

RESEARCH ARTICLE

Finite subgroups of automorphisms of K3 surfaces

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Abstract

We give a complete classification of finite subgroups of automorphisms of K3 surfaces up to deformation. The classification is in terms of Hodge theoretic data associated to certain conjugacy classes of finite subgroups of the orthogonal group of the K3 lattice. The moduli theory of K3 surfaces, in particular the surjectivity of the period map and the strong Torelli theorem allow us to interpret this datum geometrically. Our approach is computer aided and involves Hermitian lattices over number fields.

Contents

1	Introduction	2
2	Preliminaries on lattices and isometries	6
2.1	Lattices	6
2.2	Primitive extensions and glue	6
2.3	Lattices with isometry	7
3	K3 surfaces with a group of automorphisms	7
3.1	H-markings	7
3.2	Moduli spaces and periods	9
3.3	Connected components	10
3.4	Saturated effective subgroups	12
3.5	Enumerating effective characters	13
4	Conjugacy classes of isometries	14
4.1	Hermitian lattices and transfer	14
4.2	Glue estimates	16
4.3	Enumeration of conjugacy classes of isometries.	20
4.4	Computation of the group G_L	25
5	Quadratic Miranda–Morrison theory	26
6	Hermitian Miranda–Morrison theory	28
6.1	Preliminaries on Hermitian lattices	29
6.2	The trace lattice	29
6.3	Discriminant form and transfer	31
6.4	Local surjectivity of $U(L) \rightarrow U(D_L)$	32

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6.5 Local to global 33
 6.6 Generation of $U^\sharp(\mathbf{L})$ by symmetries 35
 6.7 Determinants of the kernel 37
 6.8 Computing in $\mathcal{F}(L_p)/\mathcal{F}^\sharp(L_p)$ 38
7 Fixed points **38**
A Finite groups with mixed action on a K3 surface **42**
B Fixed loci of purely nonsymplectic automorphisms **45**

1. Introduction

We work over the field \mathbb{C} of complex numbers. A *K3 surface* is a compact, complex manifold X of dimension 2 with a nowhere vanishing symplectic, holomorphic 2-form $\sigma_X \in H^0(X, \Omega_X^2)$ and vanishing irregularity $h^1(X, \mathcal{O}_X)$.

Since a K3 surface does not admit nontrivial global vector fields, its automorphism group is discrete. For a very general K3 surface it is even trivial. However, there are (families of) K3 surfaces with a nontrivial and even infinite automorphism group. Typical examples of groups that appear as automorphism groups are $\mathbb{Z} * \mathbb{Z}$, \mathbb{Z}' or $\mathbb{Z}/2\mathbb{Z}$.

K3 surfaces with a finite automorphism group have been classified by Nikulin [61, 62], Vinberg [75] and Kondo [47] with a further recent refinement due to Roulleau [70]. The purpose of this work is to classify finite subgroups $G \leq \text{Aut}(X)$, more precisely, pairs (X, G) consisting of a K3 surface X and a finite subgroup of automorphisms $G \leq \text{Aut}(X)$.

Let X be a K3 surface with cotangent sheaf Ω_X . Its automorphisms act on the symplectic forms $\mathbb{C}\sigma_X = H^0(X, \Omega_X^2)$ by scalar multiplication. We call the ones with trivial action *symplectic* and the ones with a nontrivial action *nonsymplectic*. The action on the symplectic form gives rise to an exact sequence

$$1 \rightarrow \text{Aut}_s(X) \rightarrow \text{Aut}(X) \rightarrow \text{GL}(\mathbb{C}\sigma_X),$$

where by $\text{Aut}_s(X)$ we denote the normal subgroup of symplectic automorphisms. Now, let $G \leq \text{Aut}(X)$ be a finite subgroup and set $G_s = G \cap \text{Aut}_s(X)$. Then for $n = |G/G_s|$, we get an exact sequence

$$1 \rightarrow G_s \rightarrow G \rightarrow \mu_n \rightarrow 1,$$

where μ_n is the cyclic group of order n . We call the index $n = [G : G_s]$ the *transcendental value* of G . Let φ denote Euler’s totient function. By a result of Oguiso and Machida [51], we know that $\varphi(n) \leq 20$ and $n \neq 60$.

The distinction between symplectic and nonsymplectic is crucial. For instance, a nonsymplectic automorphism of finite order may fix smooth curves whereas a symplectic automorphism of finite order k fixes only finitely many points and their number n_k depends only on k . Let M_{24} denote the Mathieu group on 24 points and M_{23} the stabilizer group of a point. Then the number of fixed points of an element of M_{23} of order k depends only on k and is equal to n_k . This observation sparked the following theorem of Mukai.

Theorem 1.1 [57]. *A finite group admits a faithful and symplectic action on some K3 surface if and only if it admits an embedding into the Mathieu group M_{23} which decomposes the 24 points into at least five orbits.*

Later, Xiao [76] gave a new proof shedding light on the combinatorics of the fixed points using the relation between X , its quotient X/G_s and its resolution which is a K3 surface again. A conceptual proof involving the Niemeier lattices was given by Kondo [48]. Finally, Hashimoto [34] classified all the symplectic actions on the K3 lattice. Since the corresponding period domains are connected, this is a classification up to deformation. See [49] for a survey of symplectic automorphisms.

In view of these results, it is fair to say that our knowledge of finite symplectic subgroups of automorphisms is fairly complete.

Similar to Hashimoto’s classification for symplectic actions, our main result is a classification up to deformation (see Definition 3.1 for a precise definition). Let X be a K3 surface and G a finite subgroup of automorphisms of X . We call the largest subgroup $S \leq \text{Aut}(X)$ such that the fixed lattices satisfy $H^2(X, \mathbb{Z})^{G_S} = H^2(X, \mathbb{Z})^S$ the *saturation* of G_S . Necessarily, the group S is finite and symplectic. The group generated by G and S is finite as well. It is called the saturation of G . We call G saturated if it is equal to its saturation. The subgroup $G \leq \text{Aut}(X)$ is called nonsymplectic, if $G_S \neq G$ and mixed if further $1 \neq G_S$. If $G_S = 1$, then it is called purely nonsymplectic. If G is nonsymplectic, then X is in fact projective. Therefore, all K3 surfaces are henceforth assumed to be projective.

Theorem 1.2. *There are exactly 4, 167 deformation classes of pairs (X, G) consisting of a complex K3 surface X and a saturated, nonsymplectic, finite subgroup $G \leq \text{Aut}(X)$ of automorphisms. For each such pair, the action of G on some lattice $L \cong H^2(X, \mathbb{Z})$ is listed in [21].*

While a list of the actions of these finite groups G is too large to reproduce here, we present a condensed version of the data in Table 3 in Appendix A. More precisely, we list all finite groups G admitting a faithful, saturated, mixed action on some K3 surface and their symplectic subgroups as well as the number $k(G)$ of deformation types.

Since the natural representation $\text{Aut}(X) \rightarrow O(H^2(X, \mathbb{Z}))$ is faithful and K3 surfaces are determined up to isomorphism by their Hodge structure, a large extent of geometric information is easily extracted from our Hodge-theoretical model of the family of surfaces and its subgroup of automorphisms. For instance, one can compute the Néron–Severi and transcendental lattice of a very-general member of the family, the invariant lattice, invariant ample polarizations, the (holomorphic and topological) Euler characteristic of the fixed locus of an automorphism, the isomorphism class of the group, its subgroup consisting of symplectic automorphisms, the number of connected components of the moduli space and the dimension of the moduli space. The pairs (X, G) with G_S among the 11 maximal groups have been classified in [20]. In many cases, projective models are listed.

Purely nonsymplectic automorphisms

On the other end of the spectrum are purely nonsymplectic groups, which are the groups G with $G_S = 1$. These groups satisfy $G \cong \mu_n$ and, by a result of Oguiso and Machida [51], we know that n satisfies $\varphi(n) \leq 20$ and $n \neq 60$. To the best of our knowledge, the following corollary completes the existing partial classifications for orders 4 [7], 6 [28], 8 [2], 16 [3], 20, 22, 24, 30 [6], n with $(\varphi(n) \geq 12)$ [18] and is completely new for orders 10, 12, 14 and 18. For order 26, it provides a missing case in [18, Thm 1.1]. For order 6, [28, Thm. 4.1] misses the case of a genus 1 curve and four isolated fixed points (0.6.2.29).

Corollary 1.3. *Let $k(n)$ be the number of deformation classes of K3 surfaces with a purely nonsymplectic automorphism acting by ζ_n on the symplectic form. The values $k(n)$ are given in Table 1.*

The most satisfying picture is for nonsymplectic automorphisms of odd prime order, where the fixed locus alone determines the deformation class; see [8]. The key tools to this result are the holomorphic and topological Lefschetz’ fixed point formulas as well as Smith theory. These relate properties of the fixed locus with the action of the automorphism on cohomology. Conversely, given the action on cohomology as per our classification, we determine its fixed locus.

Table 1. *Counts of purely nonsymplectic automorphisms.*

n	$k(n)$	n	$k(n)$	n	$k(n)$	n	$k(n)$	n	$k(n)$	n	$k(n)$	n	$k(n)$
2	75	8	38	14	12	20	7	27	2	36	3	50	1
3	24	9	13	15	8	21	4	28	3	38	2	54	1
4	79	10	37	16	7	22	4	30	10	40	1	60	0
5	7	11	3	17	1	24	9	32	2	42	3	66	1
6	150	12	48	18	16	25	1	33	1	44	1		
7	5	13	1	19	1	26	3	34	2	48	1		

Table 2. Exceptional types of purely nonsymplectic automorphisms.

n	\mathcal{G}_1	\mathcal{G}_2	\mathcal{G}_3	n	\mathcal{G}_1	\mathcal{G}_2	\mathcal{G}_3
6	$\Pi_{1,8}2_1^9$	$\Pi_{1,17}3^{-3}$	$\Pi_{1,8}2_1^9$	6	$\Pi_{1,7}2_2^8$	$\Pi_{1,15}3^3$	$\Pi_{1,7}2_2^8$
6	$\Pi_{1,5}2_4^6$	$\Pi_{1,15}3^{-4}$	$\Pi_{1,7}2_4^6$	6	$\Pi_{1,3}2_6^4$	$\Pi_{1,11}3^5$	$\Pi_{1,3}2_6^4$
6	$\Pi_{1,2}2_7^3$	$\Pi_{1,9}3^{-6}$	$\Pi_{1,3}2_7^3$	4	$\Pi_{1,9}2^8$	$\Pi_{1,17}2^2$	-

In what follows, σ is an automorphism of order n on a K3 surface acting by multiplication with $\zeta_n = \exp(2\pi i/n)$ on the holomorphic 2-form of X . We denote by

$$X^\sigma = \{x \in X \mid \sigma(x) = x\}$$

the fixed point set of σ on X . A curve $C \subseteq X$ is fixed by σ if $C \subseteq X^\sigma$, and it is called invariant by σ if $\sigma(C) = C$. Let $P \in X^\sigma$ be a fixed point. By [24, lemme 1], σ can be linearized locally at P . Hence, there are local coordinates (x, y) in a small neighborhood centered at P such that

$$\sigma(x, y) = (\zeta_n^{i+1}x, \zeta_n^{-i}y) \quad \text{with} \quad 0 \leq i \leq s = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We call P a fixed point of type i and denote the number of fixed points of type i by a_i . If $i = 0$, then P lies on a smooth curve fixed by σ . Otherwise, P is an isolated fixed point. Note that for $n = 2$ there are no isolated fixed points and at most 2 invariant curves pass through a fixed point.

In general, the fixed point set X^σ is a disjoint union of $N = \sum_{i=1}^s a_i$ isolated fixed points, k smooth rational curves and either a curve of genus > 1 or 0, 1, 2 curves of genus 1. Denote by l the number of genus $g \geq 1$ curves fixed by σ . If no such curve is fixed, set $g = 1$. We describe the fixed locus by the tuple $((a_1, \dots, a_s), k, l, g)$. It is a deformation invariant. To sum up:

$$X^\sigma = \{p_1, \dots, p_N\} \sqcup R_1 \sqcup \dots \sqcup R_k \sqcup C_1 \sqcup \dots \sqcup C_l,$$

where the R_i 's are smooth rational curves and the C_j 's smooth curves of genus $g \geq 1$.

Let L be a \mathbb{Z} -lattice and $f \in O(L)$ an isometry of order n . Set $L_k := \{x \in L \mid f^k(x) = x\}$. The small local type of f is the collection of genera $\mathcal{G}(L_k)_{k|n}$. If the genus of L is understood, then we omit $\mathcal{G}(L_n) = \mathcal{G}(L)$ from notation. Let $\Phi_k(x) \in \mathbb{Z}[x]$ denote the k -th cyclotomic polynomial. The global type of f consists of the small local type as well as the isomorphism classes of the \mathbb{Z} -lattices $\ker \Phi_k(f)$, where $k \mid n$. A genus of \mathbb{Z} -lattices is denoted by its Conway–Sloane symbol [26]. The type of an automorphism σ of a K3 surface X is defined as the type of the isometry $\sigma^{*-1} | H^2(X, \mathbb{Z})$.

Theorem 1.4. *Let X be a K3 surface and $\sigma \in \text{Aut}(X)$ of order n acting by ζ_n on $H^0(X, \Omega_X^2)$. The deformation class of (X, σ) is determined by the small local type of σ unless σ is one of the six exceptional types in Table 2. For each deformation class, the invariants $((a_1, \dots, a_s), k, l, g)$ of the fixed locus are given in Appendix B.*

Remark 1.5. For each of the 5 exceptional types of order 6, there are exactly two deformation classes. They are separated by the global type. For order 4, the two classes have the same global-type. They are separated by the isometry class of the glue between L_2 and L_4 . It is given by the image $L_4 \rightarrow D_{L_2}$ induced by orthogonal projection.

Remark 1.6. The first 5 exceptional types in Table 2 are due to a failure of the local to global principle for conjugacy of isometries. For the last one, the example shows that the small local type is not fine enough to determine local conjugacy.

Enriques surfaces

Since the universal cover X of a complex Enriques surface S is a K3 surface, our results apply to classify finite subgroups of automorphisms of Enriques surfaces. The kernel of $\text{Aut}(S) \rightarrow \text{GL}(H^0(2K_S))$ consists of the so called semisymplectic automorphisms of S . They lift to automorphisms acting by ± 1 on $H^0(X, \Omega_X^2)$. Cyclic semisymplectic automorphisms are studied by Ohashi in [65]. Mukai’s theorem on symplectic actions and the Mathieu group has an analogue for Enriques surfaces; see [58]. However, not every semisymplectic action is of ‘Mathieu type’.

Corollary 1.7. *A group H_S admits a faithful semisymplectic action on some complex Enriques surface if and only if H_S embeds into one of the following 6 groups*

G	id	G	id	G	id
A_6	(360, 118)	H_{192}	(192, 955)	S_5	(120, 34)
$A_{4,4}$	(288, 1026)	2^4D_{10}	(160, 234)	N_{72}	(72, 40)

A group H admits a faithful action on some complex Enriques surface if and only if it embeds into one of the following 9 groups:

G	id	G	id	G	id
A_{6,μ_2}	(720, 765)	H_{192}	(192, 955)	$\Gamma_{25}a_1 \cdot \mu_2$	(128, 929)
$A_{4,4,\mu_2}$	(576, 8652)	$N_{72} \cdot \mu_2$	(144, 182)	S_5	(120, 34)
$2^4D_{10} \cdot \mu_2$	(320, 1635)	$(Q_8 * Q_8) \cdot \mu_4$	(128, 135)	$(C_2 \times D_8) \cdot \mu_4$	(64, 6)

Proof. Let S be an Enriques surface and X its covering K3 surface. Let ϵ be the covering involution of $X \rightarrow S$. Let $\text{Aut}(X, \epsilon)$ denote the centralizer of ϵ . Then $1 \rightarrow \langle \epsilon \rangle \rightarrow \text{Aut}(X, \epsilon) \rightarrow \text{Aut}(S) \rightarrow 1$ is exact. In particular, if $H \leq \text{Aut}(S)$ is a finite group, then it is the image of a finite group $G \leq \text{Aut}(X)$ containing the covering involution. Conversely, $\epsilon \in \text{Aut}(X)$ is the covering involution of some Enriques surface if and only if $H^2(X, \mathbb{Z})^\epsilon \in \Pi_{1,9}2^{10}$. Thus, we can obtain the list of all finite groups acting on some Enriques surface by taking the corresponding list for K3 surfaces. For each group G in the list, one computes the Enriques involutions ϵ , their centralizer $C(\epsilon)$ in G , $C(\epsilon)_S \cong H_S$ and the quotient $H \cong C(\epsilon)/\langle \epsilon \rangle$. \square

Our method of classification applies as soon as a Torelli-type theorem is available, for instance to supersingular K3 surfaces in positive characteristic and compact hyperkähler manifolds.

Outline of the paper

In Section 2, we recall basic notions of lattices with an emphasis towards primitive extensions and lattices with isometry. The geometric setting of K3 surfaces is treated in Section 3. We set up a coarse moduli space parametrizing K3 surfaces together with finite subgroups of automorphisms. Next, we determine the connected components of the respective moduli spaces. We show that this translates the problem of classifying pairs of K3 surfaces and finite subgroups of automorphisms into a classification problem for lattices with isometry and extensions thereof.

The next sections deal with these algorithmic problems related to lattices, where it is shown that practical solutions exist. In particular, in Section 4 it is described how isomorphism classes of lattices with isometry can be enumerated. This leads to questions related to canonical images of orthogonal and unitary groups, which are addressed in the final sections. For the classical case of \mathbb{Z} -lattices, we review Miranda–Morrison theory in Section 5. For Hermitian lattices we develop the necessary tools in Section 6.

Finally, in Section 7 we classify the fixed point sets of purely nonsymplectic automorphisms of finite order on complex K3 surfaces.

2. Preliminaries on lattices and isometries

In this section, we fix notation on lattices, and refer the reader to [60, 26, 46] for standard facts and proofs.

2.1. Lattices

Let R be an integral domain of characteristic 0 and K its field of fractions. We denote by R^\times its group of units. In this paper, an R -lattice consists of a finitely generated projective R -module M and a nondegenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle_M \times M \rightarrow K$.

We call it integral if the bilinear form is R -valued, and we call it even if the square-norm of every element with respect to the bilinear form is in $2R$. If confusion is unlikely, we drop the bilinear form from notation and denote for $x, y \in M$ the value $\langle x, y \rangle$ by xy and $\langle x, x \rangle$ by x^2 . The associated quadratic form is $Q(x) = x^2/2$. We denote the dual lattice of M by M^\vee . We call M unimodular if $M = M^\vee$. For two lattices M and N , we denote by $M \perp N$ their orthogonal direct sum. The scale of M is $\mathfrak{s}(M) = \langle M, M \rangle$, and its norm $\mathfrak{n}(L)$ is the fractional ideal generated by $\langle x, x \rangle$ for $x \in M$. The set of self-isometries of M is the orthogonal group $O(M)$ of M .

We fix the following convention for the spinor norm: Let L be an R -lattice and $V = L \otimes K$. Let $v \in V$ with $v^2 \neq 0$. The reflection $\tau_v(x) = x - 2xv/v^2 \cdot v$ is an isometry of V . The spinor norm of τ_v is defined to be $Q(v) = v^2/2 \in k^\times/(k^\times)^2$. By the Cartan–Dieudonné theorem $O(V)$ is generated by reflections. One can show that this defines a homomorphism $\text{spin} : O(V) \rightarrow k^\times/(k^\times)^2$ by using the Clifford algebra of (V, Q) .

An embedding $M \rightarrow L$ of lattices is said to be primitive if its cokernel is torsion-free. For $M \subseteq L$, we denote by $M^\perp = \{x \in L \mid \langle x, M \rangle = 0\}$ the maximal submodule of L orthogonal to M . If confusion is unlikely, we denote it simply by M^\perp . The minimum number of generators of a finitely generated R -module A will be denoted by $l(A)$.

Let L be an even integral R -lattice. Its discriminant group is the group $D_L = L^\vee/L$ equipped with the discriminant quadratic form $q_L : L \rightarrow K/2R$. Note that $l(D_L) \leq l(L) = \text{rk } L$. Denote by $O(D_L)$ its orthogonal group, that is, the group of linear automorphisms preserving the discriminant form. If $f : L \rightarrow M$ is an isometry of even \mathbb{Z} -lattices, then it induces an isomorphism $D_f : D_L \rightarrow D_M$. Likewise, we obtain a natural map $O(L) \rightarrow O(D_L)$, whose kernel is denoted by $O^\sharp(L)$. For an isometry $f \in O(L)$ and H some subquotient of $L \otimes K$ preserved by f , we denote by $f|_H$ the induced automorphism of H . Let $G \leq O(L)$ be a subgroup. We denote the fixed lattice by $L^G = \{x \in L \mid \forall g \in G : g(x) = x\}$ and its orthogonal complement by $L_G = (L^G)^\perp$.

For $R = \mathbb{R}$, let s_+ be the number of positive eigenvalues of a gram matrix and s_- the number of negative eigenvalues. We call (s_+, s_-) the signature pair or just signature of L .

2.2. Primitive extensions and glue

Let $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ and L be an even integral R -lattice. We call $M \perp N \subseteq L$ a primitive extension of $M \perp N$ if M and N are primitive in L and $\text{rk } L = \text{rk } M + \text{rk } N$. Since L is integral, we have a chain of inclusions

$$M \perp N \subseteq L \subseteq L^\vee \subseteq M^\vee \perp N^\vee.$$

The projection $M^\vee \perp N^\vee \rightarrow M^\vee$ induces a homomorphism $L/(M \perp N) \rightarrow D_M$. This homomorphism is injective if and only if N is primitive in L . Let H_M denote its image, and define H_N analogously. The composition

$$\phi : H_M \rightarrow L/(M \perp N) \rightarrow H_N$$

is called a *glue map*. It is an anti-isometry, that is, $q_M(x) = -q_N(\phi(x))$ for all $x \in H_M$. Note that $L/(M \perp N) \leq H_M \perp H_N \leq D_M \perp D_N$ is the graph of $\phi: H_M \rightarrow H_N$.

Conversely, any anti-isometry $\phi: H_M \rightarrow H_N$ between subgroups $H_M \subseteq D_M$ and $H_N \subseteq D_N$ is the glue map of a primitive extension: $M \perp N \subseteq L_\phi$, where L_ϕ is defined by the property that $L_\phi/(M \perp N)$ is the graph of ϕ .

The determinants of the lattices in play are related as follows:

$$|\det L| = |D_M/H_M| \cdot |D_N/H_N| = |\det M| \cdot |\det N|/[L : (M \perp N)]^2.$$

If $f_M \in O(M)$ and $f_N \in O(N)$ are isometries, then $g = f_M \oplus f_N$ preserves the primitive extension L_ϕ if and only if $\phi \circ D_{f_M} = D_{f_N} \circ \phi$. We call ϕ an *equivariant glue map* with respect to f_M and f_N .

2.3. Lattices with isometry

We are interested in classifying conjugacy classes of isometries of a given \mathbb{Z} -lattice. If the lattice in question is definite, its orthogonal group is finite. Using computer algebra systems, one can compute the group as well as representatives for its conjugacy classes. This approach breaks down if L is indefinite.

Definition 2.1. A *lattice with isometry* is a pair (L, f) consisting of a lattice L and an isometry $f \in O(L)$. We frequently omit f from the notation and denote the lattice with isometry simply by L . Its isometry f is then denoted by f_L . We say that two lattices with isometry M and N are *isomorphic* if they are equivariantly isometric, that is, if there exists an isometry $\psi: M \rightarrow N$ with $\psi \circ f_M = f_N \circ \psi$. We view L as a $\mathbb{Z}[x, x^{-1}]$ module via the action $x \cdot m = f(m)$, $x^{-1} \cdot m = f^{-1}(m)$.

The unitary group $U(L)$ of the lattice with isometry L is the centralizer of f_L in $O(L)$. This is nothing but $\text{Aut}(L)$ in the category of lattices with isometry.

Terminology for lattices applies to lattices with isometry verbatim. For instance, the discriminant group D_L of a lattice with isometry comes equipped with the induced isometry $D_{f_L} \in O(D_L)$ and $U(D_L)$ is the centralizer of D_{f_L} in $O(D_L)$. We denote by $G_L = \text{Im}(U(L) \rightarrow U(D_L))$.

Proposition 2.2. Let M, N be lattices with isometry. Suppose that the characteristic polynomials of f_M and f_N are coprime. Then the double coset

$$U(N) \backslash \{ \text{equivariant glue maps } D_M \supseteq H_M \xrightarrow{\phi} H_N \subseteq D_N \} / U(M)$$

is in bijection with the set of isomorphism classes of lattices with isometry (L, f) with characteristic polynomial $\chi_f(x) = \chi_{f_M}(x)\chi_{f_N}(x)$ and $M \cong (\ker \chi_{f_M}(f))$ and $N \cong (\ker \chi_{f_N}(f))$.

Proof. We work in the category of lattices with isometry. Let L_ϕ and L_ψ be primitive extensions of $M \perp N$ and $h: L_\phi \rightarrow L_\psi$ an isomorphism. Then $h|_M \in U(M)$ and $h|_N \in U(N)$. We have $D_{h|_N} \phi D_{h|_M}^{-1} = \psi$, so $\psi \in U(M)\phi U(N)$. Conversely, if $D_a \phi D_b = \psi$. Then $a \oplus b: L_\phi \rightarrow L_\psi$ is an isomorphism. □

3. K3 surfaces with a group of automorphisms

See [11] or [36] for generalities on K3 surfaces.

3.1. H-markings

Let X be a K3 surface. We denote by

$$\rho_X: \text{Aut}(X) \rightarrow O(H^2(X, \mathbb{Z})), g \mapsto (g^{-1})^*$$

the natural representation of the automorphism group $\text{Aut}(X)$ on the second integral cohomology group of X . It is faithful.

Definition 3.1. Let X, X' be K3 surfaces and $G \leq \text{Aut}(X), G' \leq \text{Aut}(X')$. We call (X, G) and (X', G') *conjugate*, if there is an isomorphism $\phi: X \rightarrow X'$ such that $\phi G \phi^{-1} = G'$. They are called *deformation equivalent* if there exists a connected family $\mathcal{X} \rightarrow B$ of K3 surfaces, a group of automorphisms $\mathcal{G} \leq \text{Aut}(\mathcal{X}/B)$ and two points $b, b' \in B$ such that the restriction of $(\mathcal{X}, \mathcal{G})$ to the fiber above b is conjugate to (X, G) and to the fiber above b' is conjugate to (X', G') .

Let L be a fixed even unimodular lattice of signature $(3, 19)$. An L -marking of a K3 surface X is an isometry $\eta: H^2(X, \mathbb{Z}) \rightarrow L$. The pair (X, η) is called an L -marked K3-surface. A family of L -marked K3 surfaces is a family $\pi: \mathcal{X} \rightarrow B$ of K3 surfaces with an isomorphism of local systems $\eta: R^2\pi_*\mathbb{Z} \rightarrow \underline{L}$. If the base B is simply connected, then a marking of a single fiber extends to the whole family.

Definition 3.2. Let $H \leq O(L)$ be a *finite* subgroup. An H -marked K3 surface is a triple (X, η, G) such that (X, η) is an L -marked K3 surface and $G \leq \text{Aut}(X)$ is a group of automorphisms with $\eta \rho_X(G) \eta^{-1} = H$. We say that (X, G) is H -markable if there exists some marking by H .

