, \ldots, \xi_m) - P(\theta_0, \theta_1, \ldots, \theta_m)| \\ &\qquad \leqslant (N+1)^{m+2} (1 + \max_{0 \leqslant h \leqslant m} |\theta_h|)^N . H . \exp\{-\psi(\operatorname{Log} H)\}. \end{aligned}$$

Since $\psi(x)/x$ tends to infinity, for H sufficiently large we obtain

$$|P(\theta_0, \xi_1, \ldots, \xi_m)| \leqslant \exp\{-\tfrac{2}{3}\psi(\operatorname{Log} H)\}.$$

Using Corollary 2.4, we find an algebraic number ξ_0 of degree at most N^{m+1} and height at most H^{C_3} such that

$$\begin{aligned} |\theta_0 - \xi_0| &\leqslant |P(\theta_0, \xi_1, \ldots, \xi_m)| . H^{mC_4} \\ &\leqslant \exp\{-\tfrac{1}{2}\psi(\operatorname{Log} H)\} \quad \text{for } H \geqslant H_0. \end{aligned}$$

Another proof of Lemma 2.6 is given in [Bi] Lemma 4.5.

LEMMA 2.7. *Let ψ: $\mathbf{R}_+ \to \mathbf{R}_+$ be a positive function such that*

$$\lim_{x \to +\infty} \psi(x)/x = +\infty$$

and

$$\psi(2x)/\psi(x) \text{ is bounded when } x \to +\infty.$$

Let $\theta_0, \ldots, \theta_k, \omega_1, \ldots, \omega_h$ be complex numbers, and N a positive integer. There exist two numbers H_0, C_6, depending only on k, h, ψ, N, $\theta_0, \ldots, \theta_k$, $\omega_1, \ldots, \omega_h$ with the following property.

Assume that for all algebraic numbers $\alpha_0, \ldots, \alpha_k$ of degree at most N^{k+h+1} and height at most H, with $H \geqslant H_0$, we have

$$|\theta_0 - \alpha_0| + \ldots + |\theta_k - \alpha_k| > \exp\{-\psi(\operatorname{Log} H)\}.$$

Assume that there exist an integer $H_1 \geqslant H_0$, and algebraic numbers $\xi_1, \ldots, \xi_m$, with $m = h+k$, of degree at most N and height at most H_1, such that

$$|\theta_1 - \xi_1| + \ldots + |\theta_k - \xi_k| + |\omega_1 - \xi_{k+1}| + \ldots + |\omega_h - \xi_m| < \exp\{-C_6 \psi(\operatorname{Log} H_1)\}.$$

Then θ_0 is transcendental over the field $Q(\theta_1, \ldots, \theta_k, \omega_1, \ldots, \omega_h)$.

PROOF OF LEMMA 2.7. Let $x_1, \ldots, x_q$ be an algebraically independent subset of $\{\theta_1, \ldots, \theta_k, \omega_1, \ldots, \omega_h\}$, and, for $1 \leqslant j \leqslant q$, let $\eta_j \in \{\xi_1, \ldots, \xi_m\}$ be such that

$$\sum_{j=1}^{q} |x_j - \eta_j| < \exp\{-C_6 \psi(\operatorname{Log} H_1)\}.$$

Let $P \in \mathbf{Z}[X_0, X_1, \ldots, X_q]$ be a polynomial such that $P(\theta_0, x_1, \ldots, x_q) = 0$. By Lemma 2.5 we have

$$|P(\theta_0, \eta_1, \ldots, \eta_q)| \leqslant \exp\{-\tfrac{1}{2} C_6 \psi(\operatorname{Log} H_1)\}.$$

