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FROBENIUS ALGEBRAS AND THEIR QUIVERS

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This paper studies the construction of Frobenius algebras. We begin with a description of when a graded k-algebra has a Frobenius algebra as a homomorphic image. We then turn to the question of actual constructions of Frobenius algebras. We give a method for constructing Frobenius algebras as factor rings of special tensor algebras. Since the representation theory of special tensor algebras has been studied intensively ([**6**], see also [**2**; **3**; **4**]), our results permit the construction of Frobenius algebras which have representations with prescribed properties. Such constructions were successfully used in [**9**].

A special tensor algebra has an associated quiver (see § 3 and 4 for definitions) which determines the tensor algebra. The basic result relating an associated quiver and the given tensor algebra is that the category of representations of the quiver is isomorphic to the category of modules over the tensor algebra [**3**; **4**; **6**]. One of the questions which is answered in this paper is: given a quiver $\mathscr{Q}$, is there a Frobenius algebra Γ such that $\mathscr{Q}$ is the quiver of Γ; that is, is there a special tensor algebra $T = R + M + \otimes_R^2 M + \ldots$ with associated quiver $\mathscr{Q}$ such that there is a ring surjection $\psi : T \rightarrow \Gamma$ with $\ker \psi \subseteq (\otimes_R^2 M + \otimes_R^3 M + \ldots)$ and Γ being a Frobenius algebra? This question was completely answered under the additional constraint that the radical of Γ cubed is zero [**8**, Theorem 2.1]. The study of radical cubed zero self-injective algebras has been used in classifying radical square zero Artin algebras of finite representation type [**11**].

We hope that the explicit construction technique given in this paper will lead to the solution of other questions involving Frobenius algebras. As shown in Section 7, most of the algebras constructed by the general technique are of infinite representation type. In a future paper, we show that under slightly restricted circumstances, one may modify the constructions to get "smaller" Frobenius algebras, many of which are of finite representation type. We also hope that the results on zero dimensional Gorenstein rings will provide a powerful new tool in the study of these rings.

We summarize the contents of the paper. The basic result, characterizing when a graded algebra has a Frobenius algebra as a factor, is given in Section 1. This result is applied in Section 2 to give a characterization of zero dimensional Gorenstein factors of commutative polynomial rings. In Section 3 the notation of a quiver is introduced and the properties of quivers of selfinjective rings

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are studied. We also define special tensor algebras in this section. Section 4 provides a constructive proof of the existence of Frobenius algebras with given quivers. Section 5 studies some properties of the algebras constructed by the techniques of Sections 1 and 4. In Section 6 we reprove some well-known results about generalized uniserial Frobenius algebras in the context of the earlier results. Section 7 shows that except for the generalized uniserial algebras, the algebras which are constructed by the technique of Section 4 are of infinite representation type. Finally, in Section 8, we raise a number of open questions.

1. Characterization of Frobenius factor rings. Throughout this paper k will denote a field. All rings will be k-algebras and all ring maps will be k-algebra maps.

Recall the following classical result [**1**, 61.3]:

THEOREM 1.1. *Let Λ be a finite dimensional k-algebra. The following statements are equivalent:*

i) *Λ is isomorphic to* $\mathrm{Hom}_k(\Lambda, k)$ *as left Λ-modules.*

ii) *There exists a linear function $\bar{\sigma} : \Lambda \rightarrow k$ such that the kernel of $\bar{\sigma}$ contains no left or right ideals.*

iii) *There exists a non-degenerate bilinear form $f : \Lambda \times \Lambda \rightarrow k$ such that $f(ab, c) = f(a, bc)$ for all $a, b, c, \in \Lambda$.*

A k-algebra Λ, satisfying conditions i)–iii) of 1.1 is called a *Frobenius algebra.* If the bilinear form f is symmetric; i.e., $f(a, b) = f(b, a)$, then Λ is called a *symmetric algebra.* It should be noted that $\bar{\sigma}$ and f are related by the formula (1.2) $f(a, b) = \bar{\sigma}(ab)$.

We also note that Frobenius algebras are selfinjective [**1**, 61.2].

We now develop a criterion to determine when a graded k-algebra has a Frobenius algebra as a factor ring.

Let Ω be a k-algebra and let $\boldsymbol{a}$ be a two-sided ideal in Ω such that

(1.3) $\Omega/\boldsymbol{a}$ is a semi-simple finite dimensional k-algebra,

(1.4) $\boldsymbol{a}/\boldsymbol{a}^2$ is a finitely generated Ω/a-module,

(1.5) Ω is isomorphic as an algebra to the graded algebra $\Omega/\boldsymbol{a} + \boldsymbol{a}/\boldsymbol{a}^2 + \boldsymbol{a}^2/\boldsymbol{a}^3 + \ldots$.

Note that (1.4) is equivalent to $\dim_k \boldsymbol{a}/\boldsymbol{a}^2 < \infty$. Before proceeding, we give two examples of rings where these conditions are satisfied that will be of interest later in the paper. Consider the tensor algebra $R + M + M \otimes_R M + \ldots$ where R is a semi-simple finite dimensional k-algebra and M is a finitely generated $R - R$ bimodule. Setting $\Omega = R + M + M \otimes_R M + \ldots$ and $\boldsymbol{a} = M + M \otimes_R M + \ldots$ we see that (1.3), (1.4) and (1.5) are verified. Another case of interest is $\Omega = k[x_1, \ldots, x_n]$, the polynomial ring in n commuting variables is considered in section 2. In this case we take $\boldsymbol{a} = (x_1, \ldots, x_n)$.

For the remainder of this section we fix a k-algebra Ω and an ideal $\boldsymbol{a}$ satisfying (1.3), (1.4) and (1.5). Note that (1.3) and (1.4) imply that $\dim_k \boldsymbol{a}^i/\boldsymbol{a}^{i+1} < \infty$, for all i. Let $\bar{B}_i$ be a k-basis for $\boldsymbol{a}^i/\boldsymbol{a}^{i+1}$ for $i \geqq 0$. Suppose $\bar{B}_i = \{\bar{w}_{i1}, \ldots, \bar{w}_{in_i}\}$. For each i, choose $w_{ij} \in \boldsymbol{a}^i$ such that $p_i(w_{ij}) = \bar{w}_{ij}$, $j = 1, \ldots, n_i$ where $p_i : \boldsymbol{a}^i \to \boldsymbol{a}^i/\boldsymbol{a}^{i+1}$ is defined by (1.5). Let $B_i = \{w_{ij}\}_{j=1}^{n_i}$. Let $B = \bigcup_{i=0}^{\infty} B_i$. Then by (1.5) B is a k-basis for Ω.

Let $k^* = k - \{0\}$ and S be a subset of B. Given a set map $\mu : S \to k^*$ we define a k-linear map $\sigma(\mu) : \Omega \to k$ by

$$\sigma(\mu)(b) = \begin{cases} \mu(b), & \text{if } b \in S \\ 0, & \text{if } b \in B - S. \end{cases}$$

Conversely, given a k-linear map $\sigma : \Omega \to k$, setting $S = \{b \in B \mid \sigma(b) \neq 0\}$, we get a set map $\mu(\sigma) : S \to k^*$ defined by

$$\mu(\sigma)(b) = \sigma(b).$$

This sets up a one to one correspondence between the set of linear maps $\sigma : \Omega \to k$ and the set of set maps $\mu : S \to k^*$, where S is a subset of B. We view this correspondence as an identification and freely write linear maps $\sigma : \Omega \to k$ as set maps $\sigma : S \to k^*$ where S is a subset of B.