Two H -marked K3 surfaces (X_1, η_1, G_1) and (X_2, η_2, G_2) are called *conjugate* if there exists an isomorphism $f: X_1 \rightarrow X_2$ such that $\eta_1 \circ f^* = \eta_2$. In particular, $f G_1 f^{-1} = G_2$. We call H *effective* if there exists at least one H -marked K3 surface.

A family of H -marked K3 surfaces is a family of L -marked K3 surfaces $\pi: \mathcal{X} \rightarrow B$ of K3 surfaces with an isomorphism of local systems $\eta: R^2\pi_*\mathbb{Z} \rightarrow \underline{L}$ and group of automorphisms $\mathcal{G} \leq \text{Aut}(\mathcal{X}/B)$ such that for each $b \in B$ the fiber $(\mathcal{X}_b, \eta_b, \mathcal{G}_b)$ is an H -marked K3 surface.

Let (X, η, G) be an H -marked K3 surface. The action of G on $H^{2,0}(X)$ induces via the marking η a character $\chi: H \rightarrow \mathbb{C}^\times$. We call such a character *effective*. We denote complex conjugation by a bar $\bar{\cdot}$. Set $H_s = \ker \chi$, and denote the χ -eigenspace by

$$L_{\mathbb{C}}^\chi = \{x \in L \otimes \mathbb{C} \mid h(x) = \chi(h) \cdot x \text{ for all } h \in H\}.$$

Similarly let

$$L_{\mathbb{R}}^{\chi+\bar{\chi}} = \{x \in L_{\mathbb{R}} \mid (h+h^{-1})(x) = \chi(h)x + \bar{\chi}(h)x \text{ for all } h \in H\}.$$

The generic transcendental lattice $T(\chi)$ is the smallest primitive sublattice of L such that $T(\chi) \otimes \mathbb{C}$ contains $(L \otimes \mathbb{C})^\chi$. We call $\text{NS}(\chi) = T(\chi)^\perp$ the generic Néron–Severi lattice. Recall that L_{H_s} is the complement of the fixed lattice L^{H_s} .

Proposition 3.3. *Let $H \leq O(L)$ be a finite group and $\chi: H \rightarrow \mathbb{C}^\times$ a nontrivial character. Recall that $H_s := \ker \chi$. Then χ is effective if and only if the following hold:*

1. L_{H_s} is negative definite;
2. L_{H_s} does not contain any (-2) -vector;
3. the signature of L^H is $(1, *)$;
4. $L_{\mathbb{R}}^{\chi+\bar{\chi}}$ is of signature $(2, *)$;
5. $\text{NS}(\chi)_H$ does not contain any vector of square (-2) .

If χ is trivial, then χ is effective if and only if (1) and (2) hold.

Proof. If χ is trivial, this is known (cf. [59, 4.2, 4.3]). So let χ be nontrivial. We show that (1)–(5) are necessary. (1) and (2) follow from [59, 4.2]. Let H be an effective subgroup and (X, η, G) an H -marked K3 surface. Since $T(\chi) \otimes \mathbb{C}$ contains the period $\eta(\omega)$ of (X, η) , we have $\eta(\omega + \bar{\omega}) \in L_{\mathbb{R}}^{\chi+\bar{\chi}}$. Recall that $\omega \cdot \bar{\omega} > 0$. Thus $L_{\mathbb{R}}^{\chi+\bar{\chi}}$ has at least two positive squares. Let $h \in \text{NS}(X)$ be ample. Then $h' = \sum_{g \in G} g^* h$ is ample and G -invariant, that is, $h' \in H^2(X, \mathbb{Z})^G = \eta^{-1}(L^H)$. Since $h'^2 > 0$, L^H has at least one positive square. Since $H^{1,1}(X)$ and $H^{2,0}(X) \oplus H^{0,2}(X)$ are orthogonal so are h and $L_{\mathbb{R}}^{\chi+\bar{\chi}} \subseteq \eta(H^{2,0}(X) \oplus H^{0,2}(X))$. Therefore, all 3 positive squares of L are accounted for. This proves (3)

and (4). For (5), we note that $\text{NS}(\chi) \subseteq \eta(\text{NS}(X))$. For $\delta \in \text{NS}(X)$ with $\delta^2 = -2$, we know that δ or $-\delta$ is effective by Riemann–Roch. Therefore, $h' \cdot \delta \neq 0$, because h' is ample. Since $\text{NS}(X)_G \subseteq h^\perp \cap \text{NS}(X)$, $\text{NS}(X)_G$ does not contain any vector of square (-2) . The same holds true for $\text{NS}(\chi)_H \subseteq \eta(\text{NS}(X)_G)$.

Since the signature of $L_{\mathbb{R}}^{\chi+\bar{\chi}}$ is $(2, *)$, we can find an element ω in $(L \otimes \mathbb{C})^\chi$ such that $\omega \cdot \bar{\omega} > 0, \omega^2 = 0$. Choosing ω general enough we achieve $\omega^\perp \cap T(\chi) = 0$. By the surjectivity of the period map [11, VII (14.1)], we can find an L -marked K3 surface (X, η) with period ω . By construction $\eta(T(X)) = T(\chi)$, so X is projective and $\eta(\text{NS}(X)) = \text{NS}(\chi)$. Since L^H is of signature $(1, *)$ and L_H does not contain any (-2) -roots, we find $h \in L^H$ with $h^2 > 0$ and $h^\perp \cap \text{NS}(\chi)$ not containing any -2 roots either and after possibly replacing h by $-h$, we can assume that h lies in the positive cone. Thus, h lies in the interior of a Weyl chamber. Since the Weyl group $W(\text{NS}(X))$ acts transitively on the Weyl chambers of the positive cone, we find an element $w \in W(\text{NS}(\chi))$ such that that $(\eta \circ w)^{-1}(h)$ is ample. Let $\eta' = \eta \circ w$. By construction, every element of $G' = \eta'^{-1}H\eta'$ preserves this ample class and the period of X . So G' is a group of effective Hodge isometries. By the strong Torelli theorem (see, e.g., [11, VIII §11]), $G' = \rho_X(G)$ for some $G \leq \text{Aut}(X)$. □

Note that for any effective character $\chi: H \rightarrow \mathbb{C}^\times$ of H ,

$$\ker \chi = H_s = \{h \in H \mid L^h \text{ is of signature } (3, *)\}$$

is independent of χ . Indeed, because L_{H_s} is negative definite, $L^{H_s} \subseteq L^h$ is of signature $(3, *)$ for any $h \in H_s$. On the other hand, if $g \in H$ is not in H_s , then $\chi(g) \neq 1$ and so L^g is orthogonal to $L_{\mathbb{R}}^{\chi+\bar{\chi}}$ and contains L^H . Therefore, L^g is of signature $(1, *)$. The kernel is a normal subgroup and $H/H_s \cong \mu_n$ is cyclic. We say that an effective group H is *symplectic* if $H = H_s$ and nonsymplectic otherwise.

Lemma 3.4. *Let $H \leq O(L)$ be effective. There are at most two effective characters $\chi: H \rightarrow \mathbb{C}^\times$. They are complex conjugate.*

Proof. Fix a generator hH_s of H/H_s , and let χ be an effective character. It is determined by its value $\chi(h)$, which is a primitive n -th root of unity. Set $T = T(\chi)$. Since χ is effective, $L_{\mathbb{R}}^{\chi+\bar{\chi}} = T_{\mathbb{R}}^{\chi+\bar{\chi}}$ is of signature $(2, *)$. It is equal to the $\chi(h) + \bar{\chi}(h)$ eigenspace of $(h + h^{-1})|_{T_{\mathbb{R}}}$. The other real eigenspaces of $(h + h^{-1})|_{T_{\mathbb{R}}}$ are negative definite. Thus, any effective character χ' is equal to χ or $\bar{\chi}$. □

In particular, this shows that $T(\chi)$ and $\text{NS}(\chi)$ do not depend on the choice of the effective character χ but only on H . We may denote them as $T(H)$ and $\text{NS}(H)$.

3.2. Moduli spaces and periods

Let \mathcal{M}_H denote the fine moduli space parametrizing isomorphism classes of H -marked K3 surfaces (X, η, G) . It is a non-Hausdorff complex manifold. It can be obtained by gluing the base spaces of the universal deformations of (X, η, G) ; see [19, §3].

Let $\chi: H \rightarrow \mathbb{C}^\times$ be an effective character. We have $\mathcal{M}_H = \mathcal{M}_H^\chi \cup \mathcal{M}_H^{\bar{\chi}}$, where \mathcal{M}_H^χ parametrizes isomorphism classes of H -marked K3 surfaces (X, η, G) with $\chi(\eta\rho_X(g)\eta^{-1}) \cdot \omega_X = (g^*)^{-1}\omega_X$ for all $g \in G$. By [19, Prop. 3.9], the forgetful map $\mathcal{M}_H \rightarrow \mathcal{M}_L, (X, \eta, G) \mapsto (X, \eta)$ into the moduli space of L -marked K3 surfaces is a closed embedding.

Definition 3.5. Let $H \leq O(L)$ be effective and $\chi: H \rightarrow \mathbb{C}^\times$ be a nontrivial effective character. We denote by

$$\mathbb{D}^\chi = \{\mathbb{C}\omega \in \mathbb{P}((L \otimes \mathbb{C})^\chi) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

the corresponding period domain and its period map by

$$\mathcal{P}: \mathcal{M}_H^\chi \rightarrow \mathbb{D}^\chi.$$

It is a local isomorphism (see, e.g., [19]). The discriminant locus is $\Delta = \bigcup \{ \mathbb{P}(\delta^\perp) \mid \delta \in L_H, \delta^2 = -2 \} \subseteq \mathbb{P}(L_C)$.

Proposition 3.6. *Let $\chi: H \rightarrow \mathbb{C}^\times$ be an effective and nontrivial character.*

1. *The image of \mathcal{P} is $\mathbb{D}^\chi \setminus \Delta$.*
2. *If (X_1, η_1, G_1) and (X_2, η_2, G_2) lie in the same fiber $\mathcal{P}^{-1}(\mathbb{C}\omega)$, then (X_1, G_1) and (X_2, G_2) are conjugate.*
3. *If χ is real, then $\mathbb{D}^\chi \setminus \Delta$ has two connected components. They are complex conjugate. If χ is not real, then $\mathbb{D}^\chi \setminus \Delta$ is connected.*

Proof. (1) Let $\mathbb{C}\omega \in \mathbb{D}^\chi$. By the surjectivity of the period map, we find a marked K3 surface (X, η) with period $\mathbb{C}\omega$. Set $N = \mathbb{C}\omega^\perp \cap L$. By Lefschetz’ theorem on $(1, 1)$ -classes, we have $\eta(\text{NS}(X)) = N$.

If $\mathbb{C}\omega \notin \Delta$, we have that $\text{NS}(H)_H = \text{NS}(H) \cap (L^H)^\perp$ does not contain any roots. This means that $\eta^{-1}(\text{NS}(H)^H) \subseteq \text{NS}(X)$ contains an ample class. Thus, $\eta^{-1}H\eta$ preserves the period and the ample cone. By the strong Torelli theorem, there is a (unique) group of automorphisms $G \leq \text{Aut}(X)$ with $\eta\rho_X(G)\eta^{-1} = H$.

If conversely $\mathbb{C}\omega \in \Delta$, then H does not preserve a Weyl chamber of N so that H cannot come from a group of automorphisms of X .

(2) Let (X_1, η_1, G_1) and (X_2, η_2, G_2) be H -marked K3 surfaces in the fiber of $\mathbb{C}\omega$. Then $\varphi = \eta_1^{-1} \circ \eta_2$ is a Hodge isometry which conjugates $\rho_{X_1}(G_1)$ and $\rho_{X_2}(G_2)$. However, it may not preserve the ample cones. By [64, Lemma 1.7 and Theorem 1.8], there exists a unique element $w \in \langle \pm 1 \rangle \times W(\text{NS}(X_1))$ such that $w \circ \varphi$ preserves the ample cones and $wg^* = g^*w$ for all $g \in G_1$. Since now $w \circ \varphi$ is an effective Hodge isometry, the strong Torelli theorem applies and provides an isomorphism $F: X_1 \rightarrow X_2$ with $F^* = w \circ \varphi$. By construction, we have $FG_1F^{-1} = G_2$ as desired.

(3) By [29, §9 & §11], the period domain \mathbb{D}^χ has 2 connected components if χ is real and one else. The discriminant locus is a locally finite union of real codimension two hyperplanes. Removing it does not affect the number of connected components. □

Remark 3.7. If $\omega \in L_C^\chi$, then we have $\omega^2 = \chi(g)^2\omega^2$ for any $g \in G$. Thus, $(1 - \chi(g)^2)\omega^2 = 0$. Let $n = [H : H_s] = |\text{Im } \chi|$. For $n > 2$, this condition implies $\omega^2 = 0$ and hence the dimension of \mathbb{D}^χ is $\dim_{\mathbb{C}} L_C^\chi - 1$, while for $n = 2$ it is $\dim_{\mathbb{C}} L_C^\chi - 2$.

Denote by $N(H) = \{f \in O(L) \mid fH = Hf\}$ the normalizer of H in $O(L)$. If (X, η, G) is an H -marked K3 surface and $f \in N(H)$, then $(X, f \circ \eta, G)$ is an H -marked K3 surface as well. In fact, all H -markings of (X, G) arise in this way. So $N(H)$ is the group of changes of marking.

Set $\mathbb{D}_H = \mathbb{D}^\chi \cup \mathbb{D}^{\bar{\chi}}$. The group $N(H)$ acts on \mathbb{D}_H via an arithmetic subgroup of $O(T(H))$, respectively $U(T(H))$. Therefore, by [10] the space $\mathbb{D}_H/N(H)$ is a quasi-projective variety with only finite quotient singularities.

Theorem 3.8. *The coarse moduli space $\mathcal{F}_H := \mathcal{M}_H/N(H)$ of H -markable K3 surfaces admits a bijective period map $\mathcal{F}_H \rightarrow (\mathbb{D}_H \setminus \Delta)/N(H)$.*

Proof. We can use the action of the normalizer to forget the marking and thus obtain a period map $\mathcal{F}_H = \mathcal{M}_H/N(H) \rightarrow (\mathbb{D}_H \setminus \Delta)/N(H)$. That it is bijective follows from Proposition 3.6. Part (1) gives surjectivity and part (2) injectivity. Indeed, if two H polarizable K3 surfaces have the same image, then we can find markings on them such that they lie in the same fiber of the period map $\mathcal{M}_H \rightarrow \mathbb{D}_H$. Then they are conjugate by Proposition 3.6 (2). □

See [4, 5] for moduli of K3 surfaces and their compactifications.

3.3. Connected components

We next show that deformation classes of K3 surfaces with finite groups of automorphisms are precisely the connected components of the coarse moduli space of H -markable K3 surfaces \mathcal{F}_H . In the following, for a topological space Y we denote by $\pi_0(Y)$ the set of (path) connected components of Y .

Theorem 3.9. *Let (X, G) and (X', G') be two pairs consisting of K3 surfaces X, X' and finite groups of automorphisms $G \leq \text{Aut}(X)$, $G' \leq \text{Aut}(X')$. Then (X, G) and (X', G') are deformation equivalent if and only if they are markable by the same effective subgroup $H \leq O(L)$ and they lie in the same connected component of \mathcal{F}_H .*

Proof. Every pair (X, G) is H -markable for some effective subgroup $H \leq O(L)$.

Let (X', G') be deformation equivalent to (X, G) . This means that we find a connected family $\mathcal{X} \rightarrow B$, a group of automorphisms $\mathcal{G} \leq \text{Aut}(\mathcal{X}/B)$ and points $b, b' \in B$ such that the fibers above b and b' are conjugate to (X, G) and (X', G') . The H -marking of (X, G) induces an H -marking of the fiber above b . By parallel transport in the local system $R^2\pi_*\mathbb{Z}$, we move the marking from (X, G) to (X', G') along some continuous path γ in B connecting b and b' . Therefore, the fiber above b' is H -markable. The isomorphism of the fiber with (X', G') allows transporting this marking to (X', G') . Therefore, (X', G') is H -markable. Its point in the moduli space \mathcal{F}_H of H -markable K3-surfaces coincides with that of the fiber above b' . Likewise, (X, G) gives the same point in \mathcal{F}_H as the fiber above b . Since the two fibers lie in the same connected component of \mathcal{M}_H , their images lie in the same connected component of \mathcal{F}_H .

Conversely, let (X, G) and (X', G') be H -markable and in the same connected component of \mathcal{F}_H . Then we can find markings η , and η' such that (X, η, G) and (X, η', G') are H -marked. Then $\pi_0(\mathcal{F}_H) = \pi_0(\mathcal{M}_H/N(H)) \cong \pi_0(\mathcal{M}_{H'})/N(H')$. Since (X, G) and (X', G') lie in the same connected component of \mathcal{F}_H , we find $n \in N(H)$ such that (X, η, G) and $(X, n \circ \eta', G')$ lie in the same connected component of \mathcal{M}_H . Since \mathcal{M}_H is a fine moduli space, it has a universal family, and this gives a deformation of (X, η, G) and $(X, n \circ \eta', G')$ as H -marked K3 surfaces. By forgetting the markings, we obtain a deformation of (X, G) with (X', G') . □

Corollary 3.10. *The set of deformation classes of pairs (X, G) with X a K3 surface and $G \leq \text{Aut}(X)$ with $G \neq G_s$ is in bijection with the set $\bigcup_{H \in T} \pi_0(\mathcal{F}_H)$, where T is a transversal of the set of conjugacy classes effective, nonsymplectic subgroups of $O(L)$.*

Let L be a K3 lattice and $H \leq O(L)$ an effective subgroup. Our next goal is to determine the connected components of the coarse moduli space \mathcal{F}_H of H -markable K3 surfaces. Since the period domain $\mathbb{D}_H \setminus \Delta$ has exactly two connected components (Proposition 3.6 (3)), \mathcal{F}_H has at most two components as well. It has only one connected component if and only if the action of $N(H)$ on \mathbb{D}_H exchanges the two components.

Let $\chi : H \rightarrow \mathbb{C}^\times$ be an effective character. For $n \in N(H)$, denote by χ^n the character defined by $\chi^n(h) = \chi(n^{-1}hn)$. Denote by $N(\chi)$ the stabilizer of χ in $N(H)$. For completeness sake, we mention the following proposition.

Proposition 3.11. *Let $\chi : H \rightarrow \mathbb{C}^\times$ be an effective character and $[H : H_s] > 2$. Then the number of connected components of \mathcal{F}_H is $2/[N(H) : N(\chi)]$.*

Proof. Since $[H : H_s] > 2$, the connected components of \mathcal{M}_H are \mathcal{M}_H^χ and $\mathcal{M}_H^{\bar{\chi}}$. If $(X, \eta, G) \in \mathcal{M}_H^\chi$ and $n \in N(H)$, then $(X, n \circ \eta, G) \in \mathcal{M}_H^{\chi^n}$. □

Let now $[H : H_s] = 2$. Then we have seen that $\chi = \bar{\chi}$ and \mathbb{D}^χ has two connected components. This can be dealt with by introducing *positive sign structures*. Our account follows [72]. A period $\mathbb{C}\omega \in \mathbb{D}^\chi$ can be seen as an oriented, positive definite real 2-plane. Indeed, the two real vectors $\text{Re } \omega, \text{Im } \omega \in L_\mathbb{R}$ give an ordered (and thus oriented), orthogonal basis of a positive definite plane in L .

Definition 3.12. Let L be a \mathbb{Z} -lattice of signature (s_+, s_-) . Then a *sign structure* on L is defined as a choice of one of the connected components θ of the manifold parametrizing oriented, s_+ -dimensional, positive definite, real subspaces S of $L_\mathbb{R}$. Unless L is negative definite, it has exactly two positive sign structures.

For $[H : H_s] > 2$, the periods in \mathbb{D}^χ all give the same sign structure. But for $[H : H_s] = 2$, there are two sign structures which give the two connected components of the period domain. Whether or not

$N(H)$ preserves the sign structure is encoded by a certain character. See section 2.1 for our conventions on the spinor norm.

Proposition 3.13 [50, 55]. *Let L be an \mathbb{R} -lattice so that the spinor norm takes values in $\{\pm 1\} \cong \mathbb{R}^\times / \mathbb{R}^{\times 2}$. The action of an isometry $g \in O(L)$ on the set of positive sign structures of L is trivial if and only if $\det(g) \cdot \text{spin}(g) > 0$.*

Let L be an \mathbb{R} -lattice. We denote by $O^+(L) = \ker(\det \cdot \text{spin})$ the subgroup of orientation preserving isometries. If $G \leq O(L)$ is a subgroup, we denote by $G^+ = G \cap O^+(L)$ its normal subgroup of orientation preserving elements.

Proposition 3.14. *Let $\chi: H \rightarrow \mathbb{C}^\times$ be an effective character and $[H : H_s] = 2$. Set $T = T(\chi)$, let $\pi: N(H) \rightarrow O(T)$ be the restriction and $N_T = \pi(N(H))$. Then the number of connected components of \mathcal{F}_H is $2/[N_T : N_T^+]$.*

Proof. The subgroup of $N(H)$ fixing the sign structures of $T(\chi)$ is by the definition $\pi^{-1}(N_T^+)$. Therefore, $2/[\pi(N(H)) : \pi(N(H))^+]$ is the number of connected components of $N(H)$. □

Lemma 3.15. *Keep the notation of Proposition 3.14. Let $D: O(T) \rightarrow O(D_T)$ be the natural map, $J = D(N_T)$, $J^+ = D(N_T^+)$ and $K = \ker D$. Then $[N_T : N_T^+] = [K : K^+][J : J^+]$.*

Proof. We claim that that $K \subseteq N_T$. By definition any element $g \in K$ acts trivially on the discriminant group D_T . Therefore, $g \oplus \text{id}_{\text{NS}(\chi)}$ extends to L . Since the restriction of any $h \in H$ to T is given by $\pm \text{id}_T$, $g \oplus \text{id}_{\text{NS}(\chi)}$ commutes with h . The claim is proven. As a consequence, we have $N_T = D^{-1}(J)$. Thus, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & K^+ & \longrightarrow & N_T^+ & \longrightarrow & J^+ & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & K & \longrightarrow & N_T & \xrightarrow{D} & J & \longrightarrow & 1,
 \end{array}$$

where the vertical arrows are inclusions of normal subgroups. Hence, the cokernels exist and so we obtain the corresponding exact sequence

$$1 \rightarrow K/K^+ \rightarrow N_T/N_T^+ \rightarrow J/J^+ \rightarrow 1$$

of the cokernels. To see this follow the proof of the snake lemma, which indeed is valid in this situation. □

Remark 3.16. We will see in Proposition 5.4 how to compute the number of components $[N_T : N_T^+] = [K : K^+][J : J^+]$ using the Miranda–Morrison theory.

3.4. Saturated effective subgroups

By Theorem 3.9, the set of connected components of \mathcal{F}_H is in bijection with the deformation classes of H -markable K3 surfaces. Our next goal is to enumerate all possible effective groups H up to conjugacy in $O(L)$. The symplectic fixed and cofixed lattices L^{H_s} and L_{H_s} turn out to be the crucial invariants for this task. Let L be a \mathbb{Z} -lattice and M a subset of $L \otimes \mathbb{C}$. We denote by $O(L, M) = \{f \in O(L) \mid f_{\mathbb{C}}(M) = M\}$ the stabilizer of M .

Definition 3.17. Let L be a K3 lattice and $H \leq O(L)$ an effective subgroup. Then we call the kernel S of $O(L, L^{H_s}) \rightarrow O(L^{H_s})$ the saturation of H_s . The group generated by H and S is called the saturation of H . For a K3 surface X , we call a finite subgroup $G \leq \text{Aut}(X)$ saturated if its image $\rho_X(G_s)$ is saturated. We call a saturated symplectic group $H_s \leq O(L)$ a *heart*.

Remark 3.18. Note that the saturation S of H_s is the largest subgroup $S \leq O(L)$ with $L^S = L^{H_s}$. Further, the saturation of an effective group $H \leq O(L)$ is effective. Indeed, if $L = H^2(X, \mathbb{Z})$ for some K3 surface X and G_s a finite group of symplectic automorphisms, then the strong Torelli theorem implies that the saturation of $\rho_X(G_s) = H_s$ is in the image of ρ_X by a finite group of symplectic automorphisms containing G_s and with the same fixed lattice. We conclude that if a pair (X, G) is markable by H , then it is also markable by the saturation of H . Therefore, it is enough to enumerate the saturated effective subgroups of $O(L)$.

Remark 3.19. For the symplectic groups, the saturated subgroups are known: By a theorem of Hashimoto [34], there are exactly 44 conjugacy classes of effective, saturated subgroups $H_s \leq O(L)$. They are determined by the isometry classes of their fixed and cofixed lattices. The fixed lattices are listed by Hashimoto while the cofixed lattices can be obtained from the permutation representation of the Mathieu group M_{24} on the type $24A_1$ Niemeier lattice. Alternatively, one may obtain them from isometries of the Leech lattice and the tables enumerated in [35].

3.5. Enumerating effective characters

Let $H_s \leq O(L)$ be a symplectic effective subgroup. We would like to enumerate conjugacy classes of effective characters $\chi: H \rightarrow \mathbb{C}^\times$ with the given heart H_s .

Definition 3.20. Let $\chi: H \rightarrow \mathbb{C}^\times$ be an effective character and $n = [H : H_s]$. The distinguished generator of H/H_s is gH_s with $\chi(g) = \zeta_n := \exp(2\pi i/n)$. Set $F = L^{H_s}$ and $C = L_{H_s}$. The distinguished generator gH_s restricts to an isometry $f = g|_F$ of F of order n . We call the lattice with isometry (F, f) the *head* of H and H_s its *heart*. The *spine* of χ is the glue map $\phi: D_F \rightarrow D_C$ with $L_\phi = L$.

Our next goal is to see how heart, head and spine determine the character. The first step is to make the definition of heart and head independent of a character.

Definition 3.21. Let $H_s \leq O(L)$ be a heart, $F = L^{H_s}$ its fixed lattice and $f \in O(F)$ of order n . We call the lattice with isometry (F, f) a *head* (of H_s) if the following hold:

1. $\ker(f + f^{-1} - \zeta_n - \bar{\zeta}_n)$ is of signature $(2, *)$,
2. $(\ker \Phi_n(f)\Phi_1(f))^\perp$ does not contain any vector of square (-2) .

By abuse of notation, we will identify $O(C)$ with $O(C) \times \{\text{id}_F\} \subseteq O(L \otimes \mathbb{Q})$. Recall that for a glue map $\phi: D_F \rightarrow D_C$, we denote by $F \perp C \subseteq L_\phi$ the corresponding primitive extension. Note that $L_\phi \cong L$ is a K3 lattice as well and H_s preserves L_ϕ because all its elements act trivially on the discriminant group of C . Suppose that $\phi D_f \phi^{-1} = D_C$ for some $c \in O(C)$. Then $g = f \oplus c$ preserves L_ϕ . Set $H_\phi = \langle g, H_s \rangle$. Since H_s is saturated, any other choice of c is in cH_s , so H_ϕ is independent of this choice. Let $\chi_\phi: H_\phi \rightarrow \mathbb{C}^\times$ be defined by $\chi(g) = \zeta_n$.