From Lemma 2.6 and from our assumption that $\theta_0, \theta_1, \ldots, \theta_k$ cannot be approximated simultaneously by algebraic numbers of bounded degree, we conclude

$$P(X_0, \eta_1, \ldots, \eta_q) \equiv 0.$$

Let us write

$$P(X_0, X_1, \ldots, X_q) = \sum_{l=0}^{N} p_l(X_1, \ldots, X_q)\, X_0^l,$$

where $p_l(X_1, \ldots, X_q) \in \mathbf{Z}[X_1, \ldots, X_q]$. Since

$$p_l(\eta_1, \ldots, \eta_q) = 0 \quad \text{for } 0 \leqslant l \leqslant N,$$

we deduce from Lemma 2.5

$$|p_l(x_1, \ldots, x_q)| \leqslant \exp\{-\psi(\operatorname{Log} H_1)\} \quad (0 \leqslant l \leqslant N).$$

The left-hand side does not depend on H_1, and therefore for $H_1 \geqslant H_0$, where H_0 depends on P,

$$p_l(x_1, \ldots, x_q) = 0 \quad \text{for } 0 \leqslant l \leqslant N.$$

This proves that

$$P(X_0, \ldots, X_q) \equiv 0.$$

3. On the difference between an algebraic number and the exponential of an algebraic number

In 1953, Mahler [Ma 2] proved that if m and p are positive integers, then

$$|e^m - p| > \exp\{-40(\operatorname{Log} m)(\operatorname{Log} p)\}.$$

In 1967 [Ma 3], he succeeded to replace 40 by 33, and in 1973, Mignotte [Mi] replaced it by 17.7. It is not yet known whether there exists an absolute constant C_7 such that

$$|e^m - p| > p^{-C_7}$$

for all positive integers m and p.

From Theorem 1.1 we deduce a lower bound for $e^\alpha - \beta$, when α and β are any non-zero algebraic numbers:

$$|e^\beta - \alpha| > \exp\{-2^{42} D^3(\operatorname{Log} A)(\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A)\},$$

where $D = [Q(\alpha, \beta): Q]$, $A = \max\{H(\alpha), e^e\}$ and $B = H(\beta)$.

For instance if m, n, p, q are positive integers with $p \geqslant 3$, then

$$(3.1) \qquad |e^{m/n} - p/q| > \exp\{-2^{42}(\operatorname{Log} p)(\operatorname{Log} m + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} p)\}.$$

This result can be improved when m is relatively small: if $m < n(\operatorname{Log} p)^{1-\varepsilon}$ with $\varepsilon > 0$, then

$$(3.2) \qquad |e^{m/n} - p/q| > \exp\left\{-C_8(\varepsilon)(\operatorname{Log} p)\left(1 + \frac{\operatorname{Log} n}{\operatorname{Log}\operatorname{Log} p}\right)\right\},$$

where $C_8(\varepsilon)$ is an easily computable constant depending only on ε. More precisely, the case $n = 1$ of Theorem 1.1 shows that

$$|e^\beta - \alpha| > \exp\left\{-2^{40} . D^3(\operatorname{Log} A_1)\left(1 + \frac{\operatorname{Log} B + \operatorname{Log}\operatorname{Log} A_1}{1 + \operatorname{Log}\operatorname{Log} A_1 - \operatorname{Log}|\beta|}\right)\right\},$$

where $A_1 = \max\{H(\alpha), \exp|\beta|, e\}$.

We can replace e by e^π in (3.1): let m, n, p, q be positive integers, with $p \geqslant 3$. Then

$$|e^{\pi m/n} - p/q| > \exp\{-2^{71} . (\operatorname{Log} p)(\operatorname{Log} m + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} p)\}.$$

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This result can be deduced from Theorem 1.1 applied to the linear form

$$-i\frac{m}{n}\log\alpha_1-\log\frac{p}{q}$$

with $\log\alpha_1 = i\pi$, $A_1 = e^{\pi}$, $A_2 = p$, $D = 2$, $B = mn$ and $E_1 = e$.

The corresponding statement (3.2) for e^{π} is not yet known, even for $m = n = 1$.