Let $\sigma : S \to k^*$ be a set map. Let $I(\sigma) = \{w \in \Omega | \sigma(\langle w \rangle)\} = 0$ where $\langle w \rangle$ denotes the two-sided ideal generated by w, namely $\Omega w \Omega$.

LEMMA 1.6. *Let $\sigma : S \to k^*$ be a set map where $S \subseteq B$.*

a) *The ideal $I(\sigma)$ is the largest two-sided ideal contained in* $\ker \sigma$.

b) *The set S is finite if and only if $\boldsymbol{a}^N \subseteq I(\sigma)$ for some N.*

The proof of (1.6) is straightforward and left to the reader. We are now in a position to prove the basic result of this section.

Definition. Let S be a subset of B and let $\sigma : S \to k^*$ be a set map. We say that (S, σ) is a *Frobenius system* (with respect to $\boldsymbol{a}$) if σ satisfies

(1.7) S is a finite set.

(1.8) For all w, $\sigma(w\Omega) = 0$ if and only if $\sigma(\Omega w) = 0$.

THEOREM 1.9. *Let Ω be a k-algebra and $\boldsymbol{a}$ an ideal in Ω such that* (1.3), (1.4) *and* (1.5) *hold. Let B be a k-basis for Ω as above.*

i) *If (S, σ) is a Frobenius system for Ω then $\Omega/I(\sigma)$ is a Frobenius algebra.*

ii) *Suppose Λ is a Frobenius k-algebra and that there is a ring surjection $\phi : \Omega \to \Lambda$. If $\boldsymbol{a}^N \subseteq \ker \phi$ for some N, then there exists a Frobenius system (S, σ) for Ω with respect to $\boldsymbol{a}$ such that* $\ker \phi = I(\sigma)$.

Proof. We first prove i). Suppose that (S, σ) is a Frobenius system for Ω with respect to $\boldsymbol{a}$. Since S is a finite set, by (1.6)b), $\boldsymbol{a}^N \subseteq I(\sigma)$. Thus $\Lambda = \Omega/I(\sigma)$ is finite dimensional. Since $I(\sigma) \subseteq \ker \sigma$, the linear map $\sigma : \Omega \to k$

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induces a linear map $\bar{\sigma} : \Lambda \rightarrow k$. By (1.1) ii) it suffices to show that ker $\bar{\sigma}$ contains no left or right ideals. Let $\boldsymbol{b}$ be a left ideal in ker $\bar{\sigma}$. Consider the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "\sigma"] \arrow[d, "\phi"'] & k \\ \Lambda \arrow[ur, "\bar{\sigma}"'] & \end{tikzcd}$$

where ϕ is the canonical ring surjection. Let $\boldsymbol{c} = \phi^{-1}(\boldsymbol{b})$. Then $\boldsymbol{c}$ is a left ideal in ker σ and $\boldsymbol{c} \supseteq I(\sigma)$. We show that $\boldsymbol{c}\Omega \subseteq I(\sigma)$ and hence $\boldsymbol{b} = 0$.

Let $w \in \boldsymbol{c}$. Then $\Omega w \subseteq \boldsymbol{c} \subseteq \ker \sigma$. By (1.8) $w\Omega \subseteq \ker \sigma$. Therefore $\boldsymbol{c}\Omega \subseteq I(\sigma)$ by (1.6) a) and we are done. A similar proof shows that if $\boldsymbol{b}$ is a right ideal in ker $\bar{\sigma}$ then $\boldsymbol{b} = 0$.

We now prove ii). Consider the ring surjection $\phi : \Omega \rightarrow \Lambda$. By (1.1) ii), there exists a linear map $\bar{\sigma} : \Lambda \rightarrow k$ such that ker $\bar{\sigma}$ contains no left or right ideals. Let $\sigma : \Omega \rightarrow k$ be defined by $\sigma = \bar{\sigma}\Omega$. Then σ is a k-linear map. Since ker $\bar{\sigma}$ contains no left or right ideals, it follows that (1.8) holds. Since $I(\sigma)$ is the largest ideal in ker σ, it follows that $\ker \phi \subseteq I(\sigma)$. Since $\boldsymbol{a}^N \subseteq \ker \phi$, by (1.6) a) it follows that (1.7) holds and hence (S, σ) is a Frobenius system. It remains to show $I(\sigma) = \ker \phi$. But $I(\sigma)/\ker \phi$ is in particular, a left ideal in ker $\bar{\sigma}$ and hence is (0).

It is the purpose of this paper to show that the above relatively simple result has some very striking consequences.

2. Zero dimensional Gorenstein rings. This section is devoted to applying (1.9) to commutative polynomial rings. We first recall that if Λ is a commutative k-algebra, the following statements are equivalent:

(2.1) Λ is a finite dimensional k-algebra and Λ is an injective Λ-module.

(2.2) Λ is a Frobenius k-algebra.

(2.3) Λ is a symmetric k-algebra.

A commutative k-algebra satisfying (2.1)–(2.3) is called a *zero dimensional Gorenstein ring.*

We first note that if Ω is a commutative k-algebra and $\boldsymbol{a}$ is an ideal in Ω satisfying (1.3)–(1.5) then if S is a finite subset of an appropriate basis B and $\sigma : S \rightarrow k^*$ is an arbitrary set map, then (S, σ) is *always* a Frobenius system (since (1.8) is automatically satisfied). We show that (1.9) leads to a new description of zero dimensional Gorenstein factor rings of the commutative polynomial ring in n variables.

Let $\Omega = k[x_1, \ldots, x_n]$. Let $\boldsymbol{a} = (x_1, \ldots, x_n)$. For the basis B we choose the set of monomials $x_1^{a_1} \ldots x_n^{a_n}$, for $a_i \geqq 0$. Then (1.9) becomes

THEOREM 2.4. *Let* $\Omega = k[x_1, \ldots, x_n]$.

i) *Let S be a finite subset* $\{m_1, \ldots, m_r\}$ *of* B. *Let* $\sigma : S \to k^*$. *Then* $\Omega/I(\sigma)$ *is a zero dimensional Gorenstein ring.*

ii) *Suppose that* Λ *is a zero dimensional Gorenstein ring and suppose there is a ring surjection* $\phi : \Omega \to \Lambda$ *with* $\ker \phi \subseteq \boldsymbol{a}$. *Then there exists a finite set of monomials* $S = \{m_1, \ldots, m_r\} \subseteq B$ *and a map* $\sigma : S \to k^*$ *such that* $\ker \phi = I(\sigma)$.

Thus (2.4) give a classification of zero dimensional Gorenstein factor rings of polynomial rings in terms of very simple data; namely, a finite set of monomials and an assignment of a nonzero scalar to each of these monomials.

A more detailed study of this classification appears in [**7**].

3. Quivers. We begin by recalling the definition of a quiver. A *quiver* $\mathscr{Q}$ is a finite directed graph; i.e. $\mathscr{Q}$ consists of a finite set of vertices and a finite set of (directed) arrows between vertices. If i and j are vertices and a is an arrow from i to j we sometimes write

$$\overset{i}{0} \xrightarrow{a} \overset{j}{0}.$$

In this case i is called the *domain* of a and j is called the *codomain* of a.