Definition 3.22. Let H_s be a heart and (F, f) a head. A glue map $\phi: D_F \rightarrow D_C$ is called a *spine* if

1. $\phi \circ D_f \circ \phi^{-1}$ is in the image of $O(C) \rightarrow O(D_C)$ and
2. $\text{NS}(\chi_\phi)_{H_\phi}$ does not contain any vector of square (-2) .

Proposition 3.23. Let H_s be a heart, (F, f) a head and $\phi: D_F \rightarrow D_C$ a spine. Then the corresponding character $\chi_\phi: H_\phi \rightarrow \mathbb{C}^\times$ is effective.

Proof. This follows immediately from Proposition 3.3 and the definitions. □

Definition 3.24. Let $i = 1, 2$, $H_i \leq O(L_i)$ two effective subgroups and $\chi_i: H_i \rightarrow \mathbb{C}^\times$ two effective characters. We say that χ_1 is isomorphic to χ_2 if and only if there is an isometry $\psi: L_1 \rightarrow L_2$ with $H_2 = \psi H_1 \psi^{-1}$ and $\chi_1(h) = \chi_2(\psi \circ h \circ \psi^{-1})$ for all $h \in H_1$.

If $L_1 = L_2$, then χ_1 and χ_2 are isomorphic if and only if they are conjugate. If moreover $H_1 = H_2$, then they are isomorphic if and only if they are conjugate by an element of $N(H)$. Recall that $U(F, f)$ denotes the centralizer of f in $O(F)$.

Theorem 3.25. *Let H_S be a heart, (F, f) a head of H_S and S the set of spines $\phi: D_F \rightarrow D_C$. Then the double coset*

$$O(C)\backslash S/U(F, f)$$

is in bijection with the set of isomorphism classes of effective characters χ with the given heart and head.

Proof. By Proposition 3.23, any spine $\phi \in S$ determines an effective character $\chi_\phi: H_\phi \rightarrow \mathbb{C}^\times$ with L_ϕ a K3 lattice. Conversely, any effective character with the given heart and head arises in this fashion.

Let ϕ and ϕ' be two spines. We have to show that χ_ϕ and $\chi_{\phi'}$ are conjugate if and only if $\phi' \in O(C)\phi U(F, f)$.

Suppose $\phi' = D_a\phi D_b$ with $a \in O(C)$ and $b \in U(F, f)$. Then $a \oplus b: L_\phi \rightarrow L_{\phi'}$ gives the desired isomorphism of the characters.

Conversely, let $\psi: L_\phi \rightarrow L_{\phi'}$ be an isomorphism of χ and χ' . By construction, $V := L_\phi \otimes \mathbb{Q} = L_{\phi'} \otimes \mathbb{Q}$. We may view ψ as an element of $O(V)$. Note that $F \perp C \subseteq V$. Since ψ preserves the common heart H_S and head (F, f) of χ_1 and χ_2 , we can write it as $\psi = a \oplus b$ with $a \in O(C)$ and $b \in U(F, f)$. Then $D_a \circ \phi \circ D_b = \phi'$. □

Remark 3.26. The previous results yield the following procedure for determining a transversal of the set of isomorphism classes of effective characters.

1. Let \mathcal{H} be the set of possible hearts up to conjugacy, which have been determined by Hashimoto [34] (see Remark 3.19).
2. For each $H_S \in \mathcal{H}$, determine a transversal of the isomorphism classes of the heads (F, f) , which amounts to classifying conjugacy classes of isometries of finite order n of a given \mathbb{Z} -lattice. This is explained in Section 4; see in particular Remark 4.25 for the possible values of n .
3. For each heart $H_S \in \mathcal{H}$ and possible head (L, f) apply Theorem 3.25 to determine a transversal of the double coset of spines and therefore a transversal of the isomorphism classes of the effective characters with the given heart and head.

Altogether, we obtain a transversal of the isomorphism classes of effective characters. Each is represented by some K3 lattice L depending on the character, a finite subgroup $H \leq O(L)$, a normal subgroup $H_S \leq H$ and a distinguished generator of H/H_S .

4. Conjugacy classes of isometries

We have seen that to classify finite subgroups of automorphisms of K3 surfaces up to deformation equivalence we need to classify conjugacy classes of isometries of finite order of a given \mathbb{Z} -lattice. So given a polynomial $\mu(x) \in \mathbb{Z}[x]$ and a \mathbb{Z} -lattice L , we seek to classify all conjugacy classes of isometries of L with the given characteristic polynomial p . By general results of Grunewald and Segal [33, Cor. 1] from the theory of arithmetic groups, we know that the number of such conjugacy classes is finite and (in theory) computable. Necessary and sufficient conditions for the existence of an isometry of some unimodular lattice L with a given characteristic polynomial have been worked out by Bayer-Fluckiger and Taelman in [14, 12, 13].

4.1. Hermitian lattices and transfer

In this subsection, E is an étale \mathbb{Q} -algebra with a \mathbb{Q} -linear involution $\bar{}: E \rightarrow E$.

Definition 4.1. Let V be an E -module. A Hermitian form on V is a sesquilinear form $h: V \times V \rightarrow E$ with $h(x, y) = \overline{h(y, x)}$. We call (V, h) a Hermitian space over E . For a \mathbb{Z} -order $\Lambda \subseteq E$, we call a Λ -module $L \subseteq V$ a Hermitian Λ -lattice. All Hermitian forms and lattices are assumed nondegenerate.

Let $\text{Tr}: E \rightarrow \mathbb{Q}$ be the trace. Note that E being étale is equivalent to the trace form $E \times E \rightarrow \mathbb{Q}, (x, y) \mapsto \text{Tr}(xy)$ being nondegenerate.

Definition 4.2 (Transfer). A \mathbb{Q} -bilinear form $b: V \times V \rightarrow \mathbb{Q}$ on an E -module V is said to be an E -bilinear form if $b(ev, w) = b(v, \bar{e}w)$ for all $v, w \in V$ and $e \in E$. Let (V, h) be a Hermitian space over E . Then $b = \text{Tr} \circ h$ is an E -bilinear form. It is called the trace form of h .

Proposition 4.3 [53]. Every E -bilinear form on V is the trace form of a unique Hermitian form on V .

Note that an E -linear map preserves a Hermitian form if and only if it preserves the respective trace form. In view of these facts, we may work with E -bilinear forms and Hermitian forms over E interchangeably.

Given $x^n - 1 = \mu(x) \in \mathbb{Z}[x]$ and a lattice L , we seek to classify the conjugacy classes of isometries $f \in O(L)$ satisfying $\mu(f) = 0$. To this aim, we put $E := \mathbb{Q}[x, x^{-1}]/(\mu) \cong \mathbb{Q}[x]/(\mu)$. This is an étale algebra and $x \mapsto x^{-1}$ defines an involution $\bar{}$ on E . For $i \in I := \{i \in \mathbb{N} \mid i \text{ divides } n\}$ set $E_i = \mathbb{Q}[x]/(\Phi_i(x))$. The algebra E splits as a direct product $E = \prod_{i \in I} E_i$ of cyclotomic fields with the induced involution. Let $(e_i)_{i \in I}$ be the corresponding system of primitive idempotents in E such that $\bar{e}_i = e_i$ and $E_i = Ee_i$.

Let $\Lambda = \mathbb{Z}[x]/(\mu)$ and $\Gamma = \prod_{i \in I} \Lambda e_i$. The conductor of Γ in Λ is

$$\mathfrak{f}_\Gamma = \{x \in \Lambda \mid \Gamma x \subseteq \Lambda\}.$$

It is the largest Γ -ideal contained in Λ . We obtain the following series of inclusions:

$$\mathfrak{f}_\Gamma \subseteq \Lambda \subseteq \Gamma = \prod_{i \in I} \Lambda e_i \tag{4.1}$$

Lemma 4.4. The conductor satisfies $\mathfrak{f}_\Gamma = \bigoplus_{i \in I} (\Lambda \cap e_i \Lambda)$.

Proof. For $x \in \Lambda$, we have $x = 1x = \sum_i e_i x$ and so $x\Gamma = \bigoplus_{i \in I} \Lambda e_i x$ is contained in Λ if and only if $e_i x \in \Lambda$ for all $i \in I$. This means that $e_i x \in \Lambda \cap e_i \Lambda$. □

Example 4.5. Let $p \in \mathbb{Z}$ be a prime. For $\mu(x) = x^p - 1 = (x - 1)\Phi_p(x)$, we obtain

$$E = \mathbb{Q}[x]/(x^p - 1) \cong \mathbb{Q}[x]/(x - 1) \times \mathbb{Q}[x]/\Phi_p(x) \cong \mathbb{Q} \times \mathbb{Q}[\zeta_p].$$

Set $g(x) = \sum_{i=0}^{p-2} (i + 1 - p)x^i$. One finds that $p = (x - 1)g(x) + \Phi_p(x)$. Hence, $e_1 = \Phi_p(x)/p$ and $e_p = (x - 1)g(x)/p$. The conductor ideal is $\mathfrak{f}_\Gamma = pe_1\Lambda + pe_p\Lambda$. It contains p .

Let b be the bilinear form of the lattice L . A given isometry $f \in O(L, b)$ with minimal polynomial μ turns (L, b) into a Hermitian Λ -lattice (L, h) by letting the class of x act as f . Note that for $x, y \in L \otimes E$ we have

$$h(e_i x, e_j x) = e_i \bar{e}_j h(x, y) = e_i e_j h(x, y) = \delta_{ij} h(x, y).$$

Thus $e_i L$ is orthogonal to $e_j L$ for $i \neq j$. Equation (4.1) yields the corresponding chain of finite index inclusions

$$\mathfrak{f}_\Gamma L \subseteq L \subseteq \Gamma L. \tag{4.2}$$

Setting $L_i = e_i \mathfrak{f}_\Gamma L = L \cap e_i L = \ker \Phi_i(f)$ and $L'_i = e_i \Gamma L$ the outermost lattices are

$$(\mathfrak{f}_\Gamma L, h) = \prod_{i \in I} (L_i, h_i) \quad \text{and} \quad (\Gamma L, h) = \prod_{i \in I} (L'_i, h_i).$$

Since $\Lambda e_i = \mathbb{Z}[x]/\Phi_i(x) = \mathbb{Z}_{E_i}$ is the maximal order in E_i , L_i and L'_i are Hermitian lattices over the ring of integers of a number field. Such lattices are well understood. We use the outermost lattices of the sandwich to study the Λ -lattice (L, h) .

Example 4.6. Let L be a \mathbb{Z} -lattice and $f \in O(L)$ an isometry of prime order p . Then $L_1 = \ker \Phi_1(f)$ and $L_p = \ker \Phi_p(f)$. Since, by Example 4.5, $p \in \mathfrak{f}_\Gamma$ we have

$$pL \subseteq \mathfrak{f}_\Gamma L = L_1 \perp L_p \subseteq L. \tag{4.3}$$

The idea for the classification is as follows: Given $\mu(x)$ and the \mathbb{Z} -lattice (L, b) , we get restrictions on the possible genera of the lattices (L_i, h_i) from (L, b) and the conductor. We take L as an overlattice of the orthogonal direct sum $\perp_{i \in I} (L_i, h_i)$ up to the action of the product of unitary groups $\prod_{i \in I} U(L_i, h_i)$. Thanks to eq. (4.2), this is a finite problem. In practice, we successively take equivariant primitive extensions.

4.2. Glue estimates

The caveat of dealing with primitive extensions $A \perp B \subseteq C$ is that we do not know how to predict the genus of C . Or more precisely, how to enumerate all glue maps such that C lies in a given genus. So we have to resort to check this in line 19 of Algorithm 2 only after constructing C . To reduce the number of glue maps that have to be checked, in this section we prove various necessary conditions.

Proposition 4.7. *Let C be an integral \mathbb{Z} -lattice and $f \in O(C)$ an isometry of prime order p with characteristic polynomial $\Phi_1^{e_1} \Phi_p^{e_p}$. Set $A = \ker \Phi_1(f)$, $B = \ker \Phi_p(f)$ and $m = \min\{e_1, e_p, l(D_A), l(D_B)\}$. Then $pC \subseteq A \perp B$ and $[C : A \perp B] \mid p^m$.*

Proof. By Equation (4.3) $pC \subseteq A \perp B$. Let $D_A \geq H_A \cong C/(A \perp B) = H \cong H_B \leq D_B$ be the glue between A and B . Note that these are isomorphisms as $\mathbb{Z}[x]$ -modules and $pH = 0$ by Example 4.6.

The polynomial Φ_p annihilates B ; hence, it annihilates B^\vee and $H_B \leq D_B = B^\vee/B$. Let $B' \leq p^{-1}B$ be defined by $B'/B = H_B$. The $\mathbb{Z}[\zeta_p]$ -module B' is a finitely generated torsion-free module of rank e_p . By the invariant factor theorem over Dedekind domains [27, (22.13)], any torsion quotient module of B' is generated by at most e_p elements. On the other hand, since $H_B \leq D_B$, it is generated by at most $l(D_B)$ elements as a \mathbb{Z} -module, in particular as $\mathbb{Z}[x]$ -module.

Similarly, H_A is annihilated by Φ_1 and generated by at most $\min\{e_1, l(D_A)\}$ elements. Since the glue map is equivariant, the minimal number n of generators of H as $\mathbb{Z}[x]$ -module satisfies $n \leq m$. As $H = C/(A \perp B)$, viewed as a $\mathbb{Z}[\zeta_p]$ -module, is annihilated by the prime ideal P generated by $\Phi_1(\zeta_p)$, we have $H \cong (\mathbb{Z}[\zeta_p]/P)^n$. Since p is totally ramified in $\mathbb{Z}[\zeta_p]$, the ideal P has norm p and thus

$$[C : A \perp B] = |\mathbb{Z}[\zeta_p]/P|^n = p^n \mid p^m. \tag{4.4} \quad \square$$

We call a torsion bilinear form $b: A \times A \rightarrow \mathbb{Q}_2/\mathbb{Z}_2$ *even* if $b(x, x) = 0$ for all $x \in A$, otherwise we call it *odd*. Imitating [54, II §2], we define the functors ρ_k .

Definition 4.8. Let N be an integral lattice over \mathbb{Z}_p . Set $G_k = G_k(N) = p^{-k}N \cap N^\vee$, and define $\rho_k(N) = G_k/(G_{k-1} + pG_{k+1})$. It is equipped with the nondegenerate torsion bilinear form $b_k(\bar{x}, \bar{y}) = p^{k-1}xy \pmod{\mathbb{Z}_p}$. If $p = 2$ and both $\rho_{k-1}(N)$ and $\rho_{k+1}(N)$ are even, then we call $\rho_k(N)$ *free*. Otherwise, we call it *bound*. If it is free, $\rho_k(N)$ carries the torsion quadratic form $q_k(\bar{x}) = 2^{k-1}x^2 \pmod{2\mathbb{Z}_2}$.

Let $L = \perp_{j=0}^l (L_j, p^j f_j)$ be a Jordan decomposition with f_j a unimodular bilinear form. Then one checks that $\rho_j(L) \cong (L_j/pL_j, \bar{f}_j)$, where \bar{f}_j is the composition of $p^{-1}f_j$ and the natural map $\mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$.

Remark 4.9. Note that \bar{f}_i determines the rank of f_i , its parity and its determinant modulo p . Thus, if p is odd, it determines f_j up to isometry. For $p = 2$, this is not the case.

Let N be an integral lattice over \mathbb{Z}_p and $l \in \mathbb{Z}$ such that $p^{l+1}N^\vee \subseteq N$. Then $G_{l+1}(N) = G_{l+2}(N) = N^\vee$ and $G_l(N) = p^{-l}N \cap N^\vee$. Using $pN^\vee \subseteq p^{-l}N$, we obtain

$$\rho_{l+1}(N) = N^\vee / (p^{-l}N \cap N^\vee).$$

Multiplication by p^l gives the isomorphism

$$\rho_{l+1}(N) \cong p^l N^\vee / (N \cap p^l N^\vee) \cong p^l D_N.$$

Proposition 4.10. *Let $N_1 \perp N_2 \subseteq L$ be a primitive extension of \mathbb{Z}_p -lattices with corresponding glue map $D_1 \supseteq H_1 \xrightarrow{\phi} H_2 \subseteq D_2$, where $D_i = N_i^\vee / N_i$ is the discriminant group of N_i , $i \in \{1, 2\}$. Suppose that $pL \subseteq N_1 \perp N_2$.*

Then $p^l L^\vee \subseteq L$ if and only if the following four conditions are met.

1. $p^{l+1}D_i = 0$, that is, $p^{l+1}N_i^\vee \subseteq N_i$,
2. $p^l D_i \subseteq H_i$,
3. $\phi(p^l D_1) = p^l D_2$,
4. $\hat{\phi}: \rho_{l+1}(N_1) \cong p^l D_1 \rightarrow p^l D_2 \cong \rho_{l+1}(N_2)$ is an anti-isometry with respect to the bilinear forms b_{l+1} .

If moreover both $\rho_{l+1}(N_i)$ are free, then $\rho_l(L)$ is even if and only if

(4') $\hat{\phi}$ is an anti-isometry with respect to the quadratic forms q_{l+1} .

Proof. Suppose that $p^l L^\vee \subseteq L$. We prove (1)–(4).

Let $i \in \{1, 2\}$. By the assumptions,

$$p^{l+1}L^\vee \subseteq pL \subseteq N_1 \perp N_2.$$

Since the extension is primitive, the orthogonal projection $\pi_i: L^\vee \rightarrow N_i^\vee$ is surjective. Applying π_i to the chain of inclusions yields

$$p^{l+1}N_i^\vee \subseteq p\pi_i(L) \subseteq N_i$$

which proves (1).

For (2), consider the inclusion $p^l L^\vee \subseteq L$. A projection yields $p^l N_i^\vee \subseteq \pi_i(L)$. Now (2) follows with $D_i = N_i^\vee / N_i$ and $H_i = \pi_i(L) / N_i$.

(3) We have $\pi_i(p^l L^\vee) = p^l N_i^\vee$. Recall that the glue map ϕ is defined by its graph $L / (N_1 \perp N_2)$. Its subset

$$p^l L^\vee / ((N_1 \perp N_2) \cap p^l L^\vee)$$

projects onto both $p^l D_1$ and $p^l D_2$. This proves the claim.

(4) Let $i \in 1, 2$, $x_i \in N_1^\vee$ and $y_i \in N_2^\vee$ with $\hat{\phi}(\bar{x}_i) = \bar{y}_i$, that is, $\phi(p^l x_i + N_1) = p^l y_i + N_2$, that is,

$$p^l(x_i + y_i) \in L.$$

In fact, from the proof of (3), we know a little more, namely that $p^l(x_i + y_i) \in p^l L^\vee$ so that $x_i + y_i \in L^\vee$. This implies that $\langle p^l(x_1 + y_1), x_2 + y_2 \rangle \in \mathbb{Z}_p$ which results in

$$b_{l+1}(\bar{x}_1, \bar{x}_2) \equiv p^l \langle x_1, x_2 \rangle \equiv -p^l \langle y_1, y_2 \rangle \equiv -b_{l+1}(\bar{y}_1, \bar{y}_2) \pmod{\mathbb{Z}_p}.$$

(4') Suppose furthermore that both $\rho_{l+1}(N_i)$ are free and that $\rho_l(L)$ is even. Take $x = x_1 = x_2 \in N_1^\vee$ and $y = y_1 = y_2 \in N_2^\vee$. Then $2^{l-1} \langle x + y, x + y \rangle \in \mathbb{Z}_2$ since $\rho_l(L)$ is even. Therefore, $2^l \langle x, x \rangle \equiv 2^l \langle y, y \rangle \pmod{2\mathbb{Z}}$.

Now, suppose that (1–4) hold for the glue map ϕ . Let $x + y \in L^\vee$. We have to show that $p^l(x + y) \in L$.

Let $w \in N_1^\vee$, and $z \in N_2^\vee$ with $\hat{\phi}(\bar{w}) = \bar{z}$. By the definition of $\hat{\phi}$ and ϕ , this implies that $p^l(w+z) \in L$. Therefore,

$$\begin{aligned} b_{l+1}(\bar{y} - \hat{\phi}(\bar{x}), \bar{z}) &= b_{l+1}(\bar{y}, \bar{z}) - b_{l+1}(\hat{\phi}(\bar{x}), \hat{\phi}(\bar{w})) \\ &= b_{l+1}(\bar{y}, \bar{z}) + b_{l+1}(\bar{x}, \bar{w}) \\ &\equiv p^l \langle y, z \rangle + p^l \langle x, w \rangle \\ &\equiv \langle x + y, p^l(w + z) \rangle \\ &\equiv 0 \pmod{\mathbb{Z}_p}. \end{aligned}$$

Since the bilinear form on $\rho_{l+1}(N_2)$ is nondegenerate, this shows that $\hat{\phi}(\bar{x}) = \bar{y}$. By the definition of a glue map, we obtain $p^l(x + y) \in L$.

Suppose furthermore that (4') holds, so $p = 2$, $\rho_{l+1}(N_i)$ is free and $\hat{\phi}$ preserves the induced quadratic forms. Let $x + y \in L^\vee$; we have to show that $q_l(\overline{x + y}) \equiv 0 \pmod{\mathbb{Z}}$. Indeed,

$$\begin{aligned} 2q_l(\overline{x + y}) &= 2^l \langle x + y, x + y \rangle \\ &\equiv q_{l+1}(\bar{x}) + q_{l+1}(\bar{y}) \\ &\equiv q_{l+1}(\bar{x}) + q_{l+1}(\hat{\phi}(\bar{x})) \\ &\equiv 0 \pmod{2\mathbb{Z}}. \end{aligned}$$

□

Definition 4.11. We call a glue map *admissible* if it satisfies Proposition 4.10 (1)–(3) and (4) respectively (4').

Example 4.12. In the special case that $l = 0$, we recover the result that the discriminant bilinear forms of N_1 and N_2 are anti-isometric. And if further L is even, that the discriminant quadratic forms are anti-isometric.

Definition 4.13. Let p be a prime number and A, B, C be integral \mathbb{Z} -lattices. Let $p^{-l}\mathbb{Z} = \mathfrak{s}(C^\vee)$. We say that the triple (A, B, C) is *p-admissible* if the following hold:

1. $(A \perp B) \otimes \mathbb{Z}_q \cong C \otimes \mathbb{Z}_q$ for all primes $q \neq p$,
2. $\det A \cdot \det B = p^{2g} \det C$, where $g \leq l(D_A)$, $(\text{rk } B)/(p - 1), l(D_B)$,
3. $\mathfrak{s}(A \perp B) \subseteq \mathfrak{s}(C)$ and $p \mathfrak{s}(A^\vee \perp B^\vee) \subseteq \mathfrak{s}(C^\vee)$
4. $\rho_{l+1}(A \otimes \mathbb{Z}_p)$ and $\rho_{l+1}(B \otimes \mathbb{Z}_p)$ are anti-isometric as torsion bilinear modules. If further $p = 2$, both are free and $\rho_l(C \otimes \mathbb{Z}_2)$ is even, then they are anti-isometric as torsion quadratic modules,
5. there exist embeddings $pC \otimes \mathbb{Z}_p \hookrightarrow (A \perp B) \otimes \mathbb{Z}_p \hookrightarrow C \otimes \mathbb{Z}_p$,
6. $\dim \rho_{l+1}(A \otimes \mathbb{Z}_p) \oplus \rho_{l+1}(B \otimes \mathbb{Z}_p) \leq \dim \rho_l(C \otimes \mathbb{Z}_p)$.

Note that $(C, 0, C)$ and $(0, C, C)$ are p -admissible for all p . We call a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of genera of \mathbb{Z} -lattices p -admissible if for any representatives A of \mathcal{A} , B of \mathcal{B} and C of \mathcal{C} the triple (A, B, C) is p -admissible.

Remark 4.14. For the existence of the (not necessarily primitive!) embeddings in (6), there is a necessary and sufficient criterion found in [66, Theorem 3]. Note that condition (V) in said theorem is wrong. The correct condition is

$$(V) \quad 2^i(1 + 4\omega) \rightarrow (2^i \oplus \mathfrak{L}_{i+1})/I_{[i]}.$$

Thus, being p -admissible is a condition that can be checked easily algorithmically.

Lemma 4.15. Let C be a \mathbb{Z} -lattice and $f \in O(C)$ an isometry of order p . Let $A = \ker \Phi_1(f)$ and $B = \ker \Phi_p(f)$. Then (A, B, C) is p -admissible.

Proof. Let $p^{-l}\mathbb{Z} = \mathfrak{s}(C^\vee)$. First note that Proposition 4.10 is applicable to $L = C$ since $p^l C^\vee \subseteq C$ by the definition of l .

- (1) From Proposition 4.7, we obtain $pC \subseteq A \perp B \subseteq C$. After tensoring with \mathbb{Z}_q for $q \neq p$, we obtain (since p is a unit in \mathbb{Z}_q) that $C \otimes \mathbb{Z}_q = (A \perp B) \otimes \mathbb{Z}_q$.
- (2) This is Proposition 4.7.
- (3) $A \perp B \subseteq C$ gives $\mathfrak{s}(A \perp B) \subseteq \mathfrak{s}(C)$. Dualizing $pC \subseteq A \perp B$ yields $p(A^\vee \perp B^\vee) \subseteq C^\vee$. Now, take the scales.
- (4) This is Proposition 4.10 (4) and (4').
- (5) We know that $pC \subseteq A \perp B \subseteq C$.
- (6) Let $x \in A^\vee$ and $y \in B^\vee$ with $\hat{\phi}(\bar{x}) = \bar{y}$, that is, $p^l(x + y) \in C$. By the proof of Proposition 4.10 (4), $x + y \in C^\vee$. Suppose that $x + y$ is zero in $\rho_l(C)$, that is, $x + y \in p^{-l+1}C \cap C^\vee$. Then $p^{l-1}(x + y) \in C \subseteq p^{-1}(A \perp B)$, therefore $x \in p^{-l}A$. Thus, $\bar{x} = 0$ in $\rho_{l+1}(A)$. Similarly, $\bar{y} = 0$ in $\rho_{l+1}(B)$. This shows that the graph Γ of $\hat{\phi}$ injects naturally into $\rho_l(C)$. Note that $pC \subseteq A^\vee \perp B^\vee$ gives $pA^\vee \subseteq C^\vee$. Suppose that $\bar{x} \neq 0$. Since b_{l+1} is nondegenerate, we find $a \in A^\vee$ with

$$1/p = b_{l+1}(\bar{x}, \bar{a}) \equiv p^l \langle x, a \rangle = p^{l-1} \langle x, pa \rangle \equiv b_l(\overline{x+y}, pa) \pmod{\mathbb{Z}_p}.$$

This shows that the span of pA^\vee and Γ in $\rho_l(C) = C^\vee / (p^{-l}C + C^\vee)$ is a nondegenerate subspace of dimension $2 \dim \rho_{l+1}(A)$. □

Definition 4.16. Let L be a \mathbb{Z} -lattice with $p^{l+1}L^\vee \subseteq L$, and let $H \leq D_L$ with $p^l D_L \leq H$. We denote by $O(H, \rho_l(L))$ the set of isometries g of H which preserve $p^l D_L$ and such that the map \hat{g} induced by g on $\rho_L(L)$ preserves the torsion bilinear (respectively quadratic) form of $\rho_l(L)$.

Algorithm 1 Admissible Triples

Input: A prime p and a \mathbb{Z} -lattice C .