4. On the simultaneous approximation of a complex number and its exponential

Let a be a non-zero complex number, and ξ_0, ξ_1 two algebraic numbers of height at most H_0, H_1 respectively, $H_j \geqslant e^e$. In [C], Cijsouw proved

$$|a-\xi_0|+|e^a-\xi_1|>\exp\{-C_9(\operatorname{Log} H_1)(\operatorname{Log} H_0)\}$$

and

$$|a-\xi_0|+|e^a-\xi_1|>\exp\{-C_{10}(\operatorname{Log} H)^2(\operatorname{Log}\operatorname{Log} H)^{-1}\},$$

where $H = \max\{H_1, H_0\}$, and C_9, C_{10} depend only on a and $[Q(\xi_0, \xi_1): Q]$.

Here we prove a slightly more general result.

THEOREM 4.1. *Let a be a non-zero complex number, and ξ_0, ξ_1 be two algebraic numbers. Let D, H_0, H_1 satisfy*

$$D \geqslant [Q(\xi_0, \xi_1): Q], \quad H_0 \geqslant H(\xi_0), \quad H_1 \geqslant \max\{H(\xi_1), e^e\}.$$

Then

$$|a-\xi_0|+|e^a-\xi_1| \geqslant \exp\left\{-C_{11}(a)\, D^3(\operatorname{Log} H_1)\left(1+\frac{\operatorname{Log} H_0}{\operatorname{Log}\operatorname{Log} H_1}\right)\right\},$$

where

$$C_{11}(a) = 2^{43}(1+|a|)^2.$$

PROOF OF THEOREM 4.1. There is no loss of generality to assume $|e^a-\xi_1| \leqslant \frac{1}{3}|e^a|$. By Lemma 2.2 we can choose $\log\xi_1$ such that

$$|a-\log\xi_1| \leqslant \tfrac{3}{2}|e^{-a}|.|e^a-\xi_1|.$$

Thus

$$|\xi_0-\log\xi_1| \leqslant (1+\tfrac{3}{2}|e^{-a}|)(|a-\xi_0|+|e^a-\xi_1|)$$

and

$$|\log\xi_1| \leqslant \tfrac{1}{2}+|a|.$$

From 1.2 we conclude

$$C_{11}(a) \leqslant \tfrac{6}{5}C_2(1, R) \quad \text{with} \quad R = \tfrac{3}{2}+|a|.$$

Finally, we remark that $R(2+\operatorname{Log} R)^2 \leqslant 10(1+|A|)^2$.

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5. On Franklin Schneider's theorem

Let a and b be two complex numbers, with $a \neq 0$, and let $\log a$ be a non-zero determination of the logarithm of a. We consider lower bounds for

$$|a-\alpha|+|b-\beta|+|a^b-\gamma|,$$

when α, β, γ are algebraic numbers. From the work of Bijlsma [Bi] we know that this number can be very small when β is rational, and we will consider here only the case of irrational β. This problem has been studied by Ricci, Franklin, Schneider, Smelev, Bundschuh, and more recently in [Bi], [C–W], [M–W] and [Wü]. The best known results were firstly [C–W]:

$$\exp\{-C_{12}(\operatorname{Log} H)^3 \operatorname{Log}\operatorname{Log} H\},$$

where H is an upper bound for the heights of α, β, γ and C_{12} depends on $\log a, b$, and $D = [Q(\alpha,\beta,\gamma):Q]$, and secondly [M–W]:

$$\exp\{-C_{13} D^4 (\operatorname{Log} H)^4 (\operatorname{Log}\operatorname{Log} H)^{-1}\},$$

where C_{13} depends only on $\log a$ and b. (This last result has been slightly improved with respect to D; see [M–W]). Here we get a sharpening of these two estimates:

THEOREM 5.1. *Let a and b be two complex numbers with $a \neq 0$, and let $\log a$ be any non-zero determination of the logarithm of a.*