Let A be a ring. If I is an ideal in A such that A/I is a semi-simple Artin ring and I/I^2 has finite length as a left A/I-module then we define *the (left) quiver of A with respect to I*, $\mathscr{Q}(A, I)$, as follows:

Let $e_1, \ldots, e_n$ be a full set of nonisomorphic primitive idempotents for A/I. Let n_{ij} be the number of copies of the simple module $(A/I)e_j$ occurring in a direct sum decomposition of $(I/I^2)e_i$. Note that $(I/I^2)e_i$ is a semi-simple left A/I-module. Then $\mathscr{Q}(A, I)$ has n vertices, $1, \ldots, n$ and exactly n_{ij} arrows from vertex i to vertex j. (We identify two quivers if they differ only by a reordering of the vertices.)

If Λ is a left Artin ring with radical $\boldsymbol{r}$ then the (left) *quiver of* Λ, $\mathscr{Q}(\Lambda)$, is defined to be $\mathscr{Q}(\Lambda, \boldsymbol{r})$.

We recall the notion of a special tensor k-algebra. For a fuller description of special tensor k-algebras and their properties see [**6**; **9**].

Let k be a fixed field and let $R = k \times \ldots \times k$ (n copies) be the product ring of n copies of k, viewed as a k-algebra via diagonal action of k. Let M be an $R - R$ bimodule with k acting centrally on M. Assume M has finite dimension over k. Let k_i denote the ith copy of k in R, we may view k_i as both a left and right simple R-module. Then, a full set of nonisomorphic simple $R - R$ bimodules is given by $k_i \otimes_k k_j$, $i, j, = 1, \ldots, n$. Letting $k_{ij} = k_i \otimes_k k_j$, we have that $k_{ij} \cong k_i$ as left R-modules and $k_{ij} \cong k_j$ as right R-modules. Since $R \otimes_k R$ is a semi-simple ring and since M can be viewed as a left $R \otimes_k R$-module via its bimodule structure, we see that $M = \coprod_{i,j} k_{ij}^{(n_{ij})}$ as $R - R$ bimodules where the n_{ij} are non-negative integers and $k_{ij}^{(n_{ij})}$ denotes the direct sum of n_{ij} copies of the simple $R - R$ bimodule k_{ij}. We keep the above notation for R and M throughout this paper.

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Let T be the tensor algebra $R + M + \otimes_R^2 M + \ldots$. Such tensor algebras are called special tensor k-algebras. More generally, if Λ is a k-algebra, we say Λ is a *special tensor k-algebra* if there exists R and M as above such that $\Lambda \cong R + M + \otimes_R^2 M + \ldots$ as k-algebras. If Λ is a special tensor k-algebra, we identify Λ with a fixed isomorphic tensor algebra $R + M + \otimes_R^2 M + \ldots$. Henceforth, T will denote the special tensor k-algebra $R + M + \otimes_R^2 M + \ldots$ unless otherwise stated.

Let $e_1, \ldots, e_n$ be a full set of primitive central idempotents of R, ordered so that $e_i k_{ij} = k_{ij} = k_{ij} e_j$ for all i and j. Since R is a subring of T, we will view the e_i's as idempotents in T also.

If $T = R + M + \otimes_R^2 M + \ldots$ is a special tensor k-algebra then the quiver of T, $\mathscr{Q}(T)$ will mean $\mathscr{Q}(T, J)$, where $J = M + \otimes_R^2 M + \ldots$. On the other hand, given a quiver $\mathscr{Q}$, with vertex set I and n_{ij} arrows from vertex i to vertex j, we see that $Q = Q(T)$ where $T = R + M + \otimes_R^2 M + \ldots$ is the special tensor k-algebra defined by

$$R = \prod_{i \in I} k \quad \text{and} \quad M = \coprod_{i,j \in I} k_{ij}{}^{(n_{ij})}.$$

In each of the three above cases, namely, a ring A together with an ideal I and $\mathscr{Q}(A, I)$, a left Artin ring Λ and $\mathscr{Q}(\Lambda)$, and a special tensor algebra T and $\mathscr{Q}(T)$, we have a correspondence between the vertices of the associated quiver and a set of idempotents. This correspondence will be implicitly referred to in what follows by the phrases "... idempotent corresponding to the vertex ..." and "... vertex corresponding to idempotent ...". A much fuller account of quivers and special tensor algebras can be found in [**6**, **4**; **9**]. In these references one will find the explicit relationship between modules over the tensor algebras and "representations" of the quivers referred to in the introduction.

The remainder of this section is devoted to the study of quivers of left Artin rings, with special attention given to left self-injective Artin rings.

We need some definitions first. A *path* p in $\mathscr{Q}$ is an ordered sequence of arrows $p = (a_1, \ldots, a_m)$ such that the codomain of a_s is the domain of a_{s+1}, for $s = 1, \ldots, m - 1$. We say the *length of* p is m. A *cycle*, c, is a path $c = (a_1, \ldots, a_m)$ such that the codomain of a_m is the domain of a_1. Let i and j be vertices. We say there is a *path from i to j* if there exists a path $p = (a_1, \ldots, a_m)$ such that $i =$ domain of a_1 and $j =$ codomain of a_m. In this case we say that p is a path from i to j.

The next two concepts are crucial to what follows. Let $\mathscr{Q}$ be a quiver. We say $\mathscr{Q}$ is *connected* if given any two vertices i and j there is a sequence of vertices $i = i_0, i_1, \ldots, i_t = j$ such that either there is a path from i_u to i_{u+1} or there is a path from i_{u+1} to i_u, for $u = 0, \ldots, t - 1$. We say that $\mathscr{Q}$ is *path connected* if given any two vertices i and j there is a path from i to j and a path from j to i.

LEMMA 3.1 (a) *If $\mathscr{Q}$ is a path connected quiver then $\mathscr{Q}$ is connected.*

(b) *Let Λ be a left Artin ring. Then Λ is an indecomposable ring if and only if $\mathscr{Q}(\Lambda)$ is connected.*

(c) *Let $\mathscr{Q}$ be a quiver. The following statements are equivalent:*

i) *$\mathscr{Q}$ is path connected.*

ii) *There is a cycle $c = (a_1, \ldots, a_m)$ such that every arrow of $\mathscr{Q}$ occurs as some a_i.*

Proof. (a) is obvious and parts (b) and (c) are left to the reader.

Before continuing we need a technical, though interesting, result.