Output: All tuples $(\mathcal{A}, \mathcal{B})$ of genera of \mathbb{Z} -lattices such that $(\mathcal{A}, \mathcal{B}, C)$ is p -admissible with C the genus of C and $\text{rk } \mathcal{B}$ divisible by $p - 1$.

- 1: $n \leftarrow \text{rk } C$
- 2: $d \leftarrow \det(C)$
- 3: Initialize the empty list $L = []$.
- 4: **for** $e_p \in \{r \in \mathbb{Z} \mid 0 \leq r \leq n/(p - 1)\}$ **do**
- 5: $r_p \leftarrow (p - 1)e_p$
- 6: $r_1 \leftarrow n - r_p$
- 7: $m \leftarrow \min\{e_p, r_1\}$
- 8: Form the set

$$D = \{(d_1, d_p) \in \mathbb{N}^2 \mid \exists g \mid \gcd(d_1, d_p, p^m) : dg^2 = d_1 d_p\}.$$

- 9: **for** $(d_1, d_p) \in D$ **do**
- 10: Form the set \mathcal{L}_1 consisting of all genera of \mathbb{Z} -lattices A with

$$\text{rk } A = r_1, \quad \det A = d_1, \quad \mathfrak{s}(A) \subseteq \mathfrak{s}(C), \quad \mathfrak{s}(A^\vee) \subseteq p \mathfrak{s}(A^\vee).$$

- 11: Form the set \mathcal{L}_p consisting of all genera of \mathbb{Z} -lattices B with

$$\text{rk } B = r_p, \quad \det B = d_p, \quad \mathfrak{s}(B) \subseteq \mathfrak{s}(C), \quad \mathfrak{s}(B^\vee) \subseteq p \mathfrak{s}(C^\vee).$$

- 12: **for** $(\mathcal{A}, \mathcal{B}) \in \mathcal{L}_1 \times \mathcal{L}_p$ **do**
- 13: **if** $(\mathcal{A}, \mathcal{B}, C)$ is p -admissible **then**
- 14: Append $(\mathcal{A}, \mathcal{B})$ to L .

```

15:         end if
16:     end for
17: end for
18: end for
19: return L

```

Remark 4.17. Genera of \mathbb{Z} -lattices can be described by the Conway–Sloane genus symbol [26, 15 §7]. We have implemented an enumeration of all such genus symbols with a given signature and bounds on the scales of the Jordan components in SAGEMATH [74] and HECKE/OSCAR [31].

4.3. Enumeration of conjugacy classes of isometries.

Let $p \neq q$ be prime numbers. In this subsection, we give an algorithm which, for a given genus of \mathbb{Z} -lattices \mathcal{L} , computes a complete set of representatives for the isomorphism classes of lattices with isometry (L, f) of order $p^i q^j$ such that L is in \mathcal{L} .

Let (L, f) be a lattice with isometry. As before, we will drop f from the notation and simply denote it by L and the corresponding isometry by f_L . If $N \leq L$ is an f -invariant sublattice, we view it as a lattice with isometry $f_N = f|_N$.

The data structure we use for lattices with isometry is a triple (L, f_L, G_L) , where G_L is the image of $U(L) \rightarrow O(D_L)$ and $U(L)$ denotes the centralizer of f_L in $O(L)$.

By abuse of terminology, we call such a triple a lattice with isometry as well. So every algorithm in this section which returns lattices with isometry actually returns such triples (L, f_L, G_L) (or at least a function which is able to compute G_L when needed). We omit f_L and G_L from notation and denote the triple simply by L .

Definition 4.18. Let A be a lattice with an isometry of finite order m . For a divisor l of m , denote by H_l the sublattice $\ker \Phi_l(f_A)$ viewed as a Hermitian $\mathbb{Z}[\zeta_l]$ -lattice with ζ_l acting as $f_A|_{H_l}$. Denote by \mathcal{H}_l its genus as Hermitian lattice. For a divisor $l \mid m$, let $A_l = \ker(f_A^l - 1)$ and denote by \mathcal{A}_l its genus as \mathbb{Z} -lattice. The type of A is the collection $(\mathcal{A}_l, \mathcal{H}_l)_{l \mid m}$ and will be denoted by $t(A) = t(A, f_A)$.

Since we can encode a genus in terms of its symbol and can check for equivalence of two given symbols efficiently, the type is an effectively computable invariant.

Algorithm 2 PrimitiveExtensions

Input: Lattices A, B, C' with isometry such that (A, B, C') is p -admissible and $k \in \mathbb{N}$.

Output: A set of representatives of the double coset $G_B \backslash S / G_A$, where S is the set of all primitive extensions $A \perp B \subseteq C$ with $pC \subseteq A \perp B$ and $t(C') = t(C, f_C^k)$.

- 1: Initialize the empty list $L = []$.
- 2: Let $g \in \mathbb{N}$ be such that $p^{2g} \det A \det B = \det C'$.
- 3: Let μ_A (resp. μ_B) be the minimal polynomial of f_A (resp. f_B).
- 4: $V_A \leftarrow \ker \mu_B(D_{f_A}|_{(p^{-1}A \cap A^\vee)/A})$
- 5: $V_B \leftarrow \ker \mu_A(D_{f_B}|_{(p^{-1}B \cap B^\vee)/B})$
- 6: Let Gr_A be the set of f_A -stable subspaces of dimension g of V_A containing $p^l D_A$, where $p^{-l}\mathbb{Z} = \mathfrak{s}(C^\vee)$. Define Gr_B analogously. Form the set R consisting of anti-isometric pairs (H_A, H_B) , that is, $(H_A, q_A|_{H_A}) \cong (H_B, -q_B|_{H_B})$, as (H_A, H_B) runs through a set of representatives of Gr_A/G_A and Gr_B/G_B , respectively.
- 7: **for** $(H_A, H_B) \in R$ **do**
- 8: Compute an admissible glue map $\psi_0: H_A \rightarrow H_B$; see Definition 4.11.
- 9: Let $S_A \leq G_A$ (resp. $S_B \leq G_B$) be the stabilizer of H_A (resp. of H_B).
- 10: $S_A^{\psi_0} \leftarrow \psi_0 \text{Im}(S_A \rightarrow O(H_B))\psi_0^{-1}$
- 11: $\tilde{f}_A^{\psi_0} \leftarrow \psi_0(f_A|_{H_A})\psi_0^{-1}$

- 12: Compute an element $g \in O(H_B, \rho_{l+1}(B))$ such that $gf_A^{\psi_0}g^{-1} = f_B|_{H_B}$. If such an element does not exist, discard ψ_0 and continue the for loop in line 7.
 - 13: $\psi \leftarrow g \circ \psi_0, S_A^\psi \leftarrow gS_A^{\psi_0}g^{-1}$
 - 14: Let $O(H_B, \rho_{l+1}(B), f_B)$ be the group of isometries of H_B preserving $\rho_{l+1}(B)$ and commuting with the action of f_B .
 - 15: **for** $S_B hS_A^\psi \in S_B \setminus O(H_B, \rho_{l+1}(B), f_B) / S_A^\psi$ **do**
 - 16: Let $\Gamma_{h\psi}$ be the graph of $h\psi$.
 - 17: Define C by $C/(A \perp B) = \Gamma_{h\psi}$.
 - 18: $f_C \leftarrow f_A \oplus f_B$
 - 19: **if** $t(C, f_C^k) \neq t(C')$ **then**
 - 20: Discard C .
 - 21: **end if**
 - 22: $S_C \leftarrow \{(a, b) \in S_A \times S_B \mid b|_{H_B} \circ h\psi = h\psi \circ a|_{H_A}\}$
 - 23: $G_C \leftarrow \text{Im}(S_C \rightarrow O(D_C))$
 - 24: Append (C, f_C, G_C) to L .
 - 25: **end for**
 - 26: **end for**
 - 27: **return** L
-

Lemma 4.19. *Algorithm 2 is correct.*

Proof. Suppose that $A \perp B \subseteq C$ is an equivariant primitive extension with $pC \subseteq A \perp B$ and $t(C, f_C^p) = t(C')$. Let $\phi: H_A \rightarrow H_B$ be the corresponding glue map. It is admissible by Proposition 4.10. The existence of g in line 2 follows from (A, B, C') being p -admissible. Further, $pC \subseteq A \perp B$, gives $H_A \subseteq (p^{-1}A \cap A^\vee)/A$.

Since ϕ is equivariant, we get that $\mu_B(G_{f_A})$ vanishes on H_A . Hence, H_A is a g -dimensional subspace of the \mathbb{F}_p -vector space V_A . It is stable under f_A . Further, by Proposition 4.10, it contains $p^l D_A$. Similarly, H_B is preserved by f_B , contains $p^l D_B$ and is contained in V_B . Therefore, (H_A, H_B) appears in the for loop in line 7.

Since (A, B, C') is p -admissible, there exists an admissible glue map $\psi_0: H_A \rightarrow H_B$. It can be computed using normal forms of quadratic or bilinear forms over finite fields. The set of admissible glue maps from H_A to H_B is given by $O(H_B, \rho_{l+1}(B))\psi_0$. There exists an admissible equivariant glue map from H_A to H_B if and only if we find $g \in O(H_B, \rho_{l+1}(B))$ with

$$g\psi_0(f_A|_{H_A}) = (f_B|_{H_B})g\psi_0.$$

Reordering we find $g\psi_0(f_A|_{H_A})\psi_0^{-1}g^{-1} = f_B|_{H_B}$. This justifies lines 8 to 12 of the algorithm. So we continue with ψ an equivariant admissible glue map. Now, the set of equivariant admissible glue maps is $O(H_B, \rho_{l+1}(B), f_B)\psi$. Let $h\psi, h \in O(H_B, \rho_{l+1}(B), f_B)$ be an equivariant admissible glue map, and let $a \in S_A$ and $b \in S_B$. Then

$$bh\psi a = (b|_{H_B})h(\psi a\psi^{-1})\psi = h'\psi.$$

Therefore, $S_B h\psi S_A \mapsto S_B hS_A^\psi \psi$ defines a bijection of

$$S_B \setminus \{\text{equivariant admissible glue maps } \psi: H_A \rightarrow H_B\} / S_A$$

with the double coset

$$S_B \setminus O(H_B, \rho_{l+1}(B), f_B) / S_A^\psi.$$

Finally, the condition on (a, b) in the equation for S_C in line 22 of the algorithm is indeed the one to preserve $C/(A \perp B) \leq D_{A \perp B}$. Thus, S_C is the stabilizer of $C/(A \perp B)$ in $G_A \times G_B$. \square

Remark 4.20. The computation of representatives and their stabilizers in Steps 6 and 9 of Algorithm 2 can be very costly. In MAGMA [17], based on the algorithms in [63], a specialized method ORBITSOFSPACES for linear actions on Grassmannians is provided.

Algorithm 3 Representatives

Input: A lattice with isometry A such that $\Phi_n(f_A) = 0$ and an integer m , or just its type $t(A)$.

Output: Representatives of isomorphism classes of lattices with isometry B of order $m \cdot n$ and minimal polynomial Φ_{mn} such that $t(B, f_B^m) = t(A)$.

Algorithm 3 relies on an enumeration of genera of Hermitian lattices over maximal orders of number fields with bounds on the determinant and level. Then for each genus a single representative is computed [42, Algorithm 3.5.6] and its type is compared with that of A . Finally, Kneser's neighbor method [42, §5] is used to compute representatives for the isometry classes of the genus.

Algorithm 4 Split

Input: Two prime numbers $p \neq q$, an integer $e \geq 0$ and a lattice with isometry C such that $\Phi_{q^e}(f_C) = 0$.

Output: Representatives of the isomorphism classes of lattices with isometry M such that (M, f_M^p) is of the same type as C .

- 1: Initialize an empty list $L = []$.
 - 2: **for** $(A_0, B_0) \in \text{AdmissibleTriples}(p, C)$ **do**
 - 3: $R_1 \leftarrow \text{Representatives}(A_0, \text{id}_{A_0}, q^e)$, where A_0 is any representative of A_0
 - 4: $R_2 \leftarrow \text{Representatives}(B_0, \text{id}_{B_0}, pq^e)$, where B_0 is any representative of B_0
 - 5: **for** $(A, B) \in R_1 \times R_2$ **do**
 - 6: $E \leftarrow \text{PrimitiveExtensions}(A, B, C, p, p)$
 - 7: Append the elements of E to L .
 - 8: **end for**
 - 9: **end for**
 - 10: **return** L
-

Lemma 4.21. *Algorithm 4 is correct.*

Proof. Let M be a lattice with isometry such that (M, f_M^p) is of the same type as C . Then the minimal polynomial of f_M is a divisor of $\Phi_{pq^e} \Phi_{q^e}$. Let $M_{p^d q} = \ker \Phi_{pq^e}(f_M)$ and $M_{p^d} = \ker \Phi_{q^e}(f_M)$ be the corresponding sublattices. Then $(M_{p^d q}, M_{q^e}, C)$ is p -admissible by Lemma 4.15 applied to $f_M^{q^e}$. Hence, their \mathbb{Z} -genera appear at some point in the for loop in line 2. Similarly, at some point in the for loop in line 5, $A \cong M_{p^d q}$ and $B \cong M_{q^e}$ as Hermitian lattices. Then some lattice with isometry isomorphic to M is a member of E by Proposition 2.2.

Conversely, only lattices with isometry (M, f_M) with (M, f_M^p) of the same type as C are contained in E . No two pairs (A, B) in line 5 are isomorphic and for a given pair the extensions computed are mutually nonisomorphic by the correctness of Algorithm 2. Thus, no two elements of L can be isomorphic. \square

Algorithm 5 FirstP

Input: Two prime numbers $p \neq q$, $b \in \{0, 1\}$, an integer $e \geq 0$ and a lattice with isometry C of order q^e .

Output: Representatives of the isomorphism classes of lattices with isometry M such that (M, f_M^p) is of the same type as C . If $b = 1$, return only M such that f_M is of order pq^e .

- 1: Initialize an empty list $L = []$.
 - 2: **if** $e = 0$ **then**
 - 3: **return** Split(C, p), where in case $b = 1$ we return only those lattices M with f_M of order p .
 - 4: **end if**
 - 5: $A_0 \leftarrow \ker(\Phi_{q^e}(f_C))$
 - 6: $B_0 \leftarrow \ker(f_C^{q^{e-1}} - 1)$
 - 7: $\mathcal{A} \leftarrow \text{Split}(A_0, p, d, q, e)$
 - 8: Let e' be defined by $q^{e'} = \text{ord}(f_{B_0})$.
 - 9: $\mathcal{B} \leftarrow \text{FirstP}(B_0, p, q, e', 0)$
 - 10: **for** $(A, B) \in \mathcal{A} \times \mathcal{B}$ **do**
 - 11: **if** $b = 1$ and $p \nmid \text{ord}(f_A)$ and $p \nmid \text{ord}(f_B)$ **then**
 - 12: Discard (A, B) and continue the for loop with the next pair.
 - 13: **end if**
 - 14: $E \leftarrow \text{PrimitiveExtensions}(A, B, C, q, p)$
 - 15: Append the elements of E to L .
 - 16: **end for**
 - 17: **return** L
-

Lemma 4.22. *Algorithm 5 is correct.*

Proof. If $e = 0$, then f_C is the identity. So the call of Algorithm 4 in line 3 is valid and returns the correct result.

Let $e \geq 1$ and M be a lattice with isometry such that $t(M, f_M^p) = t(C)$. Then $f_M^{pq^e} = 1$. Set $f = f_M^{pq^{e-1}}$ which is of order q . We see that $A_0 = \ker(\Phi_q(f))$ and $B_0 = \ker(\Phi_1(f))$. Therefore, $A_0 \perp B_0 \subseteq M$ is a primitive extension and (A_0, B_0, M) is q -admissible.

Note that $\Phi_{q^e}(f_{A_0}) = 0$ so the input to Split (Algorithm 4) is valid and \mathcal{A} is a transversal of the classes of lattices A with isometry f_A such that (A, f_A^p) has the same type as A_0 . The order of f_{B_0} is a divisor of q^{e-1} . By induction on e , \mathcal{B} consists of all lattices B with isometry f_B such that (B, f_B^p) is of the same type as B_0 .

Thus, in line 10 we have $(\Phi_{pq^e} \Phi_{q^e})(f_A) = 0$ and $f_B^{pq^{e-1}} = 1$ (possibly $p \nmid \text{ord } f_B$). The condition in line 11 assures that if $b = 1$, then the order of $f_A \oplus f_B$ is indeed pq^e . Note that (A, B, C) in line 10 is indeed q -admissible because (A_0, B_0, C) is. Therefore, the call to PrimitiveExtensions is valid and returns a transversal of the isomorphism classes of lattices with isometry M with (M, f_M^p) of the same type as C . □

Algorithm 6 PureUp

Input: Two prime numbers $p \neq q$, integers $d > 0$, $e \geq 0$ and a lattice with isometry C such that $\prod_{i=0}^e \Phi_{p^d q^i}(f_C) = 0$.

Output: Representatives of the isomorphism classes of lattices with isometry M such that (M, f_M^p) is of the same type as C .

- 1: Initialize an empty list L
- 2: **if** $e = 0$ **then**
- 3: **return** Representatives(C, p)
- 4: **end if**
- 5: $A_0 \leftarrow \ker(\Phi_{p^d q^e}(f_C))$

```

6:  $B_0 \leftarrow A_0^\perp$ 
7: Let  $e'$  be defined by  $p^d q^{e'} = \text{ord}(f_{B_0})$ .
8:  $\mathcal{A} \leftarrow \text{Representatives}(A_0, p)$ 
9:  $\mathcal{B} = \text{PureUp}(B_0, p, d, q, e')$ 
10: for  $(A, B) \in \mathcal{A} \times \mathcal{B}$  do
11:    $E \leftarrow \text{PrimitiveExtensions}(A, B, C, q, p)$ 
12:   Append the elements of  $E$  to  $L$ .
13: end for
14: return  $L$ 

```

Lemma 4.23. *Algorithm 6 is correct.*

Proof. If $e = 0$, then $\Phi_{p^d}(f_C) = 0$, so `Representatives` does the job. Let M be in the output of `PureUp`. Since $d > 0$, we have $\prod_{i=0}^e \Phi_{p^{d+1}q^i}(f_M) = 0$. Therefore, M is a valid input to `PureUp` and we can proceed by induction on e . The details are similar to the proof of Algorithm 5. \square

Algorithm 7 NextP

Input: Two prime numbers $p \neq q$, integers $d, e \geq 0$ and a lattice with isometry C of order $p^d q^e$.
Output: Representatives of the isomorphism classes of lattices with isometry M of order $p^{d+1} q^e$ such that (M, f_M^p) is of the same type as C .

```

1: Initialize an empty list  $L$ 
2: if  $d = 0$  then
3:   return FirstP( $C, p, q, e, 1$ )
4: end if
5:  $B_0 \leftarrow \ker(f_C^{p^{d-1}q^e} - 1)$ 
6:  $A_0 \leftarrow B_0^\perp = \ker \prod_{i=0}^e \Phi_{p^d q^i}(f_C)$ 
7: Let  $e'$  be defined by  $p^d q^{e'} = \text{ord}(f_{A_0})$ .
8:  $\mathcal{A} \leftarrow \text{PureUp}(A_0, p, d, q, e')$ 
9: Let  $d'$  and  $e''$  be defined by  $p^{d'} q^{e''} = \text{ord}(f_{B_0})$ .
10:  $\mathcal{B} \leftarrow \text{NextP}(B_0, p, d', q, e'')$ 
11: for  $(A, B) \in \mathcal{A} \times \mathcal{B}$  do
12:    $E \leftarrow \text{PrimitiveExtensions}(A, B, C, p, p)$ 
13:   Append the elements of  $E$  to  $L$ .
14: end for
15: return  $L$ 

```

Lemma 4.24. *Algorithm 7 is correct.*

Proof. If $d = 0$, then f_C has order q^e . Hence, we can call `FirstP`. For $d > 0$, A_0 is a valid input for `PureUp`. The proof proceeds by induction on d since f_{B_0} has order at most $p^{d-1} q^e$. \square

By calling `NextP` on a complete set of representatives of the types of lattices with isometry of order $p^d q^e$, we can obtain a complete set of representatives for the isomorphism classes of lattices with isometry of order $p^{d+1} q^e$. By iterating this process, we have an algorithm to enumerate representatives for all isomorphism classes of lattices with isometries of a given order $p^d q^e$.

Remark 4.25. For the application to classifying finite groups of automorphisms of K3 surfaces, we note the following:

1. We only enumerate those lattices with the correct signatures and discard at each stage the lattices which are negative definite and contain (-2) -vectors since they do not lead to isometries preserving the ample cone.

- Let G be a finite subgroup of automorphisms of a complex K3 surface. Recall that $[G : G_s] = n$ satisfies $\varphi(n) \leq 20$ and $n \neq 60$. The integers with $\varphi(n) \leq 20$ and three prime factors are 30, 42, 60, 66 with $\varphi(n) = 8, 12, 16, 20$. Suppose $n = 66$. Since $\varphi(66) = 20 > 12$, we have $G_s = 1$. So G is cyclic and we know by [40] that the pair is unique. We can treat 42 with similar arguments. Finally, 30 is treated by hand with the help of some of the algorithms described above.

The actual computation was carried out using SAGEMATH [74], PARI [67], GAP [32] and MAGMA [17].

4.4. Computation of the group G_L

The algorithms of the previous section for enumerating isomorphism classes of lattices with isometry require as input for each lattice with isometry L the group G_L , which is the image of the natural map $U(L) \rightarrow U(D_L)$. Recall that we use a recursive approach and for primitive extensions C of lattices with isometry A and B , the group G_C can be determined using G_A and G_B (see Algorithm 2). It is therefore sufficient to explain how G_L can be computed for the lattices constructed in Algorithm 3, which form the base case of the recursive strategy. We therefore consider lattices with isometry L such that f_L has irreducible minimal polynomial and $\mathbb{Z}[f_L]$ is the maximal order of $\mathbb{Q}[f_L]$. We distinguish the following four cases:

- The lattice L is definite. Then $O(L)$ is finite and can be computed using an algorithm of Plesken and Souvignier [68].
- The lattice L is indefinite of rank 2 and $f_L = \pm 1$. For this situation, the computation of G_L will be explained in the remainder of this section.
- The lattice L is indefinite of rank ≥ 3 and $f_L = \pm 1$. In this case, Miranda–Morison theory [55, 56] along with some algorithms by Shimada [72] solve the problem. A short account of this is given in Section 5.
- The automorphism satisfies $f_L \neq \pm 1$. This will be addressed in Section 6, where we extend the theory of Miranda–Morison to the Hermitian case.

We end this section by describing the computation of G_L in case (2). Therefore, let L be an indefinite binary lattice over \mathbb{Z} and $V = L \otimes \mathbb{Q}$ the ambient quadratic space of discriminant $d \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$.

It follows from [30, §5] that we may assume that V is a two-dimensional étale \mathbb{Q} -algebra and $L \subseteq V$ is a \mathbb{Z} -lattice of rank 2. More precisely, V is isomorphic to the Clifford algebra C^+ , which in turn is isomorphic to $\mathbb{Q}(\sqrt{d})$. The \mathbb{Q} -algebra V is a quadratic extension of \mathbb{Q} if and only if d is not a square, which is the case if and only if V is anisotropic. If d is a square, then $V \cong \mathbb{Q} \times \mathbb{Q}$. If $\sigma : V \rightarrow V$ denotes the nontrivial automorphism of V as a \mathbb{Q} -algebra, then the quadratic form q on V is given by $q(x) = x\sigma(x)$ for $x \in V$. Note that $\sigma \in O(V)$ and $\det(\sigma) = -1$.

Every element $y \in V$ induces an endomorphism $\tau_y : V \rightarrow V, x \mapsto yx$ of determinant $\det(\tau_y) = y\sigma(y) = q(y)$. For a subset $X \subseteq V$, set $X^1 = \{x \in X \mid q(x) = 1\}$. The proper automorphism group $SO(V)$ of V is equal to $\{\tau_y \mid y \in V^1\}$ and $O(V) = \{\tau_y, \sigma\tau_y \mid y \in V^1\}$. We call two \mathbb{Z} -lattices I, J of V equivalent, if there exists $\alpha \in V^1$ such that $I = \alpha J$. Finally, set

$$\Lambda = \{x \in V \mid xL \subseteq L\},$$

which is a \mathbb{Z} -order of V .

Proposition 4.26. *The following hold:*

- We have $SO(L) = \{\tau_y \mid y \in (\Lambda^\times)^1\}$.
- If L is not equivalent to $\sigma(L)$, then $O(L) = SO(L)$.
- If L is equivalent to $\sigma(L)$, say $L = \alpha\sigma(L)$, then $O(L) = \langle SO(L), \sigma\tau_{\sigma(\alpha)} \rangle$.

Proof. First, note that for $y \in V$ we have $\tau_y(L) \subseteq L$ if and only if $y \in \Lambda$. This shows part (1).

Any isometry of L extends uniquely to an isometry of V and is—if the determinant is not 1—thus of the form $\sigma\tau_\alpha$ for some $\alpha \in V$. Hence, $L = (\sigma\tau_\alpha)(L) = \sigma(\alpha)\sigma(L)$, that is, L and $\sigma(L)$ are equivalent.

This shows part (2).

Now, assume that $L = \alpha\sigma(L)$. Then $(\sigma\tau_{\sigma(\alpha)})(L) = L$ and thus $\sigma\tau_{\sigma(\alpha)} \in O(L) \setminus SO(L)$. If $\sigma\tau_{\sigma(\beta)} \in O(L)$ is any nonproper isometry, then $\sigma(\beta)L = \sigma(L) = \sigma(\alpha)L$ and thus $\sigma(\beta\alpha^{-1}) \in (\Lambda^\times)^1$. This shows part (3) since $\sigma\tau_{\sigma(\beta)} = \sigma\tau_{\sigma(\alpha)}\tau_{\sigma(\beta\alpha^{-1})}$. □

Remark 4.27. We briefly describe how the previous result can be turned into an algorithm for determining generators of $O(L)$ for an indefinite binary lattice. We may assume that the ambient space V is an étale \mathbb{Q} -algebra of dimension two. The group Λ^\times is a finitely generated abelian group and generators can be computed as described in [16, 44]. Given generators of Λ^\times , determining generators of $(\Lambda^\times)^1$ is just a kernel computation. Finally, testing whether two \mathbb{Z} -lattices of V are equivalent can be accomplished using [16, 52].

5. Quadratic Miranda–Morrison theory

In this section, we review classical Miranda–Morrison theory for even indefinite \mathbb{Z} -lattices L of rank at least 3, as introduced by Miranda and Morrison in [55, 56]. Akyol and Degtyarev [1] incorporated sign structures to study connected components of the moduli spaces of plane sextics. We follow their example.

The purpose of this is twofold. First, this allows us to sketch the computation of the image of

$$O(L) \rightarrow O(D_L)$$

settling case (3) in Section 4.4. Second by incorporating the action on the sign structure, we obtain a way to compute the image of

$$O^+(L) \rightarrow O(D_L)$$

which yields the number of connected components of the moduli space \mathcal{F}_H (see Proposition 3.14 and Remark 3.16).

We denote by \mathbb{A} the ring of finite adeles and by $\mathbb{Z}_\mathbb{A}$ the ring of finite integral adeles. For a ring R , set $\Gamma_R = \{\pm 1\} \times R^\times / (R^\times)^2$. We define $O^\sharp(L \otimes R)$ as the kernel of $O(L \otimes R) \rightarrow O(D_L \otimes R)$. We note that $\Gamma_\mathbb{Q}$ has a natural diagonal embedding into $\Gamma_\mathbb{A}$ and $D_L \cong D_L \otimes \mathbb{Z}_\mathbb{A} \cong D_{L \otimes \mathbb{Z}_\mathbb{A}}$ naturally.