Let α, β, γ be algebraic numbers of height at most H, with $H \geqslant e^e$, and let D be the degree of the field $Q(\alpha,\beta,\gamma)$ over Q. Assume that β is irrational. Then

$$|a-\alpha|+|b-\beta|+|a^b-\gamma| > \exp\{-C_{14} D^4 (\operatorname{Log} H)^3 (\operatorname{Log}\operatorname{Log} H)^{-2}\}$$

with

$$C_{14} = C_{14}(\log a, b) \leqslant 2C_2(2, \tfrac{3}{2}+|\log a|+|b \log a|).$$

Several generalizations of the Franklin Schneider problems have been studied by Wallisser, Meyer, Bundschuh, and more recently in [C–W], [Bi] and [Wü]. Our Theorem 1.1 leads to several improvements of these results. Here is one example, which generalizes Theorem 5.1.

THEOREM 5.2. *Let $a_1, \dots, a_m, b_0, \dots, b_m$ be complex numbers, with $a_j \neq 0$ $(1 \leqslant j \leqslant m)$, and let $\log a_j$ denote an arbitrary value of the logarithm of a_j such that $\log a_j \neq 0$. Define*

$$R = \tfrac{3}{2} + \sum_{j=1}^{m} |\log a_j| + |b_0| + \sum_{j=1}^{m} |b_j \log a_j| \quad \text{and} \quad C_{15} = 2C_2(m+1, R).$$

Let $\alpha_1, \dots, \alpha_m, \beta_0, \dots, \beta_m, \gamma$ be algebraic numbers of height $\leqslant H$, with $H \geqslant e^e$, generating a field of degree D. Assume either $b_0 \neq 0$ or $1, \beta_1, \dots, \beta_m$ linearly independent

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over Q. Then

$$\sum_{j=1}^{m}|a_j-\alpha_j|+\sum_{j=0}^{m}|b_j-\beta_j|+|e^{b_0}a_1^{b_1}\dots a_m^{b_m}-\gamma|$$
$$>\exp\{-C_{15}D^{m+3}(\operatorname{Log}H)^{m+2}(\operatorname{Log}\operatorname{Log}H)^{-m-1}\}.$$

PROOF OF THEOREM 5.2. This result is a straightforward consequence of Lemma 2.2 and Corollary 1.2, with $n=m+1$, $H_1=\dots=H_n=B=H$.

6. Simultaneous approximations and algebraic independence

The first connection between diophantine approximations and algebraic independence goes back to Mahler in 1932 [Ma 1]. More recent results have been obtained in special cases by Bijlsma [Bi], Laurent [L] and Väänänen [V].

In [V], Väänänen gives a lower bound for $|a-\xi_0|+|b-\xi_1|+|P(a,e^b)|$ in terms of the heights of ξ_0, ξ_1 and P, when a and b are non-zero complex numbers. Similarly, in his thesis [Bi], Bijlsma gives lower bounds for several expressions like

$$|a-\xi_0|+|b-\xi_1|+|P(a,b,a^b)-\xi_2|$$

in terms of the heights of ξ_0, ξ_1, ξ_2 and P, when a,b are non-zero complex numbers.

As remarked in [V], Väänänen's result shows that if a and b are Liouville numbers of a certain type, then a and e^b are algebraically independent. In [L], Laurent shows that the lower bound for linear forms of [M–W] implies the following result of Feldman: if a is a Liouville number of a certain type, and if β is algebraic irrational, then a and a^β are algebraically independent.

We give here a rather straightforward consequence of Theorems 4.1 and 5.2 and Lemma 2.6.

THEOREM 6.1. *Let $m\geqslant 0$, $h\geqslant 0$ be non-negative integers, $a_1,\dots,a_m$, $b_0,\dots,b_m$, $c_1,\dots,c_h$ be complex numbers, and N a positive integer. There exists an easily computable number C_{16}, depending only on m, h, N and*

$$\max\{|\log a_1|,\dots,|\log a_m|,|b_0|,\dots,|b_m|,|c_1|,\dots,|c_h|\},$$

with the following property.