PROPOSITION 3.2. *Let Λ be a left Artin ring with radical $\boldsymbol{r}$ and let $e_1, \ldots, e_n$ be a full set of nonisomorphic primitive idempotents of Λ. If the simple module $(\Lambda/\boldsymbol{r})e_j$ occurs as a composition factor of $\boldsymbol{r}e_i$ then there is a path in $\mathscr{Q}(\Lambda)$ from the vertex corresponding to e_i to the vertex corresponding to e_j.*

Proof. Let $S_t = (\Lambda/\boldsymbol{r})e_t$, $P_t = \Lambda e_t$, for $t = 1, \ldots, n$. Let i be the vertex in $\mathscr{Q}(\Lambda)$ corresponding to e_i. Assume S_j is a composition factor of $\boldsymbol{r}P_i$. Then there is a positive integer m such that S_j is a summand of $\boldsymbol{r}^mP_i/\boldsymbol{r}^{m+1}P_i$. We proceed by induction on m. For $m = 1$ then S_j is a summand of $\boldsymbol{r}P_i/\boldsymbol{r}^2P_i = (\boldsymbol{r}/\boldsymbol{r}^2)e_i$. In this case there is at least one arrow from i to j and hence a path from i to j. Now suppose that S_j is a summand of $\boldsymbol{r}^mP_i/\boldsymbol{r}^{m+1}P_i$, $m > 1$. Let $f : Q \to \boldsymbol{r}P_i$ be a Λ-projective cover of $\boldsymbol{r}P_i$. Note that P_t is isomorphic to a summand of Q if and only if S_t is a summand of $\boldsymbol{r}P_i/\boldsymbol{r}^2P_i$. Now f induces a surjection $\boldsymbol{r}^{m-1}Q \to \boldsymbol{r}^mP_i$, which in turn induces a surjection $\boldsymbol{r}^{m-1}Q/\boldsymbol{r}^mQ \to \boldsymbol{r}^mP_i/\boldsymbol{r}^{m+1}P_i$. Thus S_j is a summand of $\boldsymbol{r}^{m-1}Q/\boldsymbol{r}^mQ$. It follows that S_j is a summand of $\boldsymbol{r}^{m-1}P_t/\boldsymbol{r}^mP_t$ for some t where S_t is a summand of $\boldsymbol{r}P_i/\boldsymbol{r}^2P_i$. By induction there is a path from t to j and by the first step in the induction there is a path from i to t. Thus there is a path from i to j.

The above proof in fact shows that if S_j is a summand of $\boldsymbol{r}^mP_i/\boldsymbol{r}^{m+1}P_i$ then there is a path of length m from i to j. We now apply this result to find sufficient conditions on a left Artin ring so that its quiver is path connected. In particular, we show that the quivers of indecomposable Frobenius algebras are always path connected.

PROPOSITION 3.3. *Let Λ be an indecomposable basic left Artin ring with radical $\boldsymbol{r}$. If the left socle of Λ,* soc (Λ), *is isomorphic to $\Lambda/\boldsymbol{r}$ as left Λ-modules then $\mathscr{Q}(\Lambda)$ is path connected.*

Proof. Let $e_1, \ldots, e_n$ be a full set of nonisomorphic primitive idempotents of Λ. Let $1, \ldots, n$ denote the corresponding vertices in $\mathscr{Q}(\Lambda)$. By 3.1 (b), $\mathscr{Q}(\Lambda)$ is connected. Thus it suffices to show that if there is a path from i to j then there is a path from j to i. We proceed by induction on the length of a path from i to j. Suppose first there is an arrow from i to j. We show there is a path from j to i. By the definition of $Q(\Lambda)$ we see that $e_j(\boldsymbol{r}/\boldsymbol{r}^2)e_i \neq 0$. Let $S_t = (\Lambda/\boldsymbol{r})e_t$ and $P_t = \Lambda e_t$, for $t = 1, \ldots, n$. Then S_j is a composition factor of P_i. Thus, there exists a nonzero map $f : P_j \to P_i$. Thus Image $(f) \neq 0$. Now $\operatorname{soc}(\Lambda) \cong \Lambda/\boldsymbol{r}$ and Λ basic imply that the socle of each indecomposable projective summand of Λ is simple. Thus $\operatorname{soc}(P_i) \subseteq$ Image (f). We conclude that $\operatorname{soc}(P_i)$ is a composition factor of P_j. Let $\operatorname{soc}(P_i) \cong S_t$. Then by 3.2 there is a path from j to t. We now apply 3.2 together with [**8**, Lemma 5.2] to see that there is a path from t to i. Thus there is a path from j to i. This concludes the case when there is a path from i to j of length 1. The rest of the induction argument is obvious.

As an immediate consequence we get:

COROLLARY 3.4. *If Λ is a left self injective k-algebra then $\mathcal{Q}(\Lambda)$ is path connected. In particular, the quivers of Frobenius algebras are path connected.*

Proof. Since the associated basic ring, Λ^* is still left self-injective, the result follows after noting $\mathcal{Q}(\Lambda) = \mathcal{Q}(\Lambda^*)$ and $\operatorname{soc}(\Lambda^*) \cong \Lambda^*/\boldsymbol{r}^*$ as left Λ^*-modules where $\boldsymbol{r}^* = \operatorname{rad}(\Lambda^*)$.

4. Symmetric algebras with given quivers. The main result of this section is to give a method of explicitly constructing a symmetric k-algebra with a given quiver. In the last section, we saw that if such an algebra exists, its quiver must be path connected (see 3.4). We now show that given a path connected quiver, there is a symmetric k-algebra with that quiver.

For the remainder of this section we fix the following notation: k is a field, $R = \coprod_{i=1}^n k$ and $M = \coprod_{i,j=1}^n k_{ij}^{(n_{ij})}$ as in § 3. We let $T = R + M + \otimes_R^2 M + \ldots$ be the special tensor algebra and $J = M + \otimes_R^2 M + \ldots$.

We now introduce the notion of monomials in T. For this we first choose k-bases $\{x_{ij}^l\}$, $l = 1, \ldots, n_{ij}$, $i, j = 1, \ldots, n$ of the summands $k_{ij}^{(n_{ij})}$ of M. Combining these k-bases we get a k-basis for M. Let $E = \{e_1, \ldots, e_n\}$, the set of idempotents of R. Let B' be the set of nonzero products of the x_{ij}'s. Then B' is a k-basis for J, and $B = B' \cup E$ is a k-basis for T. Scalar multiples of elements of B will be called *monomials in* $\{x_{ij}^l\}$. When no confusion can arise, we will omit any reference to the chosen basis $\{x_{ij}^l\}$ of M. Finally, let $\mathcal{Q} = \mathcal{Q}(T)$ be the quiver of T. We let i be the vertex in $\mathcal{Q}$ corresponding to e_i. Then there are n_{ij} arrows from j to i. It is not hard to see that there is a one to one correspondence between elements of B' and paths in $\mathcal{Q}$. Explicitly, let $\beta : \{x_{ij}^l\} \to \{\text{arrow in } \mathcal{Q}\}$ be an isomorphism such that $\beta(x_{ij})$ is an arrow from j to i. Then β induces a set isomorphism $\hat{\beta} : B' \to \{\text{paths in } \mathcal{Q}\}$ by $\hat{\beta}(f) = (a_1, \ldots, a_m)$ where $f = \beta^{-1}(a_m) \cdots \beta^{-1}(a_1)$. We let $\gamma = \hat{\beta}^{-1}$.

The following definition will be used frequently in what follows. We say a subset S of B is *cyclic* if there is a cycle $c = (a_1, \ldots, a_m)$ such that

(1) every arrow in $\mathcal{Q}$ occurs as some a_i, and

(2) if $f \in B$ then

$$f \in S \iff \hat{\beta}(f) = \begin{cases} c, & \text{or} \\ (a_j, \ldots, a_m, a_1, \ldots, a_{j-1}) & \text{for some } j = 2, \ldots, m. \end{cases}$$

Note that implicit in the definition of a cyclic set S are the choices of $\{x_{ij}^l\}$

and β and that $\mathcal{Q}$ is path connected. If S is a cyclic set and c is a cycle as above, we say that c is a cycle *associated* to S.

We now prove the main result of this section.