The homomorphisms

$$\sigma_p : O(L \otimes \mathbb{Q}_p) \rightarrow \Gamma_{\mathbb{Q}_p}, \quad g \mapsto (\det(g), \text{spin}(g))$$

induce a homomorphism

$$\sigma : O(L \otimes \mathbb{A}) \rightarrow \Gamma_\mathbb{A}.$$

Let $\Sigma^\sharp(L \otimes \mathbb{Z}_p)$ be the image of $O^\sharp(L \otimes \mathbb{Z}_p)$ under σ_p . We set $\Sigma(L) = \sigma(O(L \otimes \mathbb{Z}_\mathbb{A})) = \prod_p \Sigma(L \otimes \mathbb{Z}_p)$ and $\Sigma^\sharp(L) = \sigma(O^\sharp(L \otimes \mathbb{Z}_\mathbb{A})) = \prod_p \Sigma^\sharp(L \otimes \mathbb{Z}_p)$. By [54, VII 12.11], we have $\Sigma(L \otimes \mathbb{Z}_p) = \Sigma^\sharp(L \otimes \mathbb{Z}_p) = \Gamma_{\mathbb{Z}_p}$ whenever $L \otimes \mathbb{Z}_p$ is unimodular. By [54, IV.2.14 and IV.5.9], the natural map $O(L \otimes \mathbb{A}) \rightarrow O(D_L)$ is surjective. The following commutative diagram with exact rows and columns summarizes the situation (where by abuse of notation we denote restriction of σ by σ as well).

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & O^\sharp(L \otimes \mathbb{Z}_A) & \longrightarrow & O(L \otimes \mathbb{Z}_A) & \longrightarrow & O(D_L) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \sigma & & \downarrow & & \\
 1 & \longrightarrow & \Sigma^\sharp(L) & \longrightarrow & \Sigma(L) & \longrightarrow & \Sigma(L)/\Sigma^\sharp(L) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
 \end{array}$$

If V is an indefinite \mathbb{Q} -lattice of rank ≥ 3 , then the restriction $O(V) \rightarrow \Gamma_{\mathbb{Q}}$ of σ is surjective by [54, VIII 3.1].

Theorem 5.1 [54, VIII 5.1]. *Let L be an indefinite \mathbb{Z} -lattice of rank at least 3. Then we have the following exact sequence*

$$1 \rightarrow O^\sharp(L) \rightarrow O(L) \rightarrow O(D_L) \xrightarrow{\bar{\sigma}} \Sigma(L)/(\Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}} \cap \Sigma(L))) \rightarrow 1.$$

We need an analogous sequence with $O(L)$ replaced by $O^+(L)$ to compute connected components of the coarse moduli space of H -markable K3 surfaces. Define $\Gamma_{\mathbb{Q}}^+$ as the kernel of $\Gamma_{\mathbb{Q}} \rightarrow \{\pm 1\}$, $(d, s) \mapsto \text{sign}(ds)$. Then for any indefinite \mathbb{Q} -lattice V of rank ≥ 3 the homomorphism $\sigma^+ : O^+(V) \rightarrow \Gamma_{\mathbb{Q}}^+$ is surjective.

Theorem 5.2. *Let L be an indefinite \mathbb{Z} -lattice of rank at least 3. Then we have the following exact sequence*

$$O^+(L) \xrightarrow{D_+} O(D_L) \xrightarrow{\bar{\sigma}^+} \Sigma(L)/(\Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}}^+ \cap \Sigma(L))) \rightarrow 1.$$

Proof. We prove $\ker \bar{\sigma}_+ \subseteq \text{Im } D_+$: Let $\bar{g} \in O(D_L)$, and suppose that $\bar{\sigma}_+(\bar{g}) = 1$. This means that \bar{g} lifts to an element $g \in O(L \otimes \mathbb{Z}_A)$ with $D_g = \bar{g}$ and $\sigma(g) \in \Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}}^+ \cap \Sigma(L))$. After multiplying g with an element in $O^\sharp(L \otimes \mathbb{Z}_A)$, we may assume that

$$\sigma(g) \in \Gamma_{\mathbb{Q}}^+ \cap \Sigma(L).$$

Hence, there exists an element $h \in O(L \otimes \mathbb{Q})$ with $\sigma(h) = \sigma(g)$.

Since $\sigma(h^{-1}g) = 1$ and $h^{-1}g(L \otimes \mathbb{Z}_p) = L \otimes \mathbb{Z}_p$ for all but finitely many primes, we can use the strong approximation theorem (see, e.g., [42, 5.1.3], [45]) to get $f \in O(L \otimes \mathbb{Q})$ with $\sigma(f) = 1$ and $f(L \otimes \mathbb{Z}_p) = h^{-1}g(L \otimes \mathbb{Z}_p)$ at all primes and approximating $h^{-1}g$ at the finitely many primes dividing the discriminant. This yields (once the approximation is good enough) $D_f = D_{h^{-1}g}$ (cf [54, VIII 2.2]).

By construction, $hf \in O(L \otimes \mathbb{Q})$ preserves L and

$$D_{hf} = D_h \circ D_f = D_h \circ D_{h^{-1}g} = D_g$$

as desired. We have $\sigma(hf) = \sigma(h) \in \Gamma_{\mathbb{Q}}^+$. So $hf \in O^+(L)$.

We prove $\ker \bar{\sigma}_+ \supseteq \text{Im } D_+$: Let $g \in O^+(L)$. Then $\sigma(g) \in \Gamma_{\mathbb{Q}}^+$ and since $O(L) \subseteq O(L \otimes \mathbb{Z}_A)$ we have $\sigma(g) \in \Sigma(L)$ as well. □

The group $\Sigma(L)$ appearing in Theorems 5.1 and 5.2 is infinite and infinitely generated. However its quotient by $\Sigma^\sharp(L)$ is a finite group. We explain how to write it in terms of finite groups only so that it may be represented in a computer. Let T be the set of primes with $\Sigma^\sharp(L \otimes \mathbb{Z}_p) = \Sigma(L \otimes \mathbb{Z}_p) = \Gamma_{\mathbb{Z}_p}$ and S its complement. We know that S is contained in the set of primes dividing $\det L$. We can project the quotient $\Sigma(L)/\Sigma^\sharp(L)$ isomorphically to a subquotient of the finite group $\Gamma'_S = \prod_{p \in S} \Gamma_{\mathbb{Q}_p}$ and are thus reduced to a finite computation.

Indeed, denote by $\pi_S: \Gamma_{\mathbb{A}} \rightarrow \Gamma'_S$ the natural projection. Set

$$\Sigma_S(L) = \pi_S(\Sigma(L)), \quad \Sigma_S^\sharp(L) = \pi_S(\Sigma^\sharp(L)), \quad \Gamma_T = \prod_{p \in T} \Gamma_{\mathbb{Z}_p},$$

$$\Gamma_S = \pi_S(\Gamma_{\mathbb{Q}} \cap \Gamma_T \times \Gamma'_S) \quad \Gamma_S^+ = \pi_S(\Gamma_{\mathbb{Q}}^+ \cap \Gamma_T \times \Gamma'_S).$$

Γ_S^+ is spanned by the images of $\{(1, p) \mid p \in S\} \cup \{(-1, -1)\} \subseteq \Gamma_{\mathbb{Q}}^+ \subseteq \Gamma_{\mathbb{A}}^+$ under π_S . Since $\Gamma_{\mathbb{Q}}/\Gamma_{\mathbb{Q}}^+$ is spanned by $(-1, 1)$, Γ_S is spanned by the generators of Γ_S^+ together with $\pi_S((-1, 1))$.

Proposition 5.3. *The projection π_S induces isomorphisms*

$$\Sigma(L)/(\Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}}^+ \cap \Sigma(L))) \cong \Sigma_S(L)/(\Sigma_S^\sharp(L) \cdot (\Gamma_S^+ \cap \Sigma_S(L)))$$

and

$$\Sigma(L)/(\Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}} \cap \Sigma(L))) \cong \Sigma_S(L)/(\Sigma_S^\sharp(L) \cdot (\Gamma_S \cap \Sigma_S(L))).$$

Proof. Note that $\pi_S(\Gamma_{\mathbb{Q}}^+ \cap \Sigma(L)) = \Gamma_S^+ \cap \Sigma_S(L)$. Therefore

$$K := \pi_S^{-1}(\Sigma_S^\sharp(L) \cdot (\Gamma_S^+ \cap \Sigma_S(L))) = \Sigma^\sharp(L) \cdot (\Gamma_{\mathbb{Q}}^+ \cap \Sigma(L)).$$

Hence, the surjection

$$\psi: \Sigma(L) \rightarrow \Sigma_S(L)/(\Sigma_S^\sharp(L) \cdot (\Gamma_S^+ \cap \Sigma_S(L)))$$

induced by π_S has kernel K . We conclude by applying the homomorphism theorem to ψ . To prove the second isomorphism remove the $+$. □

The groups $\Sigma_S(L)$ and $\Sigma_S^\sharp(L)$ are found in the tables in [54, VII] in terms of the discriminant form of L and its signature pair.

Proposition 5.4. *Let L be an indefinite \mathbb{Z} -lattice of rank at least 3 and J a subgroup of the image of the natural map $D: O(L) \rightarrow O(D_L)$. Set $J^+ = D(O^+(L)) \cap J$ and let $K = \ker D = O^\sharp(L)$. Then $[J : J^+] = |\sigma_+(J)|$ and*

$$[K : K^+] = [\Gamma_{\mathbb{Q}} \cap \Sigma^\sharp(L) : \Gamma_{\mathbb{Q}}^+ \cap \Sigma^\sharp(L)] = [\Gamma_S \cap \Sigma_S^\sharp(L) : \Gamma_S^+ \cap \Sigma_S^\sharp(L)].$$

Proof. We have $J^+ = \ker(\bar{\sigma}_+) \cap J = \ker(\bar{\sigma}_+|_J)$. Therefore, $J/J_+ \cong \sigma_+(J)$.

The strong approximation theorem implies the equality $\sigma(O^\sharp(L)) = \Sigma^\sharp(L) \cap \Gamma_{\mathbb{Q}}$ and that $\sigma(O^\sharp(L)^+) = \Sigma^\sharp(L) \cap \Gamma_{\mathbb{Q}}$. □

Remark 5.5. The theorems allow us to compute the image of $O(L) \rightarrow O(D_L)$ for L an indefinite \mathbb{Z} -lattice of rank at least 3. Namely, one computes generators of $O(D_L)$ and lifts them p -adically to elements of $L \otimes \mathbb{Z}_p$ with sufficient precision. Then one can use these lifts to compute their spinor norm. See the work of Shimada [72] for further details. An algorithm for p -adic lifting and generators for $O(D_L)$ are given in [23].

6. Hermitian Miranda–Morrison theory

Let L be a lattice with isometry with irreducible minimal polynomial. In this section, we use the transfer construction to compute the image of $U(L) \rightarrow D(D_L)$, thus settling case (4) of Section 4.4. To this end, we develop the analogue of Miranda–Morrison theory for Hermitian lattices over the ring of integers of a number field.

6.1. Preliminaries on Hermitian lattices

In this section, we recall some basics on Hermitian lattices over the ring of integers of a number field or a local field. See [42] for an overview of the theory.

Let K be a finite extension of $F = \mathbb{Q}$ (global case) or $F = \mathbb{Q}_p$ (local case) and E an étale K -algebra of dimension 2. Let \mathcal{O} be the maximal order of E and \mathfrak{o} be the maximal order of K .

Definition 6.1. For E an étale K -algebra, we denote by $\text{Tr}_K^E : E \rightarrow K$ the trace and by $\mathfrak{D}_{E/K}^{-1} = \{x \in E \mid \text{Tr}_K^E(x\mathcal{O}) \subseteq \mathfrak{o}\}$ the inverse of the different.

In the local case, we let $\mathfrak{B} \subseteq \mathcal{O}$ be the largest ideal invariant under the involution of E/K and \mathfrak{p} the maximal ideal of \mathfrak{o} . Define e by $\mathfrak{B}^e = \mathfrak{D}_{E/K}$ and a by $\mathfrak{B}^a = \mathfrak{D}_{E/F}$.

Definition 6.2. Let (L, h) be a Hermitian \mathcal{O} -lattice. Its scale is the ideal $\mathfrak{s}(L) = h(L, L) \subseteq \mathcal{O}$, and its norm is the ideal $\mathfrak{n}(L) = \sum \{h(x, x)\mathfrak{o} \mid x \in L\} \subseteq \mathfrak{o}$.

It is known that

$$\mathfrak{D}_{E/K} \mathfrak{s}(L) \subseteq \mathfrak{n}(L) \subseteq \mathfrak{s}(L). \tag{6.1}$$

6.2. The trace lattice

In this subsection, (L, h) is a Hermitian \mathcal{O} -lattice. By transfer, we obtain its trace \mathbb{Z}_F -lattice (L, b) with $b = \text{Tr}_F^E \circ h$. Our primary interest is in even \mathbb{Z} -lattices. So our next goal is to give necessary and sufficient conditions for the trace lattice to be integral and even.

The Hermitian dual lattice is

$$L^\sharp = \{x \in L \otimes E \mid h(x, L) \subseteq \mathcal{O}\}$$

and $L^\vee = (L, b)^\vee$ is the dual lattice with respect to the trace form.

Proposition 6.3. We have

$$L^\vee = \mathfrak{D}_{E/F}^{-1} L^\sharp.$$

The trace form on L is integral if and only if $\mathfrak{s}(L)\mathfrak{D}_{E/F} \subseteq \mathcal{O}$.

The proof is left to the reader. We continue by determining the parity of the trace lattice. To this end, we first establish that the transfer construction behaves well with respect to completions, as expected.

For a place v of K , we use the following notation: For an \mathcal{O} -module M denote by $M_v = M \otimes \mathfrak{o}_v$ the completion of M at v and similar for K -vector spaces.

Proposition 6.4. Let $F = \mathbb{Q}$, E/K a degree two extension of number fields and p a prime number. Then

$$(L, \text{Tr}_\mathbb{Q}^E \circ h) \otimes \mathbb{Z}_p = \prod_{v|p} (L_v, \text{Tr}_{\mathbb{Q}_p}^{E_v} \circ h_v), \tag{6.2}$$

where v runs over all places extending the p -adic place of \mathbb{Q} .

Proof. To this end, consider the canonical isomorphism $K \otimes \mathbb{Q}_p \cong \prod_{v|p} K_v$, where the product runs over the prolongations of the p -adic valuation to K . By [71, Chap. II, §3, Prop. 4], this induces a canonical isomorphism

$$\mathcal{O} \otimes \mathbb{Z}_p \cong \prod_{v|p} \mathfrak{o}_v,$$

where \mathcal{O}_ν is the maximal order of K_ν . We obtain a corresponding canonical decomposition (using a system of primitive idempotents)

$$(L, h) \otimes_{\mathbb{Z}_p} = \bigsqcup_{\nu|p} (L_\nu, h_\nu), \tag{6.3}$$

where each summand is a Hermitian \mathcal{O}_ν -lattice. Note that \mathcal{O}_ν is indeed the maximal order of E_ν . The decomposition $E \otimes_{\mathbb{Q}_p} \cong \prod_{\nu|p} E_\nu$ and viewing Tr as the trace of the left multiplication endomorphism shows that

$$\text{Tr}_{\mathbb{Q}_p}^{E \otimes \mathbb{Q}_p} = \sum_{\nu|p} \text{Tr}_{\mathbb{Q}_p}^{E_\nu}.$$

Therefore, the trace commutes with the decomposition in eq. (6.3). □

Lemma 6.5. *Let K be a non-Archimedean dyadic local field of characteristic 0. Assume that $B \subseteq \mathcal{O}$ is a \mathbb{Z} -module such that $\text{Tr}_K^E(\mathcal{O}) \subseteq B$, $1 \in B$ and $N_K^E(\mathcal{O})B \subseteq B$. Then $\mathcal{O} \subseteq B$.*

Proof. If E/K is split or unramified, then $\text{Tr}_K^E(\mathcal{O}) = \mathcal{O}$. So let E/K be ramified. By [38, §6], there exists $u_0 \in K$ such that $\mathcal{O}^\times = N_K^E(\mathcal{O}^\times) \cup (1 + u_0)N_K^E(\mathcal{O}^\times)$ and $u_0\mathcal{O} = \mathfrak{p}^{e-1}$. Since $\mathfrak{p}^{e-1} \subseteq \mathfrak{p}^{\lfloor \frac{e}{2} \rfloor} = \text{Tr}_K^E(\mathcal{O})$ by [71, Ch. V, §3, Lemma 3], we have $(1 + u_0)N_K^E(\mathcal{O}^\times) \subseteq B$. Thus, $\mathcal{O}^\times \subseteq B$ and therefore $N_K^E(\mathcal{O})\mathcal{O}^\times \subseteq B$. As E/K is ramified it follows that $\mathcal{O} = N_K^E(\mathcal{O})\mathcal{O}^\times$. □

The following proof is inspired by [39, 3.1.9].

Proposition 6.6. *Let K be a non-Archimedean local field of characteristic 0 and (L, h) a Hermitian \mathcal{O} -lattice. The trace form $\text{Tr}_F^E \circ h$ is even if and only if $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$.*

Proof. Suppose that $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$. We have

$$\text{Tr}_F^E(h(x, x)) = 2 \text{Tr}_F^K(h(x, x)) \in 2 \text{Tr}_F^K(\mathfrak{n}(L)) \subseteq 2 \text{Tr}_F^K(\mathfrak{D}_{K/F}^{-1}) = 2\mathbb{Z}_F.$$

Now, suppose that the trace form is even. In particular, it is integral. We may assume that $F = \mathbb{Q}_2$ is dyadic. Let B be the set of all $\omega \in K$ such that $\text{Tr}_F^E(\omega h(x, x)) \in 2\mathbb{Z}_2$ for all $x \in L$. Then $\lambda \bar{\lambda} \omega h(x, x) = \omega h(\lambda x, \lambda x)$ for $\lambda \in \mathcal{O}$ gives $N(\mathcal{O})B \subseteq B$ and $1 \in B$. We calculate

$$\text{Tr}_F^E((\lambda + \bar{\lambda})h(x, x)) = \text{Tr}_F^E(\lambda h(x, x)) + \text{Tr}_F^E(\overline{\lambda h(x, x)}) = 2 \text{Tr}_F^E(h(\lambda x, x)) \in 2\mathbb{Z}_2$$

for all $\lambda \in \mathcal{O}$. This gives $\text{Tr}_K^E(\mathcal{O}) \subseteq B$. By Lemma 6.5 $\mathcal{O} \subseteq B$. Therefore, $\text{Tr}_F^K(\mathcal{O}h(x, x)) \subseteq \mathbb{Z}_2$ which means $h(x, x) \in \mathfrak{D}_{K/F}^{-1}$. □

We show that the same result holds in the global setting.

Corollary 6.7. *Let $F = \mathbb{Q}$ and E/K a degree two extension of number fields. Let (L, h) be a Hermitian \mathcal{O} -lattice. The trace form $\text{Tr}_F^E \circ h$ is even if and only if $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$.*

Proof. We use Proposition 6.4 and note that the orthogonal sum is even if and only if each summand is even. We apply Proposition 6.6 to each summand and obtain the condition that $\mathfrak{n}(L_\nu) \subseteq \mathfrak{D}_{\mathbb{Q}_2}^{K_\nu}$ for all $\nu \mid 2$. We conclude with $\nu(\mathfrak{n}(L)) = \nu(\mathfrak{n}(L_\nu))$. □

Note that even implies integral and similarly $\mathfrak{D}_{E/K} \mathfrak{s}(L) \subseteq \mathfrak{n}(L)\mathcal{O} \subseteq \mathfrak{D}_{K/F}^{-1}\mathcal{O}$ implies that $\mathfrak{s}(L) \subseteq \mathfrak{D}_{E/F}^{-1}$ which matches up perfectly with Proposition 6.3 and Corollary 6.7.

6.3. Discriminant form and transfer

Suppose that $L \subseteq L^\vee := \mathfrak{D}_{E/F}^{-1}L^\sharp$. Define a torsion Hermitian form on $D_L = L^\vee/L$ as follows

$$\bar{h}: D_L \times D_L \rightarrow E/\mathfrak{D}_{E/F}^{-1}, \quad ([x], [y]) \mapsto [h(x, y)].$$

Suppose further that $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$, that is, the trace form on L is even. Then we define the torsion quadratic form

$$\bar{q}: D_L \rightarrow K/\mathfrak{D}_{K/F}^{-1}, \quad [x] \mapsto [h(x, x)].$$

Note that $\text{Tr}_K^E(\mathfrak{D}_{E/F}^{-1}) = \mathfrak{D}_{K/F}^{-1}$. For a lattice L with even trace form, we set $U(D_L)$ to be the group of \mathcal{O} -linear automorphisms of D_L preserving \bar{q} .

Proposition 6.8. *Let (L, h) be a Hermitian \mathcal{O} -lattice with even trace form and $g \in U(D_L)$. Then g preserves \bar{h} .*

Proof. For $x \in L^\vee$, let gx denote a representative of $g(x + L)$. Let $x, y \in L^\vee$. Then

$$\text{Tr}_K^E h(x, y) = h(x + y, x + y) - h(x, x) - h(y, y).$$

Set $\delta(x, y) = h(x, y) - h(gx, gy)$. We have to prove that $\delta(x, y) \in \mathfrak{D}_{E/F}^{-1}$. Since g preserves \bar{q} ,

$$\text{Tr}_K^E(\delta(x, y)) \in \mathfrak{D}_{K/F}^{-1} = \text{Tr}_K^E(\mathfrak{D}_{E/F}^{-1}).$$

By the \mathcal{O} -linearity of g , we have $\alpha gx - g\alpha x \in L$ for all $\alpha \in \mathcal{O}$. Therefore, using $L^\vee = \mathfrak{D}_{E/F}^{-1}L^\sharp$, we have for any $\alpha \in \mathcal{O}$ that $\alpha\delta(x, y) \equiv \delta(\alpha x, y) \pmod{\mathfrak{D}_{E/F}^{-1}}$. Hence,

$$\text{Tr}_K^E(\mathcal{O}\delta(x, y)) \subseteq \mathfrak{D}_{K/F}^{-1}.$$

This means that $\delta(x, y) \in \mathfrak{D}_{K/F}^{-1}\mathfrak{D}_{E/K}^{-1} = \mathfrak{D}_{E/F}^{-1}$. □

Recall that for an even lattice with isometry (L, b, f) we have defined $U(D_L)$ as the centralizer of D_f in $O(D_L)$. By the transfer construction in Section 4.1, we may view L as a Hermitian lattice as well. The following proposition reconciles the two definitions of $U(D_L)$.

Proposition 6.9. *Let (L, b, f) be an even lattice with isometry with irreducible minimal polynomial and (L, h) the corresponding Hermitian $\mathbb{Z}[f]$ -lattice. Let $E \cong \mathbb{Q}[f]$, $K \cong \mathbb{Q}[f + f^{-1}]$ and \mathcal{O} be the maximal order of E . Suppose that (L, h) is invariant under \mathcal{O} , that is, (L, h) is a Hermitian \mathcal{O} -lattice (this is true if $\mathbb{Z}[f] = \mathcal{O}$). Then $U(D_L)$ is the centralizer of D_f in $O(D_L)$.*

Proof. It is clear that $U(D_L)$ centralizes D_f . So let $g \in O(D_L)$ centralize D_f . This implies that g is \mathcal{O} -linear. It remains to show that g preserves \bar{q} . Since $D_L = \perp_\nu D_{L_\nu}$ and $U(D_L) = \prod_\nu U(D_{L_\nu})$, we may assume that K is complete.

For $x + L \in D_L$, write gx for a representative of $g(x + L)$. Set $\delta = h(x, x) - h(gx, gx)$. We have to show that $\delta \in \text{Tr}_K^E(\mathfrak{D}_{E/F}^{-1}) = \mathfrak{D}_{K/F}^{-1}$. Since g preserves the discriminant form $q_{(L,b)}$, we have $2 \text{Tr}_F^K(\delta) = \text{Tr}_F^E(\delta) \in 2\mathbb{Z}_F$. Let B be the set of all $\omega \in \mathcal{O}$ such that $\text{Tr}_F^K(\omega\delta) \in \mathbb{Z}_F$. As in the proof of Proposition 6.6 one sees that $N(\mathcal{O})B \subseteq B$, $1 \in B$ and $\text{Tr}_K^E(\mathcal{O}) \subseteq B$. Then Lemma 6.5 provides $B = \mathcal{O}$. Thus, $\text{Tr}_F^K(\mathcal{O}\delta) \subseteq \mathbb{Z}_F$, that is, $\delta \in \mathfrak{D}_{K/F}^{-1}$. □

Remark 6.10. Proposition 6.9 provides a practical way to compute $U(D_L)$. We can write down a system of generators for $O(D_L)$. Then the computation of a centralizer is a standard task in computational group theory.

6.4. Local surjectivity of $U(L) \rightarrow U(D_L)$

In this subsection, we assume that K is a non-Archimedean local field of characteristic 0, π a prime element of \mathcal{O} , $p = \pi\bar{\pi}$, and L a Hermitian \mathcal{O} -lattice with $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$, that is, its trace lattice is even. Recall that $\mathfrak{P} \subseteq \mathcal{O}$ is the largest ideal invariant under the involution of E/K , \mathfrak{p} the maximal ideal of \mathcal{o} and the integers a, e satisfy $\mathfrak{P}^e = \mathfrak{D}_{E/K}$ and $\mathfrak{P}^a = \mathfrak{D}_{E/F}$.

If E/K is a ramified field extension, then by [71, Ch. V, §3, Lemma 3] we have for all $i \in \mathbb{Z}$ that $\text{Tr}(\mathfrak{P}^i) = \mathfrak{p}^{\lfloor \frac{i+e}{2} \rfloor}$. Therefore, $\text{Tr}(\mathfrak{P}^{1-e}) = \mathcal{o}$ and $\text{Tr}(\mathfrak{P}^{2-e}) = \mathfrak{p}$. So there exists $\rho \in E$ with $\rho\mathcal{O} = \mathfrak{P}^{1-e}$ and $\text{Tr}(\rho) = 1$. If E/K is an unramified field extension, then we find $\rho \in \mathcal{O}^\times$ with $\text{Tr}(\rho) = 1$. If $E = K \times K$, then we can take $\rho = (1, 0) \in \mathcal{O} \times \mathcal{O} = \mathcal{O}$ which satisfies $\text{Tr}(\rho) = 1$ as well.

For a Hermitian matrix $G \in E^{n \times n}$, set $\mathfrak{s}(G) = \mathfrak{s}(L)$ and $\mathfrak{n}(G) = \mathfrak{n}(L)$, where L is the free \mathcal{O} -lattice with gram matrix G .

Algorithm 8 Hermitian lift

Input: $0 \leq l \in \mathbb{Z}, \rho \in E, G = \bar{G}^t \in E^{n \times n}, F \in \text{GL}_n(\mathcal{O})$ such that

- $\text{Tr}_K^E(\rho) = 1$,
- $\mathfrak{s}(G^{-1}) \subseteq \mathfrak{P}^{1+a}, \rho \mathfrak{n}(G^{-1}) \subseteq \mathfrak{P}^{1+a}$,
- $R := G - FG\bar{F}^t$ with $\mathfrak{s}(R) \subseteq \mathfrak{P}^{l-a}, \rho \mathfrak{n}(R) \subseteq \mathfrak{P}^{l-a}$.