Let H be an integer, $H\geqslant e^e$, let $\alpha_1,\dots,\alpha_m$, $\beta_0,\dots,\beta_m$, $\gamma_1,\dots,\gamma_h$ be algebraic numbers of degree at most N and height at most H, and let $P\in\mathbf{Z}[X_0,\dots,X_{2m+h+1}]$ be a polynomial of degree at most N and height at most H. We assume either $b_0\neq 0$ or $1,\beta_1,\dots,\beta_m$ $\mathbf{Q}$-linearly independent. Moreover, we assume that the polynomial

$$P(X_0,\alpha_1,\dots,\alpha_m,\beta_0,\dots,\beta_m,\gamma_1,\dots,\gamma_h)\in\mathbf{C}[X_0]$$

is not identically zero.

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Then

$$\sum_{i=1}^{m}|a_i-\alpha_i|+\sum_{j=0}^{m}|b_j-\beta_j|+\sum_{k=1}^{h}|c_k-\gamma_k|$$
$$+|P(e^{b_0}a_1^{b_1}\dots a_m^{b_m},a_1,\dots,a_m,b_0,\dots,b_m,c_1,\dots,c_h)|$$
$$>\exp\{-C_{16}(\operatorname{Log}H)^{m+2}(\operatorname{Log}\operatorname{Log}H)^{-m-1}\}.$$

A more careful estimation of the constants involved in the proofs of Lemma 2.6 and Theorem 4.1 shows the following. Let $P\in\mathbf{Z}[X, Y]$ be a polynomial of height at most H_0 and degree at most d_0, d_0' with respect to X, Y. Let x, y be complex numbers, and α,β be algebraic numbers of degree at most d_1, d_2 and height at most H_1, H_2 respectively, satisfying $P(X,\beta)\neq 0$ and $\alpha\neq 0$. For convenience we assume $H_0\geqslant 16$ and $H_1\geqslant 16$. Then

$$|x-\alpha|+|y-\beta|+|P(e^x,y)|$$
$$>\exp\left\{-C_{17}(\operatorname{Log}H_0+\operatorname{Log}H_2)\left(1+\frac{\operatorname{Log}H_1}{\operatorname{Log}\operatorname{Log}H_0+\operatorname{Log}\operatorname{Log}H_2}\right)\right\},$$

where

$$C_{17}=2^{45}.(1+|x|)^2(d_0+d_0')(d_0d_1)^3d_2^4.$$

(Compare with [V].)

Similarly, it is easy to derive from the previous estimates the following results. Let a_1, a_2, b be complex numbers, and N a positive integer. Assume $a_2\neq 0$, and let $\log a_2$ be a determination of the logarithm of a_2. Let α_1,α_2,β be algebraic numbers of degree at most N and height at most H_1, H_2, H_3 respectively. Let $P\in\mathbf{Z}[X, Y]$ be a polynomial of (total) degree at most N and height at most H_0. Assume

$$H_i\geqslant 16\quad (i=0,1,2),\qquad \beta\notin Q\qquad P(\alpha_1, Y)\neq 0.$$

We define

$$H^*=\max\{H_0H_1,H_2\},\quad H_*=\min\{H_0H_1,H_2\}$$

and

$$C_{18}=2^{67}.(1+|\log a_2|)^2.(1+|b\log a_2|)^2N^{19}.$$

Then

$$|a_1-\alpha_1|+|a_2-\alpha_2|+|b-\beta|+|P(a_1,a_2^b)|$$
$$>\exp\{-C_{18}(\operatorname{Log}H_0+\operatorname{Log}H_1)(\operatorname{Log}H_2)(\operatorname{Log}H_3+\operatorname{Log}\operatorname{Log}H^*)(\operatorname{Log}\operatorname{Log}H_*)^{-2}\}.$$

Finally, we give a result of algebraic independence which is an easy consequence of Theorem 6.1 (see Lemma 2.7).