THEOREM 4.1. *Let $\mathcal{Q}$ be a connected quiver with at least one arrow. Let T be a special tensor algebra such that $\mathcal{Q}(T) = \mathcal{Q}$. The following statements are equivalent:*

i) *There exists a self-injective k-algebra Λ such that $\mathcal{Q}(\Lambda) = \mathcal{Q}$.*

ii) *$\mathcal{Q}$ is path connected.*

iii) *There is a Frobenius system (S, σ) for T such that S is cyclic.*

iv) *There is a symmetric k-algebra Λ such that $\mathcal{Q}(\Lambda) = \mathcal{Q}$.*

Proof. i) $\Rightarrow$ ii) follows by (3.4). iv) $\Rightarrow$ i) is clear.

ii) $\Rightarrow$ iii). Since $\mathcal{Q}$ is path connected, by (3.1)c), there is a cycle $c = (a_1, \ldots, a_m)$. Let S be a cyclic subset of B having c as its associated cycle. Define $\sigma : S \to k^*$ by $\sigma(f) = 1$ for $f \in S$. We show that (S, σ) is a Frobenius system for T. Clearly S is finite, so that (1.7) holds.

Since S is cyclic, it follows that $f \cdot g \in S \Leftrightarrow g \cdot f \in S$ for all $f, g \in B$. It follows that

(4.2) $\quad \sigma(f \cdot g) = 0 \Leftrightarrow \sigma(g \cdot f) = 0$ for all $f, g \in B$.

Since σ is identically 1 on S and 0 on $B - S$, we see that the linearity of σ and (4.2) imply

(4.3) $\quad \sigma(t \cdot t') = \sigma(t' \cdot t)$ for all $t, t' \in T$.

Property (1.8) easily follows from (4.3) and we conclude that (S, σ) is a Frobenius system for T.

iii) $\Rightarrow$ ii) is clear.

ii) $\Rightarrow$ iv). Let (S, σ) be a Frobenius system for T with S cyclic. As above we choose σ to be identically 1 on S and 0 on $B - S$, so that (4.3) holds. By (1.9), $T/I(\sigma)$ is a Frobenius algebra. Let $\Lambda = T/I(\sigma)$. Let $\bar{\sigma} : \Lambda \to k$ be the linear map induced from σ. We see that $\bar{\sigma}(\lambda \cdot \lambda') = \bar{\sigma}(\lambda' \cdot \lambda)$ for all $\lambda, \lambda' \in \Lambda$. By the discussion in § 1, it follows that Λ is a symmetric algebra.

It remains to show that $\mathcal{Q}(\Lambda) = \mathcal{Q}$. For this, by § 3, it suffices to show that $I(\sigma) \subseteq J^2$ since, if so, then $\operatorname{rad}(\Lambda) = J/I(\sigma)$ and hence $\operatorname{rad}(\sigma)/\operatorname{rad}(\sigma)^2 \cong J/J^2 \cong M$. Thus $\mathcal{Q}(\Lambda) = \mathcal{Q}(R \ltimes M) = \mathcal{Q}$, where $R \ltimes M$ is the trivial extension of R by M.

Let $t \in T - J^2$. We show that $t \notin I(\sigma)$. Let $\delta : T \to \{0, 1, 2, \ldots\}$ be defined by $\delta(e_i) = 0$, $\delta(f) = m$, f is a monomial with $f = \alpha g$, $g \in B'$ and $\hat{\beta}(g)$ is of length m and $\delta(\sum_{i=1}^{r} f_i) = \max(\delta(f_1), \ldots, \delta(f_r))$, where the f_i's are monomials. Now suppose that S is cyclic with associated cycle $c = (a_1, \ldots, a_m)$. First note that if $f \in S$ then $\delta(f) = m$. It follows that $J^{m+1} \subseteq I(\sigma)$. Suppose that

$$t = \sum_{i=1}^{n} \alpha_i e_i + \sum_{x \in \{x_{ij}{}^l\}} \alpha_x x + \sum_i \beta_i h_i$$

where $\alpha_i, \alpha_x, \beta_i \in k$ and $h_i \in B \cap J^2$.

First assume some $\alpha_i \neq 0$. We may assume that $\alpha_1 \neq 0$. There is an $f \in S$ such that $e_1 f = f$. (Choose a reordering of $(a_1, \ldots, a_m)$ so that the domain of a_1 is the vertex corresponding to e_1. Then $f = \beta^{-1}(a_1, \ldots, a_m)$ works.) Now consider tf. Then, since $e_j f = 0$ for $j \neq 1$ and $\delta(f) = m$ it follows that $tf - \alpha_1 e_1 f \in I(\sigma)$. Thus, $\sigma(tf) = \sigma(\alpha_1 f) = \alpha_1 \neq 0$. We conclude that $t \notin I(\sigma)$.

Thus we may assume that $\alpha_i = 0$ for $i = 1, \ldots, n$. Now choose $x \in \{x_{ij}{}^l\}$ such that $\alpha_x \neq 0$. Since the cycle c contains all arrows, for some j, $a_j = \beta(x)$. By reordering the a_i's, we may assume that $a_1 = \beta(x)$. Let $f^* = \gamma(a_1, \ldots, a_m)$ and $g^* = \gamma(a_2, \ldots, a_m)$. Consider $\sigma(g^* t)$. We show that $\sigma(g^* t) \neq 0$, and hence $t \notin I(\sigma)$. Note that $\delta(g^*) = m - 1$. We have

$$g^* \cdot t = \alpha_x f^* + \sum_{\substack{y \in \{x_{ij}{}^l\} \\ y \neq x}} \delta \alpha_y g^* y + \sum_j \beta_j g^* h_j.$$

Now $J^{m+1} \subseteq I(\sigma) \Rightarrow \beta_j g^* h_j \in I(\sigma)$. Thus $\sigma(g^* t) = \alpha_x \sigma(f^*) + \sum \alpha_y \sigma(g^* y)$. It suffices to show that $g^* y \notin S$ for $y \in \{x_{ij}{}^l\} - \{x\}$, since, if so $\sigma(g^* t) = \alpha_x \sigma(f^*) \neq 0$. Assume $g^* y \neq 0$. Now $\hat{\beta}(f^*) = (a_1, \ldots, a_m)$. Since $y \neq x$ we conclude that $\beta(y) \neq a_1$. Say $\beta(y) = a_j$, $j \neq 1$. Then $\hat{\beta}(g^* y) = (a_j, a_2, \ldots, a_m)$. The number of times a_1 occurs in $(a_j, a_2, \ldots, a_m)$ is one less than the number of times a_1 occurs in c. Thus $(a_j, a_2, \ldots, a_m)$ is not a reordering of c and we conclude that $\gamma(a_j, a_2, \ldots, a_m) = g^* y \notin S$.

This completes the proof.

Note that the proof of the theorem explicitly gives a method for constructing symmetric algebras with given quivers. Namely, given a path connected quiver $\mathcal{Q}$ and a field k, choose a cycle c passing through all the arrows of $\mathcal{Q}$. Let T be the special tensor k-algebra with quiver $\mathcal{Q}$. Choose a basis $\{x_{ij}{}^l\}$ of M and an isomorphism $\beta : \{x_{ij}{}^l\} \to \{\text{arrows in } \mathcal{Q}\}$. Then (S, σ), where S is cyclic corresponding to c, $\sigma \equiv 1$ on S, is a Frobenius system that works; i.e. $T/I(\sigma)$ is the desired symmetric algebra.

In the next section we study properties of the Frobenius system (S, σ). We also investigate the uniqueness of $I(\sigma)$ with respect to (S, σ).