Output: $F' \in \text{GL}_n(\mathcal{O})$ such that for $l' = 2l + 1$ and $R' = G - F'G\bar{F}'^t$ the following hold

- $F' \equiv F \pmod{\mathfrak{P}^l \mathcal{O}^{n \times n} \pi^{-a} G^{-1} \subseteq \mathfrak{P}^{l+1} \mathcal{O}^{n \times n}}$,
- $\mathfrak{s}(R') \subseteq \mathfrak{P}^{l'-a}, \rho \mathfrak{n}(R') \subseteq \mathfrak{P}^{l'-a}$.

1: $R \leftarrow G - FG\bar{F}^t$

2: Write $R = U + D + \bar{U}^t$ with U upper triangular and D diagonal.

3: **return** $F + (U + \rho D)\bar{F}^{-t}G^{-1}$

Theorem 6.11. *Algorithm 8 is correct.*

Proof. With $X = (U + \rho D)\bar{F}^{-t}$ and $F' = F + XG^{-1}$, we calculate

$$\begin{aligned} F'G\bar{F}'^t &= FG\bar{F}^t + U + \bar{U}^t + \text{Tr}_K^E(\rho)D + XG^{-1}\bar{X}^t \\ &= G + XG^{-1}\bar{X}^t. \end{aligned}$$

Hence, $R' = -XG^{-1}\bar{X}^t$. Since $R \equiv 0 \pmod{\mathfrak{s}(R)}$, $D \equiv 0 \pmod{\mathfrak{n}(R)}$, we have $U + \rho D \equiv 0 \pmod{\mathfrak{s}(R) + \rho \mathfrak{n}(R) \subseteq \mathfrak{P}^{l-a}}$, hence $U + \rho D \equiv 0 \pmod{\mathfrak{P}^{l-a}}$. Together with $F \in \text{GL}_n(\mathcal{O})$ this implies

$$X \equiv 0 \pmod{\mathfrak{P}^{l-a}}.$$

Hence,

$$\mathfrak{s}(R') = \mathfrak{s}(XG^{-1}\bar{X}^t) \subseteq \mathfrak{P}^{2l-2a} \mathfrak{s}(G^{-1}) \subseteq \mathfrak{P}^{2l+1-a}$$

and

$$\rho \mathfrak{n}(R') = \mathfrak{P}^{2l-2a} \rho \mathfrak{n}(G^{-1}) \subseteq \mathfrak{P}^{2l+1-a} = \mathfrak{P}^{2l+1-a}.$$

It remains to show that $F' \in \text{GL}_n(\mathcal{O})$. Since $\mathfrak{s}(G^{-1}) \subseteq \mathfrak{P}^{1+a}$, we have $\mathfrak{P}^l \mathcal{O}^{n \times n} \pi^{-a} G^{-1} \subseteq \mathfrak{P}^{l+1} \mathcal{O}^{n \times n}$. Therefore, $F \equiv F' \pmod{\mathfrak{P}^{l+1}}$. □

Theorem 6.12. *Let K be a non-Archimedean local field of characteristic 0 and L a Hermitian \mathcal{O} -lattice with even trace lattice. Then $U(L) \rightarrow U(D_L)$ is surjective.*

Proof. We take an orthogonal splitting $L = M \perp N$ with M being $\mathfrak{D}_{E/F}^{-1}$ -modular and $\mathfrak{s}(N) \subseteq \mathfrak{D}_{E/F}^{-1}$. Then $\mathfrak{D}_{E/F}^{-1}L^\sharp/L \cong \mathfrak{D}_{E/F}^{-1}N^\sharp/N$. After replacing L with N , we may and will assume that $\mathfrak{s}(L) \subseteq \mathfrak{D}_{E/F}^{-1}$.

Recall that $\mathfrak{P}^a = \mathfrak{D}_{E/F}$. Identify $L^\vee = \mathfrak{D}_{E/F}^{-1}L^\sharp$ with $\mathcal{O}^n = \mathcal{O}^{1 \times n}$ by choosing a basis. Let G be the respective gram matrix of $\mathfrak{D}_{E/F}^{-1}L^\sharp$. We have $L = \mathcal{O}^n \pi^{-a} G^{-1}$ and $p^{-a}G^{-1}$ is the corresponding Gram matrix of L . Therefore

$$\mathfrak{s}(G^{-1}) \subseteq \mathfrak{P}^{1+a} \text{ and } \mathfrak{n}(G^{-1}) \subseteq \mathfrak{P}^{e+a}.$$

If E/K is unramified or split, then $e = 0$, $\mathfrak{s}(G^{-1}) = \mathfrak{n}(G^{-1})$ by eq. (6.1) and we find $\rho \in \mathcal{O}$ with $\text{Tr}(\rho) = 1$. Therefore $\rho \mathfrak{n}(G^{-1}) = \rho \mathfrak{s}(G^{-1}) \subseteq \mathfrak{s}(G^{-1}) \subseteq \mathfrak{P}^{1+a}$ holds. If E/K is ramified, we find $\rho \in E$ with $\rho \mathcal{O} = \mathfrak{P}^{1-e}$. Then $\rho \mathfrak{n}(G^{-1}) \subseteq \mathfrak{P}^{1-e+e+a} = \mathfrak{P}^{1+a}$ as well.

Let $f \in U(D_L)$ be represented by $F \in \text{GL}_n(\mathcal{O})$, that is,

$$f(x + L) = xF + L = xF + \mathcal{O}^n \pi^{-a} G^{-1}.$$

Set $R = G - FG\bar{F}^t$. Since f preserves \bar{h} and \bar{q} , we have

$$\mathfrak{s}(R) \subseteq \mathfrak{P}^{-a} \text{ and } \rho \mathfrak{n}(R) \subseteq \rho \mathfrak{D}_{K/F}^{-1} \subseteq \rho \mathfrak{P}^{e-a} \subseteq \mathfrak{P}^{-a},$$

where in the last equality we used that $\rho \mathfrak{P}^e \subseteq \mathcal{O}$ irrespective of E/K being inert, split or ramified. Set $F_0 = F$. We inductively define a sequence by setting F_{i+1} to be the output of Algorithm 8 with $l = 2^i - 1$, $F \leftarrow F_i$, $G \leftarrow G$ as given and $R \leftarrow R_i := G - F_i G \bar{F}_i^t$. Then $\mathfrak{s}(R_i) \subseteq \mathfrak{P}^{2^i - 1 - a}$ and $\rho \mathfrak{n}(R_i) \subseteq \mathfrak{P}^{2^i - 1 - a}$. Since $F_i \equiv F_{i+1} \pmod{\mathfrak{P}^{2^i}}$, the sequence $(F_i)_{i \in \mathbb{N}}$ converges. Its limit is the desired lift. \square

For a Hermitian lattice L with even trace lattice, we denote by $U^\sharp(L)$ the kernel of $U(L) \rightarrow U(D_L)$.

6.5. Local to global

Let E/K be a quadratic extension of number fields with nontrivial automorphism $\bar{\cdot} : E \rightarrow E$. Let \mathfrak{o} be the maximal order of K and \mathcal{O} the maximal order of E . In this section, L is a Hermitian \mathcal{O} -lattice with even trace lattice. The goal of this subsection is to compute the image of the natural map

$$D : U(L) \rightarrow U(D_L).$$

Denote by \mathbb{A}_K the ring of finite adèles of K . Denote by $\mathfrak{o}_{\mathbb{A}}$ the ring of integral finite adèles of K . We have natural isomorphisms $D_L \cong D_L \otimes \mathfrak{o}_{\mathbb{A}} \cong D_L \otimes \mathfrak{o}_{\mathbb{A}}$. Via the diagonal embedding we view K as a subring of \mathbb{A}_K . This induces the inclusion $U(V) \subseteq U(L \otimes \mathbb{A}_K)$. Let $\det : U(L \otimes \mathbb{A}_K) \rightarrow \prod_{\mathfrak{p}} E_{\mathfrak{p}}$ denote the componentwise determinant. Set

$$\mathcal{F}(E) = \{(x)_{\mathfrak{p}} \in \prod_{\mathfrak{p}} E_{\mathfrak{p}} \mid x \in E, x\bar{x} = 1\},$$

$$\begin{aligned} \mathcal{F}(L_{\mathfrak{p}}) &= \det(U(L_{\mathfrak{p}})), & \mathcal{F}^\sharp(L_{\mathfrak{p}}) &= \det(U^\sharp(L_{\mathfrak{p}})), \\ \mathcal{F}(L) &= \det(U(L \otimes \mathfrak{o}_{\mathbb{A}})), & \mathcal{F}^\sharp(L) &= \det(U^\sharp(L \otimes \mathfrak{o}_{\mathbb{A}})). \end{aligned}$$

Note that $U(L \otimes \mathcal{O}_A) \rightarrow U(D_L)$ is surjective by Theorem 6.12. The following commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & U^\sharp(L \otimes \mathcal{O}_A) & \longrightarrow & U(L \otimes \mathcal{O}_A) & \longrightarrow & U(D_L) \longrightarrow 1 \\
 & & \downarrow \det & & \downarrow \det & & \downarrow \\
 1 & \longrightarrow & \mathcal{F}^\sharp(L) & \longrightarrow & \mathcal{F}(L) & \longrightarrow & \mathcal{F}(L)/\mathcal{F}^\sharp(L) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1.
 \end{array}$$

with exact rows and columns summarizes the situation.

Proposition 6.13. *Let V be a nondegenerate Hermitian space over E/K . Then $\det(U(V)) = \mathcal{F}(E)$.*

Proof. We know $\det(U(V)) \subseteq \mathcal{F}(E)$. The other inclusion is clear when $\dim_E V$ is one. Since we can always split a subspace of dimension one, the statement follows. \square

Set $\mathcal{O}_A := \mathcal{O} \otimes \mathcal{O}_A$. For an isometry $f : L \rightarrow M$ of Hermitian \mathcal{O}_A -lattices, we denote by $D_f = (D_{f_p})_p$ the induced map on the discriminant forms. Let $L \subseteq L^\vee$ be a Hermitian \mathcal{O}_A -lattice. Then $\sigma \in O(L \otimes \mathbb{A}_K)$ induces an isometry $\sigma : L \rightarrow \sigma(L)$ of \mathcal{O}_A -lattices and an isometry $D_\sigma : D_L \rightarrow D_{\sigma(L)}$ of the respective discriminant groups.

Proposition 6.14. *Let L be an indefinite Hermitian \mathcal{O} -lattice with $\text{rk}(L) \geq 2$. For $\sigma \in U(L \otimes \mathbb{A}_K)$, the following are equivalent:*

1. *There is a map $\varphi \in U(L \otimes K)$ such that $D_\varphi = D_\sigma$ and $\varphi(L \otimes \mathcal{O}_A) = \sigma(L \otimes \mathcal{O}_A)$.*
2. *$\det(\sigma) \in \mathcal{F}(E) \cdot \mathcal{F}^\sharp(L)$.*

Proof. First, suppose that a map φ as in (1) exists. Since $\varphi(L \otimes \mathcal{O}_A) = \sigma(L \otimes \mathcal{O}_A)$ and $D_\varphi = D_\sigma$, we have $\varphi^{-1} \circ \sigma \in U^\sharp(L \otimes \mathcal{O}_A)$. Thus,

$$\det(\sigma) \in \mathcal{F}(E) \cdot \mathcal{F}^\sharp(L).$$

Now, suppose that $\det(\sigma) \in \mathcal{F}(E) \cdot \mathcal{F}^\sharp(L)$. Then there exists $u \in \mathcal{F}(E)$ and $\rho \in U^\sharp(L \otimes \mathcal{O}_A)$ such that $\det(\sigma) = u \det(\rho)$. By Proposition 6.13, there exists $\psi \in U(L \otimes K)$ with $\det(\psi) = u$. Let $\phi := \psi^{-1} \circ \sigma \circ \rho^{-1}$. Then $\det(\phi) = 1$. By the strong approximation theorem [45], there exists $\eta \in U(L \otimes K)$ with $\eta(L \otimes \mathcal{O}_A) = \phi(L \otimes \mathcal{O}_A)$ and $D_\eta = D_\phi$ (approximate ϕ at the finitely many primes dividing the discriminant and those with $\phi_p(L_p) \neq L_p$). Set $\varphi := \psi \circ \eta \in U(L \otimes K)$. Then

$$\varphi(L \otimes \mathcal{O}_A) = (\psi \circ \eta)(L \otimes \mathcal{O}_A) = (\psi \circ \phi)(L \otimes \mathcal{O}_A) = (\sigma \circ \rho^{-1})(L \otimes \mathcal{O}_A) = \sigma(L \otimes \mathcal{O}_A).$$

Further,

$$D_\varphi = D_\psi \circ D_\eta = D_\sigma \circ D_\rho^{-1} = D_\sigma$$

since D_ρ is the identity because $\rho \in U^\sharp(L \otimes \mathcal{O}_A)$. \square

Theorem 6.15. *Let L be an indefinite Hermitian \mathcal{O} -lattice with $\text{rk}(L) \geq 2$. Then there is an exact sequence*

$$U(L) \rightarrow U(D_L) \xrightarrow{\delta} \mathcal{F}(L)/(\mathcal{F}(E) \cap \mathcal{F}(L)) \cdot \mathcal{F}^\sharp(L) \rightarrow 1,$$

where δ is induced by the determinant.

Proof. Let $\hat{\gamma} \in U(D_L)$, and lift it to some $\gamma \in U(L \otimes \mathcal{O}_A)$ with $D_\gamma = \bar{\gamma}$. By Proposition 6.14, γ lies in $U(L \otimes K)$ if and only if $\det(\gamma) \in \mathcal{F}(E) \cdot \mathcal{F}^\sharp(L)$ which is equivalent to

$$\det(\gamma) \in (\mathcal{F}(E) \cdot \mathcal{F}^\sharp(L)) \cap \mathcal{F}(L) = (\mathcal{F}(E) \cap \mathcal{F}(L)) \cdot \mathcal{F}^\sharp(L)$$

and this does not depend on the choice of lift γ of $\bar{\gamma}$. We conclude with the general fact that $U(L) = U(L \otimes K) \cap U(L \otimes \mathcal{O}_A)$. □

In order to make Theorem 6.15 effective, we will compute the groups $\mathcal{F}^\sharp(L_p)$ and $\mathcal{F}(L_p)$ in Section 6.6. See Theorem 6.25 for the exact values.

Remark 6.16. Let S be the set of primes of K dividing the order of D_L . For practical purposes, we note that $\mathcal{F}(L_p) = \mathcal{F}^\sharp(L_p)$ for the primes not in S . Hence, $\mathcal{F}(L)/\mathcal{F}^\sharp(L) \cong \prod_{p \in S} \mathcal{F}(L_p)/\mathcal{F}^\sharp(L_p)$ and it is enough to compute δ_p for the primes in S . This can be achieved by lifting $\bar{\gamma} \in U(D_{L_p})$ to some $\gamma \in U(L_p)$ with sufficient precision using Algorithm 8.

6.6. Generation of $U^\sharp(L)$ by symmetries

Let K be a finite extension of $F = \mathbb{Q}_p$ and E/K a ramified quadratic extension. Let $\text{Tr} = \text{Tr}_K^E$ be the trace. Recall that $\mathfrak{P} \subseteq \mathcal{O}$ is the largest ideal invariant under the involution of E/K , \mathfrak{p} the maximal ideal of \mathcal{O} and the integers a, e satisfy $\mathfrak{P}^e = \mathfrak{D}_{E/K}$ and $\mathfrak{P}^a = \mathfrak{D}_{E/F}$. Note that as E/K is ramified we have $a \equiv e \pmod{2}$. Let $\pi \in \mathcal{O}$ be a prime element and $p = \pi\bar{\pi}$. For any $v \equiv e \pmod{2}$, there exists a skew element $\omega \in E^\times$ with $v_{\mathfrak{P}}(\omega) = v$.

Let V be a nondegenerate Hermitian space over E . In what follows, L is a full \mathcal{O} -lattice in V with even trace form. Therefore, its scale and norm satisfy

$$\mathfrak{s}(L) =: \mathfrak{P}^i \subseteq \mathfrak{P}^{-a} \text{ and } \mathfrak{n}(L) =: \mathfrak{p}^k \subseteq \mathfrak{D}_{K/F}^{-1} = \mathfrak{P}^{e-a}.$$

This gives the inequalities $0 \leq i + a$ and $0 \leq 2k + a - e$ and by eq. (6.1) $i \leq 2k \leq i + e$. We say that L is *subnormal* if $\mathfrak{n}(L)\mathcal{O} \subseteq \mathfrak{s}(L)$, that is, $i < 2k$. A sublattice of rank two is called a *plane* and a sublattice of rank one a *line*. By [37, Propositions 4.3, 4.4], the lattice L decomposes into an orthogonal direct sum of lines and subnormal planes.

The group $U^\sharp(L)$ is the kernel of the natural map

$$U(L) \rightarrow U(D_L).$$

For $\varphi \in U(L)$, we have $\varphi \in U^\sharp(L)$ if and only if $(\varphi - \text{id}_L)(L^\vee) \subseteq L$. For $x, y \in L$, we write $x \equiv y \pmod{\mathfrak{P}^i}$ if $x - y \in \mathfrak{P}^i L$.

We single out the elements of $U(V)$ fixing a hyperplane—the symmetries.

Definition 6.17. Let V be a Hermitian space, $s \in V$ and $\sigma \in E^\times$ with $h(s, s) = \text{Tr}(\sigma)$. We call the linear map

$$S_{s,\sigma}: V \rightarrow V, \quad x \mapsto x - h(x, s)\sigma^{-1}s$$

a *symmetry* of V . It preserves the Hermitian form $h(\cdot, \cdot)$. If s is isotropic, then we have $\det(S_{s,\sigma}) = 1$ and otherwise $\det(S_{s,\sigma}) = -\bar{\sigma}/\sigma$. The inverse is given by $S_{s,\sigma}^{-1} = S_{s,\bar{\sigma}}$. Note that the symmetry $S_{s,\sigma}$ of V preserves L if $s \in L$ and $h(L, s) \subseteq \mathcal{O}\sigma$. We denote the subgroup of $U(L)$ generated by the symmetries preserving L by $S(L)$ and set $S^\sharp(L) = U^\sharp(L) \cap S(L)$.

By [22], symmetries generate the unitary group $S(L) = U(L)$ if $\mathcal{O}/\mathfrak{P} \neq \mathbb{F}_2$. Otherwise, one has to include so called rescaled Eichler isometries, which are isometries fixing subspaces of codimension 2. Fortunately, as we will see, symmetries suffice to generate $U^\sharp(L)$. The condition that the trace form on L is even eliminates all the technical difficulties of [22].

Lemma 6.18. *Let $\varphi \in U^\sharp(L)$ and $x \in L$. Then $\varphi(x) - x \in h(x, L)\mathfrak{P}^a L$.*

Proof. For any $x \in L$, the inclusion $h(x, L)^{-1}x \subseteq L^\sharp$ gives

$$\mathfrak{P}^{-a}h(x, L)^{-1}x \subseteq \mathfrak{P}^{-a}L^\sharp = L^\vee.$$

Hence, $(\varphi(x) - x)h(x, L)^{-1}\mathfrak{P}^{-a} \subseteq (\varphi - \text{id}_V)(L^\vee) \subseteq L$. Multiply by the ideal $h(x, L)\mathfrak{P}^a$ to reach the conclusion. □

Lemma 6.19. *Let $S_{s,\sigma}$ be a symmetry of V with $s \in \mathfrak{P}^{i+a}L$. Then $S_{s,\sigma} \in U^\sharp(L)$ if $\mathfrak{P}^{2i+a} \subseteq \sigma\mathcal{O}$.*

Proof. We have $(\text{id}_V - S_{s,\sigma})(\mathfrak{P}^{-a}L^\sharp) = h(\mathfrak{P}^{-a}L^\sharp, s)\sigma^{-1}s \subseteq \mathfrak{P}^i\sigma^{-1}s \subseteq L$. □

Lemma 6.20. *Let $x, x' \in L$ with $h(x, x) = h(x', x')$, $h(x, L) = h(x', L) = \mathfrak{P}^i$ and $x \equiv x' \pmod{\mathfrak{P}^{i+a}}$. Then there is an element $\varphi \in S^\sharp(L)$ with $\varphi(x) = x'$.*

Proof. Note that $h(x, x - x') \in \mathfrak{P}^{2i+a}$. If $h(x, x - x')\mathcal{O} = \mathfrak{P}^{2i+a}$, then with $\sigma = h(x, x - x')$ and $s = x - x'$ we have $S_{s,\sigma}(x) = x'$, and Lemma 6.19 implies that $S_{s,\sigma} \in U^\sharp(L)$.

If $h(x, x - x')\mathcal{O} \subseteq \mathfrak{P}^{2i+a+1}$, choose $s \in \mathfrak{P}^{i+a}L$ with

$$h(s, x)\mathcal{O} = h(s, x')\mathcal{O} = \mathfrak{P}^{2i+a}$$

which is possible since $h(x, L) = h(x', L) = \mathfrak{P}^i$. We have $v_{\mathfrak{P}}(h(s, s)\rho) \geq 2i + 2a + 2k + 1 - e > 2i + a$ and $2i + a \equiv e \pmod{2}$. With $\omega \in E$ a skew element of valuation $2i + a$, $\sigma := h(s, s)\rho + \omega$ satisfies $v_{\mathfrak{P}}(\sigma) = 2i + a$. By Lemma 6.19, we have $S_{s,\sigma} \in S^\sharp(L)$. Then

$$h(x, x - S_{s,\sigma}(x')) = h(x, x - x') + h(x, s)h(s, x')\bar{\sigma}^{-1}$$

gives $h(x, x - S_{s,\sigma}(x'))\mathcal{O} = \mathfrak{P}^{2i+a}$. Further, $x \equiv S_{s,\sigma}(x') \pmod{\mathfrak{P}^{i+a}}$. Thus, by the first case we can map x to $S_{s,\sigma}(x')$. □

Lemma 6.21. *Let $L = P \perp M$, with P a subnormal plane. Then $U^\sharp(L) = S^\sharp(L)U^\sharp(M)$.*

Proof. By [37], there exists a basis $u, v \in P$ with $h(u, u) = p^k$, $h(v, v) \in \mathfrak{p}^k$ and $h(u, v) = \pi^i$. Note that L subnormal implies $i < 2k$. Let $\varphi \in U^\sharp(L)$. By Lemma 6.20, there exists a symmetry $S \in S^\sharp(L)$ with $S(u) = \varphi(u)$. Therefore, we may and will assume that $\varphi(u) = u$.

Write $\varphi(v) = \gamma u + \delta v + m$ for some $m \in \mathfrak{P}^{i+a}L$ and $\gamma, 1 - \delta \in \mathfrak{P}^{i+a}$. Then we have

$$\begin{aligned} h(v, v - \varphi(v))\mathcal{O} &= (-\bar{\gamma}\bar{\pi}^i + (1 - \bar{\delta})h(v, v))\mathcal{O} \\ h(u, v - \varphi(v)) &= h(u, v) - h(\varphi(u), \varphi(v)) = 0. \end{aligned} \tag{6.4}$$

The symmetry $S_{s,\sigma} \in U(L \otimes E)$ with $s = v - \varphi(v)$ and $\sigma = h(v, v - \varphi(v))$ preserves u and maps v to $\varphi(v)$. If $v_{\mathfrak{P}}(\gamma) = i + a$, then

$$v_{\mathfrak{P}}((1 - \bar{\delta})h(v, v)v) \geq i + a + 2k > 2i + a = v_{\mathfrak{P}}(-\bar{\gamma}\bar{\pi}^i).$$

Thus, $h(v, v - \varphi(v))\mathcal{O} = \mathfrak{P}^{2i+a}$. It follows that $S_{s,\sigma} \in S^\sharp(L)$ by Lemma 6.19 and we are done.

Let now $v_{\mathfrak{P}}(\gamma) > i + a$. We consider $v' = u - \pi^i p^{k-i}v \in L$. It satisfies

$$h(u, v') = 0, \quad h(v', v) \equiv \pi^i \pmod{\mathfrak{P}^{i+1}} \quad \text{and} \quad v' \equiv u \pmod{\mathfrak{P}}.$$

In particular, $v_{\mathfrak{P}}(h(v', v)) = i$. Set $s = \pi^{i+a}v'$, and let $\omega \in E$ be a skew element such that $v_{\mathfrak{P}}(\omega) = 2i + a$. Since $v_{\mathfrak{P}}(h(s, s)\rho) > 2i + a$, the element $\sigma = \rho h(s, s) + \omega$ satisfies $\text{Tr}(\sigma) = h(s, s)$ and $v_{\mathfrak{P}}(\sigma) = 2i + a$.

We have $S_{s,\sigma} \in S^\sharp(L)$, $S_{s,\sigma}(u) = u$ and $S_{s,\sigma}(\varphi(v)) = \gamma'u + \delta'v + w$ with $\gamma', \delta' \in \mathcal{O}$ and

$$\gamma' = \gamma - h(\gamma u + \delta v, \pi^{i+a}v')\sigma^{-1}\pi^{i+a} = \gamma - \delta h(v, v')p^{i+a}\sigma^{-1}.$$

Since $\delta \in \mathcal{O}^\times$ by eq. (6.4) and $v_{\mathfrak{P}}(\gamma) > i+a$, we have $v_{\mathfrak{P}}(\gamma') = i+a$. We conclude as in the first case. \square

Theorem 6.22. *Let E/K be ramified. Then we have $U^\sharp(L) = S^\sharp(L)$.*

Proof. We proceed by induction on the rank of L . We know that $L = M \perp N$ with M a line or a subnormal plane. By Lemmas 6.20 and 6.21, we have $U^\sharp(L) = S^\sharp(L)U^\sharp(N)$. By induction, $U^\sharp(N) = S^\sharp(N)$. \square

Remark 6.23. For E/K unramified or $E = K \times K$, one can prove that $U^\sharp(L) = S^\sharp(L)$ as well. Since we do not need this result for the computation of $\det(U^\sharp(L))$, the proof is omitted.

6.7. Determinants of the kernel

We use the same assumptions and notation as in Section 6.4. In particular, we are in the local setting. Let $\delta \in E$ be of norm $\delta\bar{\delta} = 1$ and $x \in V$ be anisotropic. A *quasi-reflection* is a map of the form

$$\tau_{x,\delta}: V \rightarrow V, y \mapsto y + (\delta - 1) \frac{h(y,x)}{h(x,x)}x.$$

We have $\tau_{x,\delta} \in U(V)$ and $\det(\tau_{x,\delta}) = \delta$. Let $s = x$ and $\sigma = h(x,x)(1 - \delta)^{-1}$. Then $\tau_{x,\delta} = S_{s,\sigma}$. Conversely, if s is anisotropic and $\sigma \in E$ with $\text{Tr}(\sigma) = h(s,s)$, set $\delta = -\bar{\sigma}/\sigma$, then $S_{s,\sigma} = \tau_{x,\delta}$. Thus, the quasi-reflections are exactly the symmetries at anisotropic vectors. The symmetries at isotropic vectors are called transvections.