THEOREM 6.2. *Let $m\geqslant 0$, $h\geqslant 0$ be non-negative integers, $a_1,\dots,a_m$, $b_0,\dots,b_m$, $c_1,\dots,c_h$ be complex numbers, and N a positive integer.*

Assume that there exists an increasing sequence H_l of positive integers and, for each l, that there exist algebraic numbers $\alpha_1^{(l)}, \ldots, \alpha_m^{(l)}$, $\beta_0^{(l)}, \ldots, \beta_m^{(l)}$, $\gamma_1^{(l)}, \ldots, \gamma_h^{(l)}$, of degree at most N and height at most H_l, such that

$$\sum_{i=1}^{m} |a_i - \alpha_i^{(l)}| + \sum_{j=0}^{m} |b_j - \beta_j^{(l)}| + \sum_{k=1}^{h} |c_k - \gamma_k^{(l)}| < \exp\{-(\operatorname{Log} H_l)^{m+2}\}.$$

Assume moreover either $b_0 \neq 0$, or $1, \beta_1^{(l)}, \ldots, \beta_m^{(l)}$ $\mathbf{Q}$-linearly independent for each l. Then the number $e^{b_0} a_1^{b_1} \ldots a_m^{b_m}$ is transcendental over the field

$$\mathbf{Q}(a_1, \ldots, a_m, b_0, \ldots, b_m, c_1, \ldots, c_h).$$

References

[Ba] A. Baker (1977), "The theory of linear forms in logarithms", in: *Transcendence Theory; Advances and Applications*, edited by A. Baker and D. W. Masser (Academic Press, New York). Chap. 1, pp. 1–27.

[Bi] A. Bijlsma (1978), *On the Simultaneous Approximation of a, b and a^b* (Thesis, Amsterdam). Math. Center Tracts no. 94.

[C] P. L. Cijsouw (1975), "On the simultaneous approximation of certain numbers", *Duke Math. J.* **42**, 249–257.

[C–W] P. L. Cijsouw and M. Waldschmidt (1977), "Linear forms and simultaneous approximations", *Compositio Math.* **34**, 173–197.

[L] M. Laurent (1977), *Indépendance algébrique de nombres de Liouville à des puissances algébriques*, Thèse de troisième cycle, Université P. et M. Curie (Paris VI).

[Ma 1] K. Mahler (1932), "Zur Approximation der Exponential Funktion und des Logarithmus", *J. reine und angew. Math.* **166**, 118–150.

[Ma 2] K. Mahler (1953), "On the approximation of logarithms of algebraic numbers", *Phil. Royal Trans. Royal Soc. London*, Ser. A **245**, 371–398.

[Ma 3] K. Mahler (1967), "Application of some formulae by Hermite to the approximation of exponentials and logarithms", *Math. Ann.* **168**, 200–227.

[Mi] M. Mignotte (1974), "Approximations rationnelles de π et quelques autres nombres", *Journées Arith.* [1973, Grenoble], *Bull. Soc. Math. France*, Mémoire **37**, 121–132.

[M–W] M. Mignotte and M. Waldschmidt (1978), "Linear forms in two logarithms and Schneider's method", *Math. Annalen.* **231**, 241–267.

[S] Th. Schneider (1957), *Einführung in die transzendenten Zahlen* (Springer-Verlag, Berlin and Gauthier-Villars, Paris).

[V] K. Väänänen (to appear), "On the arithmetic properties of certain values of the exponential function".

[Wa] M. Waldschmidt (1978), "Transcendence measures for exponentials and logarithms", *J. Austral. Math. Soc.* (Ser. A) **25**, 445–465.

[Wü] Wüstholz (1976), *Simultane Approximationen, die mit Potenzen bzw. Logarith men zusammenhängen* (Dissertation, Freiburg).

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