5. Properties of $T/I(\sigma)$. Let T be a special tensor k-algebra and let B be a k-basis as in Section 4. In this section we investigate when two Frobenius systems (S, σ) and (S', σ') have the property that $I(\sigma) = I(\sigma')$. We also give some information on the structure of the algebra $T/I(\sigma)$. Our results though give no insight into the question of when two Frobenius systems give rise to isomorphic factor rings.

Let (S, σ) be a Frobenius system. Let $S_{\max}$ be the set of $f \in S$ such that $\langle f \rangle \cap S - \{f\} = \emptyset$.

LEMMA 5.1. *If $f \in B$ such that $\langle f \rangle \cap S_{\max} = \emptyset$, then $f \in I(\sigma)$.*

We leave the proof to the reader. The converse is not true in general.

LEMMA 5.2. *Let (S, σ) and (S', σ') be Frobenius systems in T. If $I(\sigma) = I(\sigma')$ then $S_{\max} = S_{\max}'$.*

Proof. Let $f \in B$. Then $f \in I(\sigma) \Leftrightarrow \sigma(\langle f \rangle) = 0 \Leftrightarrow \langle f \rangle \cap S = \emptyset \Leftrightarrow \langle f \rangle \cap S_{\max} = \emptyset$. The result follows.

We again note that the converse is not true in general.

We say a *monomial has degree m* if it is a product of exactly m elements of $\{x_{ij}{}^l\}$ and a nonzero scalar. If $t \in T$ we say that *t has degree m* if $t = \sum_{i=1}^r f_i$, where the f_i's are nonzero monomials and $m = \max_i\{\text{degree of } f_i\}$. Note that if $t \in T$ then the degree of $t = \delta(t)$, where $\delta : T \to \{0, 1, 2, \ldots\}$ is the function defined in the proof of (4.1).

The condition that $S = S_{\max}$ occurs frequently. For example, if (S, σ) is a Frobenius system with S cyclic then $S = S_{\max}$. This can be easily seen from the fact that $f \in S$ implies that the degree of f is m where m is the length of the cycle associated to S.

PROPOSITION 5.3. *Let (S, σ) and (S', σ') be Frobenius systems in T, with $S = S_{\max}$ and $S' = S_{\max}{}'$. Then $I(\sigma) = I(\sigma')$ if and only if $S = S'$ and for each idempotent $e \in E$ there is a nonzero constant c_e such that $\sigma(fe) = c_e\sigma'(fe)$ for all $f \in B$.*

Proof. First suppose that $I(\sigma) = I(\sigma')$. Then by (5.2) $S = S'$. For each $e \in E$, choose $g_e \in S$ such that $g_e \cdot e = g_e$. (If no such g_e exists then $\sigma(fe) = \sigma'(fe) = 0$ and we may take $c_e = 1$.) Let $h \in S$. If $h \in B - S$ then $\sigma(he) = 0 = c_e\sigma'(he)$, for any c_e. If $he = 0$ we get equality for any choice of c_e. Suppose that $he \neq 0$. Then $he = h$. Let $\phi : T \to T/I(\sigma)$ be the canonical surjection. Let $c_e = \sigma(g_e)/\sigma'(g_e)$. We claim $\sigma(h) = c_e\sigma'(h)$. Now $\phi(h) \in$ socle of $T/I(\sigma)$ since $S = S_{\max}$. Thus $\sigma(h) = \alpha\sigma(g_e)$ since both $\sigma(h)$ and $\sigma(g_e)$ are in the socle of $(T/I(\sigma))e$ which is simple since $T/I(\sigma)$ is a Frobenius algebra. Thus $h - \alpha g_e \in I(\sigma)$. Thus $\sigma(h - \alpha g_e) = 0 = \sigma'(h - \alpha g_e)$. We get $\sigma(h) = \alpha\sigma(g_e)$ and $\sigma'(h) = \alpha\sigma'(g_e)$. But $\sigma(g_e) = c_e\sigma'(g_e)$ and thus $\sigma(h) = c_e\sigma'(h)$.

Now suppose that $S = S'$ and for each $e \in E$ there is a $c_e \in k^*$ such that $\sigma(fe) = c_e\sigma'(fe)$, for all $f \in B$. We only show that $I(\sigma) \subseteq I(\sigma')$. A similar argument gives the other containment. Let $t \in I(\sigma)$ and suppose $t \notin I(\sigma')$. Then $\sigma'(\langle t \rangle) \neq 0$. Choose $f, g \in B$ such that $\sigma'(ftg) \neq 0$. Since $g \in B$ there is $e \in E$ such that $ge = g$. Thus $\sigma'(ftg) = \sigma'(ftge)$. Thus $\sigma(ftge) = c_e\sigma'(ftge) \neq 0$ by the linearity of σ. We conclude that $\sigma(ftge) \neq 0$, a contradiction.

We now turn to the structure of $T/I(\sigma)$ for a Frobenius system (S, σ). We say that S is *homogeneous of degree m* if each element of S has degree m. Note that if S is homogeneous of degree m then $S = S_{\max}$. Furthermore, if S is cyclic then S is homogeneous of degree m where m is the length of a cycle associated to S. We now state, without proof, a result showing that the "homogeneity" of S is reflected in $T/I(\sigma)$.

PROPOSITION 5.4. *Let T be a special tensor k-algebra with a path connected quiver. Suppose that (S, σ) is a Frobenius system in T with S homogeneous of degree m. Let $t \in T$ with $t = \sum_{i=0}^N t_i$, where t_i are sums of monomial of degree i. Then*

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(1) *$t \in I(\sigma) \Leftrightarrow t_i \in I(\sigma)$ for all i.*

(2) *If j is the smallest integer such that $t_j \in I(\sigma)$ then the image of t in $T/I(\sigma)$ is in* $\operatorname{rad}(T/I(\sigma))^j - \operatorname{rad}(T/I(\sigma))^{j+1}$.

(3) *Each indecomposable projective $T/I(\sigma)$-module has Loewy length m; that is,* $\operatorname{rad}(T/I(\sigma))^{m+1}$ *annihilates $T/I(\sigma)$ but* $\operatorname{rad}(T/I(\sigma))^m e \neq 0$ *for each nonzero idempotent e in $T/I(\sigma)$.*

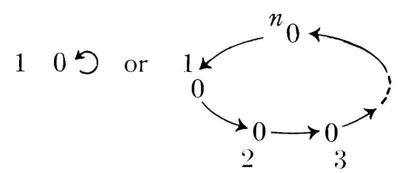
6. Classification of generalized uniserial quasi-Frobenius algebras. We recall that a left and right Artin ring Λ is called *left* (respectively, *right*) *generalized uniserial* if for each primitive idempotent e, Λe (respectively, $e\Lambda$) has a unique composition series. A ring is called *generalized uniserial* if it is left and right generalized uniserial. The following well-known result classifies generalized uniserial Artin rings.

PROPOSITION 6.1 [**10**]. *Let Λ be an Artin ring with radical r. Then the following statements are equivalent:*

(1) *Λ is generalized uniserial.*

(2) *For each primitive idempotent e of Λ, we have* (i) *$\mathbf{r}e/\mathbf{r}^2e$ is a simple left Λ-module and* (ii) *$e\mathbf{r}/e\mathbf{r}^2$ is a simple right Λ-module.*

Our aim is to translate (6.1) in the setting of special tensor k-algebras and quivers. Recall that a k-algebra Λ is called *quasi-Frobenius* if Λ is an Artin ring and Λ is selfinjective as a left and as a right Λ-module. A quiver $\mathcal{Q}$ is said to be of *type Z_n* if it is either



for $n \geqq 2$. The following result is an immediate consequence of (6.1).