Lemma 6.24. *Let $x \in L$ be primitive, anisotropic and $\delta \in E$ of norm $\delta\bar{\delta} = 1$. Then $\tau_{x,\delta} \in U^\sharp(L)$ if and only if $(\delta - 1) \in \mathfrak{P}^a h(x,x)$.*

Proof. We have $\tau_{x,\delta} \in U^\sharp(L)$ if and only if $(\tau_{x,\delta} - \text{id}_V)(L^\vee) \subseteq L$. This amounts to $(\delta - 1)\mathfrak{P}^{-a}h(x,x)^{-1}x \in L$. The lemma follows since x is primitive. \square

For $i \geq 0$, set

$$\begin{aligned} \mathcal{E}_0 &= \{u \in \mathcal{O}^\times \mid u\bar{u} = 1\} \\ \mathcal{E}^i &= \{u \in \mathcal{E}_0 \mid u \equiv 1 \pmod{\mathfrak{P}^i}\}. \end{aligned}$$

Note that $\mathcal{E}_0 = \mathcal{E}^0 = \mathcal{E}^{e-1}$, $\mathcal{E}_1 := \{u\bar{u}^{-1} \mid u \in \mathcal{O}^\times\} = \mathcal{E}^e$ and $[\mathcal{E}_0 : \mathcal{E}_1] = 2$ by [43, 3.4, 3.5].

Theorem 6.25. *Let $F = \mathbb{Q}_p$, K/F a finite field extension, E/K an étale K -algebra of dimension 2 with absolute different $\mathfrak{P}^a := \mathfrak{D}_{E/F}$. Suppose that L is a Hermitian \mathcal{O} -lattice with $\mathfrak{p}^k := \mathfrak{n}(L) \subseteq \mathfrak{D}_{K/F}^{-1}$. Then $\det(U^\sharp(L)) =: \mathcal{F}^\sharp(L) = \mathcal{E}^{2k+a}$.*

Proof. Let $x \in L$ be a norm generator, that is, $h(x,x)\mathcal{O} = \mathfrak{n}(L) = \mathfrak{p}^k$. Let $\delta \in \mathcal{E}^{2k+a}$. Then

$$(\delta - 1) \in \mathfrak{P}^a h(x,x) \subseteq \mathfrak{P}^a \mathfrak{n}(L).$$

By Lemma 6.24, we have $\tau_{x,\delta} \in U^\sharp(L)$ and so $\delta \in \det(U^\sharp(L))$. Hence,

$$\mathcal{E}^{2k+a} \subseteq \det(U^\sharp(L)).$$

Let $\mathfrak{P}^i := \mathfrak{s}(L)$ and $\varphi \in U^\sharp(L)$. By Lemma 6.18, $\varphi \equiv \text{id} \pmod{\mathfrak{P}^{i+a}}$. Thus, $\det(\varphi) \equiv 1 \pmod{\mathfrak{P}^{i+a}}$, that is, $\det(\varphi) \in \mathcal{E}^{i+a}$.

If E/K is unramified, then $\mathfrak{s}(L) = \mathfrak{n}(L)\mathcal{O}$, so that $i = 2k$ and

$$\det(U^\sharp(L)) \subseteq \mathcal{E}^{i+a} = \mathcal{E}^{2k+a}.$$

Now, suppose that E/K is ramified. By Theorem 6.22, the group $U^\sharp(L) = S^\sharp(L)$ is generated by symmetries. Since transvections have determinant one, it is enough to consider the determinants of

the quasi-reflections in $U^\sharp(L)$. Let $\tau_{x,\delta} \in U^\sharp(L)$ be a quasi-reflection. Recall that $\det(\tau_{x,\delta}) = \delta$. By Lemma 6.24, we have $(\delta - 1) \in \mathfrak{P}^a h(x, x) \subseteq \mathfrak{P}^a \mathfrak{n}(L)$. This proves $\det(U^\sharp(L)) \subseteq \mathcal{E}^{2k+a}$. \square

6.8. Computing in $\mathcal{F}(L_{\mathfrak{p}})/\mathcal{F}^\sharp(L_{\mathfrak{p}})$

Let E/K be a quadratic extension number fields with rings of integers \mathcal{O} and \mathfrak{o} , respectively. Let L a Hermitian \mathcal{O} -lattice. Determining the image of δ in Theorem 6.15 requires the computation in the finite quotient $\mathcal{F}(L_{\mathfrak{p}})/\mathcal{F}^\sharp(L_{\mathfrak{p}})$, where \mathfrak{p} is a prime ideal of \mathfrak{o} (see also Remark 6.16). To simplify notation, we now assume that K is a local field of characteristic 0, E/K an étale K -algebra of dimension 2 and the notation as in Section 6.6. Hence, our aim is to be able to do computations in $\mathcal{F}(L)/\mathcal{F}^\sharp(L)$. As we are only interested in computing in the abelian group as opposed to determining it completely, it is sufficient to describe the computation of the supergroup $\mathcal{E}_0/\mathcal{F}^\sharp(L)$. By Theorem 6.25, we know that $\mathcal{F}^\sharp(L) = \mathcal{E}^i$ for some $i \in \mathbb{Z}_{\geq 0}$. It is thus sufficient to describe the computation of $\mathcal{E}_0/\mathcal{E}^i$. By definition, this group is isomorphic to $\ker(\bar{N}_i)$, where

$$\bar{N}_i: \mathcal{O}^\times/(1 + \mathfrak{P}^i) \longrightarrow \mathfrak{o}^\times/\mathfrak{N}(1 + \mathfrak{P}^i), \bar{u} \mapsto \overline{\mathfrak{N}(u)}.$$

Depending on the structure of the extension E/K , this kernel can be described as follows:

- If $E \cong K \times K$, then $\mathcal{E}_0/\mathcal{E}^i$ is isomorphic to $(\mathfrak{o}/\mathfrak{p}^i)^\times$.
- If E/K is an unramified extension of local fields, then $\mathcal{E}_0/\mathcal{E}^i$ is isomorphic to the kernel of the map $(\mathcal{O}/\mathfrak{P}^i)^\times \rightarrow (\mathfrak{o}/\mathfrak{p}^i)^\times, \bar{u} \mapsto \overline{\mathfrak{N}(u)}$.
- If E/K is a ramified extension of local fields, then the situation is more complicated due to the norm not being surjective. Using the fact that by definition we have

$$\mathcal{E}^i/\mathcal{E}^{i+1} \cong \ker\left((1 + \mathfrak{P}^i)/(1 + \mathfrak{P}^{i+1}) \longrightarrow \mathfrak{N}(1 + \mathfrak{P}^i)/\mathfrak{N}(1 + \mathfrak{P}^{i+1}), \bar{u} \mapsto \overline{\mathfrak{N}(u)}\right),$$

this quotient can be determined using explicit results on the image of the multiplicative groups $1 + \mathfrak{P}^i$ under the norm map, found, for example, in [71, Chap. V]. Applying this iteratively, we obtain $\mathcal{E}_0/\mathcal{E}^i$.

In all three cases, the computations of the quotient groups $\mathcal{E}_0/\mathcal{E}^i$ reduce to determining unit groups of residue class rings or kernels of morphisms between such groups. These unit groups are finitely generated abelian groups, whose structure can be determined using classical algorithms from algebraic number theory; see, for example, [25, Sec. 4.2].

7. Fixed points

We classify the fixed point sets of purely nonsymplectic automorphisms of finite order n on complex K3 surfaces. We only use the description of the fixed locus for $n = p$ a prime (see, e.g., [8]); hence providing an independent proof in the known cases and completing the classification in all other cases.

Given the action of σ on some lattice $L \cong H^2(X, \mathbb{Z})$, we want to derive the invariants $((a_1, \dots, a_s), k, l, g)$ of the fixed locus X^σ as defined in the introduction. The topological and holomorphic Lefschetz' fixed point formula [9, Thm. 4.6] yield the following relations

$$\sum_{i=1}^s a_i - 2k + l(2 - 2g) = 2 + \text{Tr } \sigma^* |H^2(X, \mathbb{C})$$

and

$$1 + \zeta_n^{-1} = \sum_{i=1}^s \frac{a_i}{(1 - \zeta_n^{i+1})(1 - \zeta_n^{-i})} + l(1 - g) \frac{1 + \zeta_n}{(1 - \zeta_n)^2}.$$

We adopt the following strategy: By induction, we know the invariants of the fixed loci of σ^p for $p \mid n$. Note that σ acts with order dividing p on X^{σ^p} and $X^\sigma \subseteq X^{\sigma^p}$. From the fixed loci of σ^p , we derive the obvious upper bounds on k, l and g . Then for each possible tuple (k, l, g) , we find all (a_1, \dots, a_s) satisfying the Lefschetz formulas, which amounts to enumerating integer points in a bounded polygon. The result is a finite list of possibilities for the invariants of X^σ .

In what follows, we derive compatibility conditions coming from the description of X^σ as the fixed point set of the action of σ on X^{σ^p} .

Lemma 7.1. *Let P be an isolated fixed point of type i of σ . Let $pm = n$ with p prime. Then P is a fixed point of σ^p of type*

$$t(i) = \min\{i + 1 \bmod m, n - i \bmod m\} - 1.$$

Moreover, we have

$$a_i(\sigma) \leq a_{t(i)}(\sigma^p) \quad \text{and} \quad \sum_{\{i \mid t(i)=j\}} a_i(\sigma) \equiv a_j(\sigma^p) \pmod{p},$$

where $1 \leq j \leq (m - 1)/2$.

Proof. In local coordinates $\sigma^p(x, y) = (\zeta_m^{i+1}x, \zeta_m^{-i}y)$. Note that σ acts on the set of fixed points of type j of σ^p . Hence, the number of fixed points $\sum_{\{i \mid t(i)=j\}} a_i(\sigma)$ of σ is congruent to the order $a_j(\sigma^p)$ of this set modulo p . □

In particular, from the invariants of σ , we can infer how many isolated fixed points of σ lie on a fixed curve of X^{σ^p} . More precisely, by Lemma 7.1 $\sum_{\{i \mid t(i)=0\}} a_i(\sigma)$ is the number of isolated fixed points of σ which lie on a fixed curve of σ^p . The number of such points is bounded above and below by the following lemma.

Lemma 7.2. *Let p be a prime number, C a smooth curve of genus g and $\sigma \in \text{Aut}(C)$ an automorphism of order p with C/σ of genus g' . Then σ fixes*

$$r = \frac{2g - 2 - p(2g' - 2)}{p - 1}$$

points. In particular, given p and g there is a finite number of possibilities for r . Note that for $g = 0$ we have $r = 2$.

Proof. The canonical map $\pi: C \rightarrow C/\sigma$ is ramified precisely in the fixed points and with multiplicity p . By the Hurwitz formula, $2g - 2 = p(2g' - 2) + (p - 1)r$. □

Inductively carrying out this strategy, we obtain a unique possibility in most cases and in the remaining 33 cases two possibilities. We call the corresponding automorphisms ambiguous. In what follows, we disambiguate by using elliptic fibrations. We say that a curve C is fixed by σ if $\sigma|_C = \text{id}_C$. If merely $\sigma(C) = C$, we say that it is invariant.

Lemma 7.3. *Let p be a prime divisor of n . Suppose that $\sigma^{n/p}$ fixes an elliptic curve E . Denote by $\pi: X \rightarrow \mathbb{P}^1$ the elliptic fibration induced by the linear system $|E|$. Suppose σ has no isolated fixed points on E and σ^m leaves invariant a section of π for some $m \mid n, m \neq n$. Then σ fixes E if and only if π admits a σ -invariant section.*

Proof. Since we assume that σ has no isolated fixed points on E , either σ fixes E entirely or no point on E at all.

Suppose that S is a σ -invariant section. Then $E \cap S$ is a fixed point. Therefore, σ fixes E .

Conversely, suppose that there is no σ -invariant section. By assumption, we find a σ^m -invariant section S which must satisfy $\sigma(S) \neq S$. If σ acts trivially on E , then $\{P\} = E \cap S \cap \sigma(S)$ is a fixed point.

The three σ^m -invariant curves E, S and $\sigma(S)$ pass through P . This contradicts the local description of the action around P . Therefore, σ must act as a translation on E . □

Lemma 7.4. *Let τ be an automorphism of prime order p of a K3 surface X acting trivially on $\text{NS}(X)$ and fixing an elliptic curve E . If $f \in \text{NS}(X)$ is isotropic, primitive and nef such that $f^\perp/\mathbb{Z}f$ is not an overlattice of a root lattice, then $f = [E]$.*

Proof. Since τ acts trivially on $\text{NS}(X)$, it lies in the center of the automorphism group $\text{Aut}(X)$ and fixes E . Hence, every automorphism leaves E invariant, that is, E is a curve canonically defined on X . Let $\pi: X \rightarrow \mathbb{P}^1$ be the genus one fibration defined by $[E]$. The fibration π is canonically defined, therefore $\text{Aut}(X)$ is virtually abelian of rank t given by the rank of the Mordell–Weil group of (the Jacobian of) π . Let R be the root sublattice of $[E]^\perp/\mathbb{Z}[E]$. Then by the Shioda–Tate formula [73, 5.2] $t = \text{rk NS}(X) - \text{rk } R - 2$. Since $\text{Aut}(X)$ is virtually abelian, there is at most one elliptic fibration of positive rank. By [69, §3], f is the class of a fiber of an elliptic fibration. Since $f^\perp/\mathbb{Z}f$ is not an overlattice of a root lattice, the Shioda–Tate formula implies that it has positive Mordell–Weil rank. Hence, it must coincide with π and so $[E] = f$. □

Lemma 7.5. *Let p be a prime divisor of n . Suppose that σ^p fixes an elliptic curve E inducing the elliptic fibration π . Set $\tau = \sigma^{n/p}$ and $N = \text{NS}(X)^\tau$. Let $f \in \text{NS}(X)^\sigma$ be isotropic and primitive such that $f^{\perp N}/\mathbb{Z}f$ is not an overlattice of a root lattice. Then π has a τ -invariant section if and only if $\langle f, N \rangle = \mathbb{Z}$. Similarly, π has a σ -invariant section if and only if $\langle f, \text{NS}(X)^\sigma \rangle = \mathbb{Z}$.*

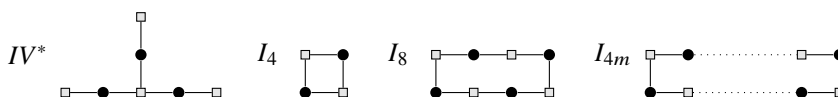
Proof. By [64, Lemma 1.7 and Theorem 1.8], there exists an element δ of the Weyl group $W(\text{NS}(X))$ commuting with σ such that $\delta(f)$ is nef. Hence, we may assume that f is nef. In order to apply Lemma 7.4, set $\tau = \sigma^{n/p}$ and consider $G = \langle \tau \rangle$. Choose a marking $\eta: H^2(X, \mathbb{Z}) \rightarrow L$, and set $H = \eta\rho_X(G)\eta^{-1}$. Then we can deform the H -marked K3 surface (X, G, η) to (X', G', η') such that $\text{NS}(X')^{G'} = \text{NS}(X')$. Let E' be the elliptic curve fixed by G' . It satisfies $\eta'([E']) = \eta([E])$. Note that $f' = \eta'^{-1} \circ \eta(f)$ is still nef since $\eta'(\text{NS}(X')) \subseteq \eta(\text{NS}(X))$. Hence, by Lemma 7.4 we get $f' = [E']$ which gives $f = [E]$.

Finally, if $\langle [E], \text{NS}(X) \rangle = \mathbb{Z}$, we can find $s \in \text{NS}(X)$ with $\langle [E], s \rangle = 1$ and $s^2 = -2$. After possibly replacing s by $-s$, we may assume that s is effective. We can write $s = \delta_1 + \dots + \delta_n$ for (-2) -curves $\delta_i \in \text{NS}(X)$. Now, $1 = \langle f, s \rangle$ and $\langle f, \delta_i \rangle \geq 0$ (f is nef) imply that $\langle f, \delta_i \rangle = 1$ for a single $1 \leq i \leq n$; δ_i is the desired section. Note that τ (respectively σ) preserves δ_i if and only if it preserves s . □

Among the 22 ambiguous automorphisms σ of order $n = 4$, there are 16 cases where σ^2 fixes a single elliptic curve, 4 cases where σ^2 fixes a curve of genus 2 and 3, 5, 7 or 9 rational curves, and in the remaining 2 cases σ^2 fixes two curves of genus one. The ambiguity is whether σ fixes some curve or not.

First, let σ^2 fix a unique elliptic curve E of genus 1. This means that σ is compatible with an elliptic fibration $\pi: X \rightarrow \mathbb{P}^1$. Moreover, σ^2 leaves invariant a section S of π because L^{σ^2} contains a copy of the hyperbolic plane U . Note that σ^2 must act nontrivially on the base \mathbb{P}^1 of the fibration, as otherwise its action at the tangent space to the point in $E \cap S$ would be trivial. Hence, σ^2 has exactly two fixed points in \mathbb{P}^1 giving two invariant fibers; one is E and the other one we call C . The rational curves fixed by σ^2 must be components of the fiber C because all fixed points lie in $E \cup C$. We fix the fiber type of C and consider each fiber type separately. We know that C is a singular fiber of Kodaira type I_{4m} , $m = 1, 2, 3, 4$ or IV^* . They correspond to 3, 3, 3, 4 and 3 ambiguous cases.

The following figure shows the dual graph of the irreducible components of C . Each node corresponds to a smooth rational curve and two nodes are joined by an edge if and only if the corresponding curves intersect. The square nodes are curves fixed pointwise by σ^2 and the round nodes are curves which are invariant but not fixed by σ^2 . The automorphism σ acts on the graph with order dividing 2 and maps squares to squares.



We start by determining the fixed locus in the case that C is of type I_{4m} . Lefschetz calculations show that for the given σ^2 we have the following possibilities for the fixed locus X^σ of σ : four isolated points $((4), 0, 0, 1)$ or four isolated points and a curve of genus one $((4), 0, 1, 1)$.

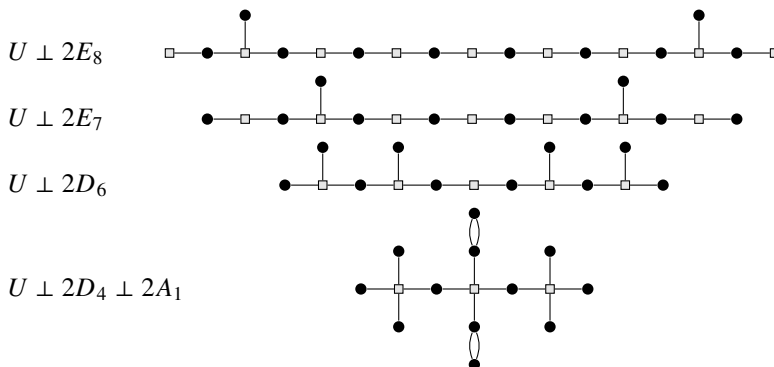
The automorphism σ^2 fixes every second node of the circular I_{4m} configuration and the zero section intersects a σ^2 -fixed curve in I_{4m} .

The action of σ on the intersection graph is visible on the lattice side since we can identify the class F of E by Lemma 7.4 and choose simple roots in F^\perp giving its components. Carrying out this computation gives the following three cases.

1. The curves in I_{4m} are rotated by σ . Then C does not have any σ fixed points and E contains four isolated fixed points for 0.4.4.9, 0.4.3.6, 0.4.2.5, 0.4.1.3.
2. The automorphism σ acts as a reflection on the graph I_{4m} leaving invariant two of the σ^2 -fixed curves and the section S passes through one of them. Then I_{4m} contains 4 isolated fixed points and E does not contain any isolated fixed points. However, the intersection $S \cap E$ is fixed but cannot be isolated. Hence, σ fixes E for 0.4.4.7, 0.4.3.8, 0.4.2.3, 0.4.1.6.
3. The automorphism σ acts as a reflection leaving invariant two of the σ^2 -fixed curves, and the section S does not pass through them. Then σ fixes four isolated points on I_{4m} and no isolated points on E . By Lemma 7.3 σ does not fix E for 0.4.4.8, 0.4.3.7, 0.4.2.4, 0.4.1.4, 0.4.1.5.

Let $C = IV^*$ and σ ambiguous. We know that σ fixes 6 isolated points, one rational curve and possibly E of genus 1. The central curve as well as the three leaves must be fixed by σ^2 . There are 3 possible actions. It can leave invariant each component of the IV^* fiber. Then the central component is fixed and the leaves carry two fixed points each. Hence, the action on E does not have an isolated fixed point. Therefore, σ fixes E if and only if some section is preserved. This is the case for 0.4.3.11 but not for 0.4.3.10. In the third case, σ swaps two of the branches. Therefore, the central component cannot be fixed by σ , so it contains 2 isolated fixed points. The invariant leaf must be the fixed rational curve. Then there are four fixed points left, they must lie on E giving 0.4.3.9. This settles the first 16 ambiguous cases.

In the next four ambiguous cases, σ^2 fixes a curve of genus 2 and 3, 5, 7 or 9 rational curves. In each case, we know that σ fixes exactly four isolated fixed points. The ambiguity is whether σ fixes 1 rational curve and the genus 2 curve, or no curve at all. For each case, we exhibit a σ -invariant hyperbolic plane U . Since σ fixes a curve of genus 2, $\text{Aut}(X)$ is finite, and so $K = U^\perp$ is a root lattice. Then $\text{NS}(X) = U \perp K$ and K determines the ADE-types of the singular fibers of the σ -equivariant fibration induced by U . The square nodes are fixed by σ^2 while the round nodes are not.



Note that σ must act nontrivially on the graph because otherwise it has too many fixed points or fixed curves. Since σ maps squares to squares, we see that σ must act as a reflection preserving the central square. However, the corresponding curve cannot be fixed, because the two adjacent ones and hence the corresponding intersection points with the central node are swapped. Thus, σ cannot fix a curve and the four ambiguous cases are settled.

Consider the ambiguous cases 0.4.5.12, 0.4.5.14, 0.6.2.29, 0.6.3.36, 0.8.1.7, 0.8.1.8, 0.8.2.8, 0.8.2.10, 0.9.1.3, 0.9.1.4 and 0.10.1.11. The question is whether or not σ fixes an elliptic curve. In each case, Lefschetz calculations show that there are no isolated fixed points on the elliptic curve in question. Hence, in view of Lemma 7.3 this can be decided by whether or not there exists a section of the corresponding fibration. This is settled by Lemma 7.5. Indeed, we can randomly search until we find $f \in L^\sigma$ corresponding to $[E]$.

For 0.10.2.1, we have the two possibilities $((0, 0, 1, 6), 0, 0, 1)$ and $((5, 0, 0, 0), 1, 0, 1)$ for the fixed locus of σ . We know that σ^2 fixes an elliptic curve E and one rational curve. It is the central component of the invariant fiber C of type \tilde{E}_7 . Since $\text{NS}(X)^\sigma$ has rank 6, σ must act nontrivially on $\tilde{E}_7 \subseteq \text{NS}(X)$. This means that it swaps two of the three arms of the configuration. Hence, it cannot act trivially on the central component and so σ cannot fix a rational curve. The fixed locus is $((0, 0, 1, 6), 0, 0, 1)$.

For the last ambiguous case 0.12.1.12, the automorphism σ has a unique fixed point and the ambiguity is whether or not it fixes an elliptic curve. We know that σ^i for $i = 2, 3, 4, 6$ fixes a unique elliptic curve E_i . Since $E_2 \subseteq E_4, E_6$, we have $E_2 = E_4 = E_6$ and similarly $E_6 \subseteq E_3$ implies $E_6 = E_3$. Therefore, $E = E_i$ is independent of i . Now, σ^3 and σ^4 fix E , hence their product σ^7 fixes E as well. But $\sigma \in \langle \sigma^7 \rangle$ and so σ fixes the elliptic curve E . The fixed locus of 0.12.1.12 is therefore $((1, 0, 0, 0, 0), 0, 1, 1)$.

A. Finite groups with mixed action on a K3 surface

The following table lists all finite groups G admitting a faithful, saturated, mixed action on some K3 surface, their symplectic subgroups $1 \neq G_s < G$ as well as the number $k(G)$ of deformation types. Note that in three cases an entry appears twice because the normal subgroup $G_s < G$ does not lie in the same $\text{Aut}(G)$ -orbit. The notation $G = G_s \cdot \mu_n$ means that G is an extension of G_s by μ_n . It may or may not split. Our notation for the groups G_s follows Hashimoto [34]. Isomorphism classes of groups will be referred to either using standard notation for classical families or using the id as provided by the library of small groups [15].

Table 3. Finite groups with faithful, saturated nonsymplectic action on some K3 surface.