COROLLARY 6.2. *Let Λ be a generalized uniserial indecomposable quasi-Frobenius Artin rnig. Then $\mathcal{Q}(\Lambda)$ is of type Z_n.*

Proof. Let $e_1, \ldots, e_n$ be a full set of nonisomorphic primitive idempotents of Λ and let $1, \ldots, n$ be the corresponding vertices in $\mathcal{Q}(\Lambda)$. Since $\mathbf{r}e_i/\mathbf{r}^2e_i$ and $e_i\mathbf{r}/e_i\mathbf{r}^2$ are simple modules, we see there is exactly one arrow entering and leaving each vertex i. By (3.4), $\mathcal{Q}(\Lambda)$ is path connected and the result easily follows.

We also have the following result.

PROPOSITION 6.3. *Let T be an indecomposable special tensor k-algebra with quiver $\mathcal{Q}$. Then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is of type Z_n.*

(2) *There is an ideal I in T such that $J^N \subseteq I \subseteq J^2$ for some N and T/I is a generalized quasi-Frobenius algebra.*

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Proof. (2) $\Rightarrow$ (1) follows fom (6.2) since $\mathcal{Q}(T/I) = \mathcal{Q}$.

(1) $\Rightarrow$ (2). Clearly, if $\mathcal{Q}$ is of type Z_n then $\dim_k(J/J^2)e = \dim_k e(J/J^2) = 1$ for all primitive idempotents e of R. By (4.1), since $\mathcal{Q}$ is path connected, we may construct $T/I(\sigma)$, where (S, σ) is a Frobenius system with S cyclic and $\sigma \equiv 1$ on S. As in (4.1), it is easy to show that $I(\sigma) \subseteq J^2$ and hence $T/I(\sigma)$ is generalized uniserial by (6.1). This completes the proof.

In fact, we will show that if $\mathcal{Q}$ is of type Z_n and I is an ideal in T such that T/I is quasi-Frobenius, generalized uniserial and $J^N \subseteq I \subseteq J^2$, then $I = I(\sigma)$ for some Frobenius system (S, σ). Before proving this we need one more definition. Keeping the notation of section 4, we say that a subset S of B is *complete and homogeneous of degree m* if $S = (B \cap J^m) - J^{m+1}$; that is, S is a k-basis of $\otimes_R^m M$. Note that if S is cyclic it need not be complete and homogeneous of degree m, for all m.

We now establish the result mentioned earlier.

THEOREM 6.4. *Let T be an indecomposable special tensor k-algebra with quiver $\mathcal{Q}$ of type Z_n. Let I be an ideal in T. The following statements are equivalent:*

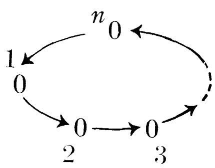
(1) *T/I is quasi-Forbenius and $J^N \subseteq I \subseteq J^2$ for some $N \geqq 2$.*

(2) *$I = J^N$ for some $N \geqq 2$.*

(3) *$I = I(\sigma)$ for some Frobenius system (S, σ) where S is complete and homogeneous of degree $N - 1$, for some $N \geqq 2$ and $\sigma \equiv 1$ on S.*

Proof. (1) $\Rightarrow$ (2). Let I be such that $J^N \subseteq I \subseteq J^2$ and T/I is quasi-Frobenius. Assume $J^{N-1} \not\subseteq I$. It follows that there is an indecomposable projective T/I-module Loewy length N, since $\mathrm{rad}(T/I)^N = 0$ but $\mathrm{rad}(T/I)^{N-1} \neq 0$.

Let $e_1, \ldots, e_n$ be a full set of nonisomorphic idempotents of R and let i be the vertex in $\mathcal{Q}$ corresponding to e_i. The quiver $\mathcal{Q}$ being of type Z_n allows us to reorder the e_i's so that $\mathcal{Q}$ is



Let $\Gamma = T/I$ and $\boldsymbol{r} = \mathrm{rad}(\Gamma)$. Let X be a Γ-module. Let $ll(X)$ denote the Loewy length of X. Suppose that $ll(\Gamma e_i) = N$. We claim that if $0^i \rightarrow 0^j$ in $\mathcal{Q}$ then $ll(\Gamma e_j) \geqq N$. Let $S_i = \Gamma e_i/\boldsymbol{r}e_i$. If $0^i \rightarrow 0^j$ then $S_j \cong \boldsymbol{r}e_i/\boldsymbol{r}^2 e_i$. Thus there is a nonzero map $\Gamma e_j \rightarrow \boldsymbol{r}e_i$. By the uniseriality of $\boldsymbol{r}e_i$, we conclude that $\Gamma e_j \rightarrow \boldsymbol{r}e_i$ is a surjection. But since Γ is quasi-Frobenius, $\mathrm{soc}(\Gamma e_j) \not\cong \mathrm{soc}(\boldsymbol{r}e_i)$ and hence $\mathrm{soc}(\Gamma e_j) \subseteq \ker(\Gamma e_j \rightarrow \boldsymbol{r}e_i)$. Thus $ll(\Gamma e_j) \geqq ll(\boldsymbol{r}e_i) + 1 = N$. This proves the claim. Since $\mathcal{Q}$ is of type Z_n, and since for some i, $ll(\Gamma e_i) = N$, we conclude that $ll(\Gamma e_i) = N$ for all i.

Now $J^N \subseteq I$ implies there is a canonical surjection $T/J^N \rightarrow T/I$. It suffices to show this is an isomorphism. Now, since $\mathcal{Q}$ is of type Z_n, we see that

$\operatorname{soc}(T/J^N) \cong J^{N-1}/J^N \cong \otimes_R^{N-1} M \cong T/J$. Thus, by [**9**, Prop. 2.5] T/J^N is quasi-Frobenius. Furthermore, it is not hard to see that the Loewy length of each indecomposable projective T/J^N-module is exactly N. It now follows that the canonical surjection $T/J^N \to T/I$ is an isomorphism, since if not, $\{\ker(T/J^N \to T/I\} \cap \operatorname{soc}(T/J^N) \neq 0$ implying that some indecomposable projective T/I-module has Loewy length less than N.

(2) $\Rightarrow$ (3). Let (S, σ) where S is complete and homogeneous of degree $N - 1$ and $\sigma \equiv 1$ on S. Noting that, since $\mathscr{Q}$ is of type Z_n, given a vertex i, there is exactly one path beginning at i of length m, it is not hard to show that (S, σ) is a Frobenius system. We omit the details since they are analogous to those in the proof of (4.1). Now $J^N \subseteq I(\sigma)$ since if $f \in B$ and the degree of f is greater than or equal to N then $f \in I(\sigma)$. Again consider $T/J^N \to T/I(\sigma)$. By (5.4) every indecomposable projective $T/I(\sigma)$-module has Loewy length N. We now apply the argument given at the end of (1) $\Rightarrow$ (2) above to show $I(\sigma) = J^N$.

(3) $\Rightarrow$ (1). By (6.3) one needs only show that $I(\sigma) \subseteq J^2$. This again is similar to arguments given in (4.1) and we omit the details.

We remark that it is not hard to show that if $\mathscr{Q}$ is path connected but not of type Z_n and S is complete and homogeneous of degree N, in general we cannot form a Frobenius system (S, σ) with $\sigma \equiv 1$.

7. The representation type of Frobenius systems. The purpose of this section is to show that except for the generalized uniserial algebras, if (S, σ) is a Frobenius system with S cyclic then $T/I(\sigma)$ is of infinite representation type. We begin with a well-known result whose proof we include for completeness.

PROPOSITION 7.1. *Let Λ be an Artin ring and let e be an idempotent in Λ. If Λ is of finite representation type then $e\Lambda e$ is also of finite representation type.*

Proof. Let $e' = 1 - e$. Let Γ be the product ring $e\Lambda e \times e'\Lambda e'$. Let $\phi : \Gamma \to \Lambda$ be defined by $\phi(x, y) = x + y$. Since ϕ is a ring map, we have two functors F: $\operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$, the forgetful functor and G: $\operatorname{mod}(\Gamma) \to \operatorname{mod}(\Lambda)$, given by $G(M) = \Lambda \otimes_\Gamma M$, where $\operatorname{mod}(\Lambda)$ and $\operatorname{mod}(\Gamma)$ are the categories of finitely generated left modules. Let $M \in \operatorname{mod}(\Gamma)$. Then $FG(M) = \Lambda \otimes_\Gamma M$, viewed as a Γ-module, decomposes into $(e\Lambda \otimes_\Gamma M) \otimes (e'\Lambda \otimes_\Gamma M)$. Furthermore $M = eM \oplus e'M$ as Γ-modules. Thus eM is a summand $e\Lambda \otimes_\Gamma M$. Similarly $e'M$ is a summand of $e'\Lambda \otimes_\Gamma M$. We conclude that M is a summand of $FG(M)$. Now, if Λ is of finite representation type, we see there are only a finite number of nonisomorphic finitely generated Γ-modules isomorphic to $F(X)$ for some indecomposable $X \in \operatorname{mod}(\Lambda)$. We conclude then that Γ must be of finite representation type. Since Γ is a product, one factor being $e\Lambda e$, we get the desired result.

COROLLARY 7.2. *Let Λ be an Artin ring and let e be a primitive idempotent in Λ. If Λ is of finite representation type then $e\,\boldsymbol{r}\,e$ is a uniserial $e\Lambda e$-module, where* $\boldsymbol{r} = \operatorname{rad}(\Lambda)$.

Proof. Suppose Λ is of finite representation type. Then, so is $e\Lambda e$. Since e is a primitive idempotent, we see that $e\Lambda e$ is a local Artin ring. A local Artin ring A is of finite representation type if and only if $\operatorname{rad}(A)/\operatorname{rad}(A)^2$ is a simple (left and right) A-module (see [**3**]). Thus, by 6.1, the result follows after noting that $\operatorname{rad}(e\Lambda e) = e\,\boldsymbol{r}\,e$.

We now prove the main result of this section.

THEOREM 7.3. *Let T be a special tensor k-algebra such that $\mathscr{Q}(T)$ is path connected. Let (S, σ) be a Frobenius system with S cyclic. Let $I(\sigma)$ be the ideal associated to σ. The following statements are equivalent:*

(1) *The Frobenius algebra $T/I(\sigma)$ is of finite representation type.*

(2) *$\mathscr{Q}(T)$ is of type Z_n.*

(3) *$T/I(\sigma)$ is a generalized uniserial quasi-Frobenius algebra.*

Proof. By (6.3) and (6.4), (2) $\Leftrightarrow$ (3). (3) $\Rightarrow$ (1) is clear.

It remains to show that (1) $\Rightarrow$ (2). Assume that $\mathscr{Q}(T)$ is not of type Z_n. We show that there is an idempotent e such that $e(\operatorname{rad}(T/I(\sigma))e$ is not uniserial. Let $\Gamma = T/I(\sigma)$ and $\boldsymbol{r} = \operatorname{rad}(\Gamma)$. The assumption that $\mathscr{Q}(T)$ is not of type Z_n implies that there is some vertex i with at least two arrows entering i or at least two arrows leaving i. Let $c = (a_1, \ldots, a_m)$ be a cycle such that each arrow occurs as some a_j. Let (S, σ) be a Frobenius system with c associated to S.

For simplicity assume that there are two arrows a and b having vertex i as domain. Then $a = a_j$ and $b = a_l$ for some j and l. By renumbering the a_i's we may assume that $a = a_1$ and $b = a_j$. Let u, v be chosen as follows:

i) $1 \leqq u, j \leqq v$,

ii) codomain of a_u = vertex i = codomain of a_v, and

iii) if $1 \leqq l < u$ or $j \leqq l < v$ then codomain $a_l \neq$ vertex i.

Using the notation of Section 4, let $f = \gamma(a_1, \ldots, a_u)$ and $g = \gamma(a_j, \ldots, a_v)$. Then we see that $efe = f$ and $ege = g$ since $(a_1, \ldots, a_u)$ and $(a_j, \ldots, a_v)$ are cycles from i to i. Furthermore $f \neq g$ since $a_1 \neq a_j$. Finally, by the choice of u and v, we see that f and g are not proper products of elements h in B' such that $ehe = h$. Let $\phi: T \rightarrow T/I(\sigma)$ be the canonical surjection. Then $\phi(f), \phi(g) \in e\,\boldsymbol{r}\,e$. Thus $e\,\boldsymbol{r}\,e/(e\,\boldsymbol{r}\,e)^2$ is not simple (since $\phi(f) \neq \phi(g) \Rightarrow \phi(f)$ and $\phi(g)$ are k-independent using the fact that $f, g \in B'$ and that B' is a k-basis of J). This completes the proof.

As mentioned in the introduction, under certain restrictions on $\mathscr{Q}(T)$, one may construct Frobenius systems so that the resulting Frobenius algebras are of finite representation type and yet are not generalized uniserial. Such considerations will be dealt with in a future paper. We should mention though at this time there is no known necessary and sufficient conditions on the quiver of a Frobenius k-algebra which imply that it is of finite representation type.

8. Concluding remarks. We first remark that there should be a generalization of the notion of a Frobenius system which would allow one to classify quasi-Frobenius k-algebras. Clearly condition (1.8) must be weakened appropriately.

As pointed out earlier, the question of isomorphic algebras has been avoided. We mention just a few open questions whose answers would be of great interest.

1) Given a special tensor k-algebra T and bases B and B^*, find necessary and sufficient conditions on Frobenius systems (S, σ) with respect to B and (S^*, σ^*) with respect to B^* so that $T/I(\sigma)$ is isomorphic to $T/I(\sigma^*)$.

2) If (S, σ) is a Frobenius system in T, is $T/I(\sigma)$ isomorphic to $T/I(\sigma^*)$ for some Frobenius system (S^*, σ^*) where either $S^* = S_{\max}{}^*$ or $\sigma^* \equiv 1$ on S^*?

Finally, we list some other open questions.

3) Given a special tensor algebra T with path connected quiver $\mathcal{Q}$, find all Frobenius systems in T with respect to some fixed basis B.

4) Are there necessary and sufficient conditions on a quiver $\mathcal{Q}$ such that there is a Frobenius algebra of finite representation type with that quiver?

As shown in Section 7, the algebras constructed by cycles are "too big".

5) Given a path connected quiver $\mathcal{Q}$ find methods of constructing Frobenius systems so that $T/I(\sigma)$ is of finite representation type.

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