G	id	$k(G)$	G	id	$k(G)$	G	id	$k(G)$
$C_2 \cdot \mu_2$	(4, 1)	5	$C_3 \cdot \mu_{15}$	(45, 2)	1	$C_4 \cdot \mu_{12}$	(48, 22)	1
$C_2 \cdot \mu_2$	(4, 2)	354	$C_3 \cdot \mu_{18}$	(54, 4)	1	$C_4 \cdot \mu_{12}$	(48, 23)	1
$C_2 \cdot \mu_3$	(6, 2)	26	$C_3 \cdot \mu_{18}$	(54, 9)	1	$C_4 \cdot \mu_{12}$	(48, 24)	1
$C_2 \cdot \mu_4$	(8, 1)	3	$C_2^2 \cdot \mu_2$	(8, 2)	4	$D_6 \cdot \mu_2$	(12, 4)	140
$C_2 \cdot \mu_4$	(8, 2)	200	$C_2^2 \cdot \mu_2$	(8, 3)	40	$D_6 \cdot \mu_3$	(18, 3)	21
$C_2 \cdot \mu_5$	(10, 2)	6	$C_2^2 \cdot \mu_2$	(8, 5)	330	$D_6 \cdot \mu_4$	(24, 5)	18
$C_2 \cdot \mu_6$	(12, 2)	11	$C_2^2 \cdot \mu_3$	(12, 3)	4	$D_6 \cdot \mu_5$	(30, 1)	1
$C_2 \cdot \mu_6$	(12, 5)	99	$C_2^2 \cdot \mu_3$	(12, 5)	11	$D_6 \cdot \mu_6$	(36, 12)	33
$C_2 \cdot \mu_7$	(14, 2)	4	$C_2^2 \cdot \mu_4$	(16, 3)	73	$D_6 \cdot \mu_8$	(48, 4)	4
$C_2 \cdot \mu_8$	(16, 5)	50	$C_2^2 \cdot \mu_4$	(16, 5)	1	$D_6 \cdot \mu_{10}$	(60, 11)	1
$C_2 \cdot \mu_9$	(18, 2)	2	$C_2^2 \cdot \mu_4$	(16, 6)	2	$D_6 \cdot \mu_{12}$	(72, 27)	2
$C_2 \cdot \mu_{10}$	(20, 2)	1	$C_2^2 \cdot \mu_4$	(16, 10)	77	$C_3^3 \cdot \mu_2$	(16, 3)	3
$C_2 \cdot \mu_{10}$	(20, 5)	12	$C_2^2 \cdot \mu_6$	(24, 10)	13	$C_3^3 \cdot \mu_2$	(16, 10)	2
$C_2 \cdot \mu_{12}$	(24, 2)	2	$C_2^2 \cdot \mu_6$	(24, 13)	19	$C_3^3 \cdot \mu_2$	(16, 11)	34
$C_2 \cdot \mu_{12}$	(24, 9)	24	$C_2^2 \cdot \mu_6$	(24, 15)	14	$C_3^3 \cdot \mu_2$	(16, 14)	72
$C_2 \cdot \mu_{14}$	(28, 4)	4	$C_2^2 \cdot \mu_8$	(32, 5)	14	$C_3^3 \cdot \mu_3$	(24, 13)	2
$C_2 \cdot \mu_{15}$	(30, 4)	3	$C_2^2 \cdot \mu_8$	(32, 36)	6	$C_3^3 \cdot \mu_4$	(32, 6)	6
$C_2 \cdot \mu_{16}$	(32, 16)	3	$C_2^2 \cdot \mu_9$	(36, 3)	2	$C_3^3 \cdot \mu_4$	(32, 7)	3
$C_2 \cdot \mu_{18}$	(36, 5)	3	$C_2^2 \cdot \mu_{12}$	(48, 21)	4	$C_2^2 \cdot \mu_4$	(32, 22)	31

(Continues)

Table 3. (Continued)

G	id	$k(G)$	G	id	$k(G)$	G	id	$k(G)$
$C_2 \cdot \mu_{20}$	(40, 9)	3	$C_2^2 \cdot \mu_{12}$	(48, 31)	3	$C_3^2 \cdot \mu_4$	(32, 45)	7
$C_2 \cdot \mu_{24}$	(48, 23)	3	$C_2^2 \cdot \mu_{12}$	(48, 44)	1	$C_3^2 \cdot \mu_6$	(48, 31)	1
$C_2 \cdot \mu_{28}$	(56, 8)	1	$C_2^2 \cdot \mu_{18}$	(72, 16)	2	$C_3^2 \cdot \mu_6$	(48, 49)	5
$C_2 \cdot \mu_{30}$	(60, 13)	2	$C_4 \cdot \mu_2$	(8, 1)	3	$C_3^2 \cdot \mu_7$	(56, 11)	1
$C_3 \cdot \mu_2$	(6, 1)	46	$C_4 \cdot \mu_2$	(8, 2)	98	$C_3^2 \cdot \mu_7$	(56, 11)	1
$C_3 \cdot \mu_2$	(6, 2)	51	$C_4 \cdot \mu_2$	(8, 3)	102	$C_3^2 \cdot \mu_8$	(64, 4)	3
$C_3 \cdot \mu_3$	(9, 2)	26	$C_4 \cdot \mu_2$	(8, 4)	3	$C_3^2 \cdot \mu_8$	(64, 87)	1
$C_3 \cdot \mu_4$	(12, 1)	15	$C_4 \cdot \mu_3$	(12, 2)	7	$C_2^3 \cdot \mu_{12}$	(96, 196)	2
$C_3 \cdot \mu_4$	(12, 2)	15	$C_4 \cdot \mu_4$	(16, 2)	21	$D_8 \cdot \mu_2$	(16, 7)	6
$C_3 \cdot \mu_5$	(15, 1)	4	$C_4 \cdot \mu_4$	(16, 4)	23	$D_8 \cdot \mu_2$	(16, 8)	2
$C_3 \cdot \mu_6$	(18, 3)	38	$C_4 \cdot \mu_4$	(16, 5)	10	$D_8 \cdot \mu_2$	(16, 11)	202
$C_3 \cdot \mu_6$	(18, 5)	36	$C_4 \cdot \mu_4$	(16, 6)	9	$D_8 \cdot \mu_2$	(16, 13)	11
$C_3 \cdot \mu_7$	(21, 2)	1	$C_4 \cdot \mu_5$	(20, 2)	1	$D_8 \cdot \mu_3$	(24, 10)	3
$C_3 \cdot \mu_8$	(24, 1)	3	$C_4 \cdot \mu_6$	(24, 2)	1	$D_8 \cdot \mu_4$	(32, 9)	10
$C_3 \cdot \mu_8$	(24, 2)	3	$C_4 \cdot \mu_6$	(24, 9)	5	$D_8 \cdot \mu_4$	(32, 11)	4
$C_3 \cdot \mu_9$	(27, 2)	3	$C_4 \cdot \mu_6$	(24, 10)	8	$D_8 \cdot \mu_4$	(32, 25)	19
$C_3 \cdot \mu_{10}$	(30, 1)	4	$C_4 \cdot \mu_6$	(24, 11)	1	$D_8 \cdot \mu_4$	(32, 38)	1
$C_3 \cdot \mu_{10}$	(30, 4)	2	$C_4 \cdot \mu_8$	(32, 3)	4	$D_8 \cdot \mu_6$	(48, 26)	1
$C_3 \cdot \mu_{12}$	(36, 6)	5	$C_4 \cdot \mu_8$	(32, 12)	4	$D_8 \cdot \mu_6$	(48, 45)	4
$C_3 \cdot \mu_{12}$	(36, 8)	4	$C_4 \cdot \mu_{10}$	(40, 9)	2	$D_8 \cdot \mu_8$	(64, 6)	5
$C_3 \cdot \mu_{14}$	(42, 6)	2	$C_4 \cdot \mu_{12}$	(48, 20)	1	$D_8 \cdot \mu_{12}$	(96, 52)	1
$Q_8 \cdot \mu_2$	(16, 8)	7	$C_2^4 \cdot \mu_4$	(64, 32)	2	$A_{3,3} \cdot \mu_2$	(36, 10)	11
$Q_8 \cdot \mu_2$	(16, 9)	1	$C_2^4 \cdot \mu_4$	(64, 60)	2	$A_{3,3} \cdot \mu_2$	(36, 13)	10
$Q_8 \cdot \mu_2$	(16, 12)	2	$C_2^4 \cdot \mu_4$	(64, 90)	1	$A_{3,3} \cdot \mu_3$	(54, 5)	2
$Q_8 \cdot \mu_2$	(16, 13)	11	$C_2^4 \cdot \mu_4$	(64, 193)	1	$A_{3,3} \cdot \mu_3$	(54, 13)	3
$Q_8 \cdot \mu_3$	(24, 3)	2	$C_2^4 \cdot \mu_5$	(80, 49)	1	$A_{3,3} \cdot \mu_4$	(72, 21)	1
$Q_8 \cdot \mu_3$	(24, 11)	1	$C_2^4 \cdot \mu_6$	(96, 70)	1	$A_{3,3} \cdot \mu_4$	(72, 39)	1
$Q_8 \cdot \mu_4$	(32, 11)	5	$C_2^4 \cdot \mu_6$	(96, 197)	1	$A_{3,3} \cdot \mu_4$	(72, 45)	3
$Q_8 \cdot \mu_4$	(32, 38)	2	$C_2^4 \cdot \mu_6$	(96, 228)	1	$A_{3,3} \cdot \mu_6$	(108, 25)	4
$Q_8 \cdot \mu_6$	(48, 26)	1	$C_2^4 \cdot \mu_6$	(96, 229)	1	$A_{3,3} \cdot \mu_6$	(108, 36)	1
$Q_8 \cdot \mu_6$	(48, 32)	2	$C_2^4 \cdot \mu_7$	(112, 41)	1	$A_{3,3} \cdot \mu_6$	(108, 38)	4
$Q_8 \cdot \mu_6$	(48, 33)	1	$C_2^4 \cdot \mu_7$	(112, 41)	1	$A_{3,3} \cdot \mu_6$	(108, 43)	1
$Q_8 \cdot \mu_6$	(48, 46)	1	$C_2^4 \cdot \mu_8$	(128, 48)	1	$A_{3,3} \cdot \mu_8$	(144, 185)	1
$D_{10} \cdot \mu_2$	(20, 3)	3	$C_2^4 \cdot \mu_{10}$	(160, 235)	1	$Hol(C_5) \cdot \mu_2$	(40, 12)	9
$D_{10} \cdot \mu_2$	(20, 4)	19	$C_2^4 \cdot \mu_{12}$	(192, 994)	1	$Hol(C_5) \cdot \mu_3$	(60, 6)	1
$D_{10} \cdot \mu_3$	(30, 2)	3	$C_2 \times D_8 \cdot \mu_2$	(32, 6)	3	$Hol(C_5) \cdot \mu_4$	(80, 30)	1
$D_{10} \cdot \mu_4$	(40, 5)	3	$C_2 \times D_8 \cdot \mu_2$	(32, 7)	1	$C_7 : C_3 \cdot \mu_2$	(42, 1)	4
$D_{10} \cdot \mu_4$	(40, 12)	4	$C_2 \times D_8 \cdot \mu_2$	(32, 27)	14	$C_7 : C_3 \cdot \mu_2$	(42, 2)	2
$D_{10} \cdot \mu_5$	(50, 3)	1	$C_2 \times D_8 \cdot \mu_2$	(32, 28)	2	$C_7 : C_3 \cdot \mu_3$	(63, 3)	1
$D_{10} \cdot \mu_6$	(60, 6)	1	$C_2 \times D_8 \cdot \mu_2$	(32, 28)	2	$C_7 : C_3 \cdot \mu_4$	(84, 2)	1
$D_{10} \cdot \mu_6$	(60, 10)	2	$C_2 \times D_8 \cdot \mu_2$	(32, 30)	1	$C_7 : C_3 \cdot \mu_6$	(126, 7)	1
$D_{10} \cdot \mu_8$	(80, 28)	1	$C_2 \times D_8 \cdot \mu_2$	(32, 34)	4	$C_7 : C_3 \cdot \mu_6$	(126, 10)	1
$D_{10} \cdot \mu_{10}$	(100, 14)	1	$C_2 \times D_8 \cdot \mu_2$	(32, 39)	2	$S_4 \cdot \mu_2$	(48, 48)	74
$D_{10} \cdot \mu_{12}$	(120, 17)	1	$C_2 \times D_8 \cdot \mu_2$	(32, 43)	2	$S_4 \cdot \mu_3$	(72, 42)	2
$A_4 \cdot \mu_2$	(24, 12)	40	$C_2 \times D_8 \cdot \mu_2$	(32, 46)	46	$S_4 \cdot \mu_4$	(96, 186)	8
$A_4 \cdot \mu_2$	(24, 13)	47	$C_2 \times D_8 \cdot \mu_2$	(32, 48)	1	$S_4 \cdot \mu_6$	(144, 188)	2
$A_4 \cdot \mu_3$	(36, 11)	7	$C_2 \times D_8 \cdot \mu_2$	(32, 49)	13	$2^4 C_2 \cdot \mu_2$	(64, 32)	2
$A_4 \cdot \mu_4$	(48, 30)	10	$C_2 \times D_8 \cdot \mu_4$	(64, 12)	1	$2^4 C_2 \cdot \mu_2$	(64, 138)	8
$A_4 \cdot \mu_4$	(48, 31)	7	$C_2 \times D_8 \cdot \mu_4$	(64, 34)	2	$2^4 C_2 \cdot \mu_2$	(64, 202)	12
$A_4 \cdot \mu_6$	(72, 42)	5	$C_2 \times D_8 \cdot \mu_4$	(64, 67)	5	$2^4 C_2 \cdot \mu_2$	(64, 215)	3
$A_4 \cdot \mu_6$	(72, 47)	4	$C_2 \times D_8 \cdot \mu_4$	(64, 71)	2	$2^4 C_2 \cdot \mu_2$	(64, 216)	1
$A_4 \cdot \mu_{12}$	(144, 123)	1	$C_2 \times D_8 \cdot \mu_4$	(64, 90)	7	$2^4 C_2 \cdot \mu_2$	(64, 226)	4
$D_{12} \cdot \mu_2$	(24, 6)	4	$C_2 \times D_8 \cdot \mu_4$	(64, 92)	4	$2^4 C_2 \cdot \mu_2$	(64, 241)	1
$D_{12} \cdot \mu_2$	(24, 8)	2	$C_2 \times D_8 \cdot \mu_4$	(64, 99)	2	$2^4 C_2 \cdot \mu_3$	(96, 70)	1
$D_{12} \cdot \mu_2$	(24, 14)	54	$C_2 \times D_8 \cdot \mu_4$	(64, 101)	1	$2^4 C_2 \cdot \mu_4$	(128, 621)	1
$D_{12} \cdot \mu_3$	(36, 12)	7	$C_2 \times D_8 \cdot \mu_4$	(64, 102)	1	$2^4 C_2 \cdot \mu_4$	(128, 645)	1
$D_{12} \cdot \mu_4$	(48, 14)	3	$C_2 \times D_8 \cdot \mu_4$	(64, 196)	2	$2^4 C_2 \cdot \mu_4$	(128, 850)	3
$D_{12} \cdot \mu_4$	(48, 35)	3	$C_2 \times D_8 \cdot \mu_4$	(64, 199)	1	$2^4 C_2 \cdot \mu_4$	(128, 853)	1
$D_{12} \cdot \mu_6$	(72, 28)	2	$C_2 \times D_8 \cdot \mu_8$	(128, 2)	1	$2^4 C_2 \cdot \mu_4$	(128, 1090)	1

(Continues)

Table 3. (Continued)

G	id	$k(G)$	G	id	$k(G)$	G	id	$k(G)$
$D_{12} \cdot \mu_6$	(72, 30)	1	$C_2 \times D_8 \cdot \mu_8$	(128, 50)	1	$2^4 C_2 \cdot \mu_6$	(192, 1000)	1
$D_{12} \cdot \mu_6$	(72, 48)	5	$C_2 \times D_8 \cdot \mu_8$	(128, 206)	1	$Q_8 * Q_8 \cdot \mu_2$	(64, 134)	2
$D_{12} \cdot \mu_8$	(96, 27)	1	$SD_{16} \cdot \mu_2$	(32, 40)	2	$Q_8 * Q_8 \cdot \mu_2$	(64, 138)	4
$D_{12} \cdot \mu_8$	(96, 106)	1	$SD_{16} \cdot \mu_2$	(32, 42)	2	$Q_8 * Q_8 \cdot \mu_2$	(64, 139)	1
$D_{12} \cdot \mu_{12}$	(144, 79)	1	$SD_{16} \cdot \mu_2$	(32, 43)	6	$Q_8 * Q_8 \cdot \mu_2$	(64, 257)	1
$C_2^4 \cdot \mu_2$	(32, 27)	6	$SD_{16} \cdot \mu_3$	(48, 26)	1	$Q_8 * Q_8 \cdot \mu_2$	(64, 264)	4
$C_2^4 \cdot \mu_2$	(32, 46)	5	$SD_{16} \cdot \mu_4$	(64, 124)	1	$Q_8 * Q_8 \cdot \mu_2$	(64, 266)	1
$C_2^4 \cdot \mu_2$	(32, 51)	7	$SD_{16} \cdot \mu_4$	(64, 125)	1	$Q_8 * Q_8 \cdot \mu_3$	(96, 201)	1
$C_2^4 \cdot \mu_3$	(48, 49)	2	$SD_{16} \cdot \mu_6$	(96, 180)	1	$Q_8 * Q_8 \cdot \mu_3$	(96, 204)	1
$C_2^4 \cdot \mu_3$	(48, 50)	1	$A_{3,3} \cdot \mu_2$	(36, 9)	2	$Q_8 * Q_8 \cdot \mu_4$	(128, 134)	1
$Q_8 * Q_8 \cdot \mu_4$	(128, 522)	1	$C_2 \times S_4 \cdot \mu_4$	(192, 1469)	1	$2^4 D_6 \cdot \mu_6$	(576, 8656)	1
$Q_8 * Q_8 \cdot \mu_4$	(128, 524)	2	$T_{48} \cdot \mu_2$	(96, 189)	1	$S_5 \cdot \mu_2$	(240, 189)	12
$Q_8 * Q_8 \cdot \mu_4$	(128, 1633)	1	$T_{48} \cdot \mu_2$	(96, 193)	2	$L_2(7) \cdot \mu_2$	(336, 208)	8
$Q_8 * Q_8 \cdot \mu_6$	(192, 201)	1	$T_{48} \cdot \mu_3$	(144, 122)	1	$L_2(7) \cdot \mu_2$	(336, 209)	4
$Q_8 * Q_8 \cdot \mu_6$	(192, 1504)	1	$T_{48} \cdot \mu_6$	(288, 900)	1	$L_2(7) \cdot \mu_4$	(672, 1046)	1
$Q_8 * Q_8 \cdot \mu_6$	(192, 1509)	1	$A_5 \cdot \mu_2$	(120, 34)	7	$4^2 A_4 \cdot \mu_2$	(384, 591)	1
$Q_8 * Q_8 \cdot \mu_8$	(256, 332)	1	$A_5 \cdot \mu_2$	(120, 35)	12	$4^2 A_4 \cdot \mu_2$	(384, 18134)	1
$Q_8 * Q_8 \cdot \mu_{12}$	(384, 5816)	1	$A_5 \cdot \mu_3$	(180, 19)	2	$4^2 A_4 \cdot \mu_2$	(384, 18135)	2
$3^2 C_4 \cdot \mu_2$	(72, 39)	2	$A_5 \cdot \mu_4$	(240, 91)	1	$4^2 A_4 \cdot \mu_2$	(384, 18235)	2
$3^2 C_4 \cdot \mu_2$	(72, 40)	6	$A_5 \cdot \mu_6$	(360, 119)	1	$4^2 A_4 \cdot \mu_2$	(384, 18236)	1
$3^2 C_4 \cdot \mu_2$	(72, 41)	1	$A_5 \cdot \mu_6$	(360, 122)	1	$4^2 A_4 \cdot \mu_3$	(576, 5129)	1
$3^2 C_4 \cdot \mu_2$	(72, 45)	5	$\Gamma_{25a1} \cdot \mu_2$	(128, 928)	3	$4^2 A_4 \cdot \mu_4$	(768, 1083510)	1
$3^2 C_4 \cdot \mu_3$	(108, 36)	1	$\Gamma_{25a1} \cdot \mu_2$	(128, 932)	1	$4^2 A_4 \cdot \mu_4$	(768, 1088651)	1
$3^2 C_4 \cdot \mu_4$	(144, 120)	2	$\Gamma_{25a1} \cdot \mu_2$	(128, 1755)	4	$4^2 A_4 \cdot \mu_4$	(768, 1088659)	1
$3^2 C_4 \cdot \mu_4$	(144, 185)	1	$\Gamma_{25a1} \cdot \mu_2$	(128, 1758)	1	$4^2 A_4 \cdot \mu_6$	(1152, 155469)	1
$3^2 C_4 \cdot \mu_6$	(216, 157)	1	$\Gamma_{25a1} \cdot \mu_2$	(128, 1759)	1	$H_{195} \cdot \mu_2$	(384, 17948)	3
$S_{3,3} \cdot \mu_2$	(72, 40)	3	$\Gamma_{25a1} \cdot \mu_3$	(192, 201)	1	$T_{192} \cdot \mu_2$	(384, 5602)	1
$S_{3,3} \cdot \mu_2$	(72, 46)	8	$\Gamma_{25a1} \cdot \mu_4$	(256, 6029)	1	$T_{192} \cdot \mu_2$	(384, 5608)	1
$S_{3,3} \cdot \mu_3$	(108, 38)	2	$\Gamma_{25a1} \cdot \mu_6$	(384, 5837)	1	$T_{192} \cdot \mu_2$	(384, 20097)	1
$S_{3,3} \cdot \mu_4$	(144, 115)	1	$A_{4,3} \cdot \mu_2$	(144, 183)	7	$T_{192} \cdot \mu_3$	(576, 8277)	1
$S_{3,3} \cdot \mu_6$	(216, 157)	1	$A_{4,3} \cdot \mu_2$	(144, 189)	3	$T_{192} \cdot \mu_6$	(1152, 157515)	1
$S_{3,3} \cdot \mu_6$	(216, 170)	1	$A_{4,3} \cdot \mu_3$	(216, 92)	1	$A_{4,4} \cdot \mu_2$	(576, 8653)	1
$2^4 C_3 \cdot \mu_2$	(96, 70)	4	$A_{4,3} \cdot \mu_3$	(216, 164)	1	$A_{4,4} \cdot \mu_2$	(576, 8654)	1
$2^4 C_3 \cdot \mu_2$	(96, 227)	9	$A_{4,3} \cdot \mu_6$	(432, 535)	1	$A_{4,4} \cdot \mu_2$	(576, 8657)	1
$2^4 C_3 \cdot \mu_2$	(96, 229)	6	$A_{4,3} \cdot \mu_6$	(432, 745)	1	$A_{4,4} \cdot \mu_4$	(1152, 157850)	1
$2^4 C_3 \cdot \mu_3$	(144, 184)	2	$N_{72} \cdot \mu_2$	(144, 182)	1	$A_6 \cdot \mu_2$	(720, 763)	2
$2^4 C_3 \cdot \mu_4$	(192, 184)	1	$N_{72} \cdot \mu_2$	(144, 186)	2	$A_6 \cdot \mu_2$	(720, 764)	6
$2^4 C_3 \cdot \mu_4$	(192, 191)	1	$N_{72} \cdot \mu_4$	(288, 841)	1	$A_6 \cdot \mu_2$	(720, 766)	4
$2^4 C_3 \cdot \mu_4$	(192, 1495)	1	$M_9 \cdot \mu_2$	(144, 182)	2	$A_6 \cdot \mu_4$	(1440, 4595)	1
$2^4 C_3 \cdot \mu_5$	(240, 191)	1	$M_9 \cdot \mu_2$	(144, 187)	1	$F_{384} \cdot \mu_2$	(768, 1086051)	1
$2^4 C_3 \cdot \mu_6$	(288, 1025)	2	$M_9 \cdot \mu_3$	(216, 153)	1	$F_{384} \cdot \mu_2$	(768, 1090134)	1
$2^4 C_3 \cdot \mu_6$	(288, 1029)	1	$M_9 \cdot \mu_6$	(432, 735)	1	$F_{384} \cdot \mu_2$	(768, 1090135)	1
$C_2 \times S_4 \cdot \mu_2$	(96, 187)	2	$2^4 D_6 \cdot \mu_2$	(192, 955)	8	$F_{384} \cdot \mu_4$	(1536, 'no id')	1
$C_2 \times S_4 \cdot \mu_2$	(96, 195)	1	$2^4 D_6 \cdot \mu_2$	(192, 1538)	9	$M_{20} \cdot \mu_2$	(1920, 240993)	1
$C_2 \times S_4 \cdot \mu_2$	(96, 226)	15	$2^4 D_6 \cdot \mu_3$	(288, 1025)	1	$M_{20} \cdot \mu_2$	(1920, 240995)	2
$C_2 \times S_4 \cdot \mu_4$	(192, 972)	2	$2^4 D_6 \cdot \mu_4$	(384, 5566)	1	$M_{20} \cdot \mu_4$	(3840, 'no id')	1

B. Fixed loci of purely nonsymplectic automorphisms

Let X be a K3 surface and $\sigma \in \text{Aut}(X)$ a purely nonsymplectic automorphism of order n acting by ζ_n on $H^0(X, \Omega_X^2)$. Recall that the fixed locus X^σ is the disjoint union of $N = \sum_{i=1}^s a_i$ isolated fixed points, k smooth rational curves and either a curve of genus > 1 or 0, 1, 2 curves of genus 1. Denote by $l \geq 0$ the number of genus $g \geq 1$ curves fixed by σ . If no such curve is fixed, set $g = 1$. Thus

$$X^\sigma = \{p_1, \dots, p_N\} \sqcup R_1 \cdots \sqcup R_k \sqcup C_1 \cdots \sqcup C_l$$

where the R_i 's are smooth rational curves and the C_j 's smooth curves of genus g . Let $P \in X^\sigma$ be an isolated fixed point. Recall that there are local coordinates (x, y) in a small neighborhood centered at P such that

$$\sigma(x, y) = (\zeta_n^{i+1}x, \zeta_n^{-i}y) \quad \text{with} \quad 1 \leq i \leq s = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We call P a fixed point of type i and denote the number of fixed points of type i by a_i .

In the following we list for each deformation class of (X, σ) the invariants $((a_1, \dots, a_s), k, l, g)$ of the fixed locus of σ and its powers. The column labeled 'K3 id' contains the label of the K3 surface in the database [21]. The following columns contain the invariants of the fixed locus of $\sigma^{n/j}$ where n is the order of σ and j the label of the column.

Table 4. Fixed loci of purely nonsymplectic automorphisms of order 2.

K3 id	$j = 2$				K3 id	$j = 2$				K3 id	$j = 2$				K3 id	$j = 2$			
	(a_i)	k	l	g		(a_i)	k	l	g		(a_i)	k	l	g		(a_i)	k	l	g
0.2.1.1	(0)	10	0	1	0.2.7.3	(0)	5	1	2	0.2.11.1	(0)	5	1	6	0.2.13.4	(0)	0	1	3
0.2.2.1	(0)	9	0	1	0.2.7.4	(0)	4	1	1	0.2.11.2	(0)	4	1	5	0.2.14.1	(0)	0	1	4
0.2.2.2	(0)	9	1	1	0.2.7.5	(0)	4	1	1	0.2.11.3	(0)	4	1	5	0.2.14.2	(0)	1	1	5
0.2.3.1	(0)	9	1	2	0.2.7.6	(0)	4	0	1	0.2.11.4	(0)	3	1	4	0.2.14.3	(0)	2	1	6
0.2.3.2	(0)	8	1	1	0.2.8.1	(0)	3	0	1	0.2.11.5	(0)	3	1	4	0.2.15.1	(0)	2	1	7
0.2.3.3	(0)	8	1	1	0.2.8.2	(0)	3	1	1	0.2.11.6	(0)	2	1	3	0.2.15.2	(0)	1	1	6
0.2.3.4	(0)	8	0	1	0.2.8.3	(0)	4	1	2	0.2.11.7	(0)	2	1	3	0.2.15.3	(0)	1	1	6
0.2.3.5	(0)	8	0	1	0.2.8.4	(0)	5	1	3	0.2.11.8	(0)	0	2	1	0.2.15.4	(0)	0	1	5
0.2.4.1	(0)	7	0	1	0.2.9.1	(0)	5	1	4	0.2.11.9	(0)	1	1	2	0.2.16.1	(0)	0	1	6
0.2.4.2	(0)	7	1	1	0.2.9.2	(0)	4	1	3	0.2.11.10	(0)	0	0	1	0.2.16.2	(0)	1	1	7
0.2.4.3	(0)	8	1	2	0.2.9.3	(0)	3	1	2	0.2.11.11	(0)	0	1	1	0.2.17.1	(0)	1	1	8
0.2.5.1	(0)	7	1	2	0.2.9.4	(0)	2	1	1	0.2.12.1	(0)	0	1	2	0.2.17.2	(0)	0	1	7
0.2.5.2	(0)	6	1	1	0.2.9.5	(0)	2	0	1	0.2.12.2	(0)	1	1	3	0.2.18.1	(0)	0	1	8
0.2.5.3	(0)	6	0	1	0.2.10.1	(0)	1	0	1	0.2.12.3	(0)	2	1	4	0.2.18.2	(0)	1	1	9
0.2.6.1	(0)	5	0	1	0.2.10.2	(0)	1	1	1	0.2.12.4	(0)	3	1	5	0.2.19.1	(0)	1	1	10
0.2.6.2	(0)	5	1	1	0.2.10.3	(0)	2	1	2	0.2.12.5	(0)	4	1	6	0.2.19.2	(0)	0	1	9
0.2.6.3	(0)	6	1	2	0.2.10.4	(0)	3	1	3	0.2.13.1	(0)	3	1	6	0.2.19.3	(0)	0	1	9
0.2.7.1	(0)	6	1	3	0.2.10.5	(0)	4	1	4	0.2.13.2	(0)	2	1	5	0.2.20.1	(0)	0	1	10
0.2.7.2	(0)	5	1	2	0.2.10.6	(0)	5	1	5	0.2.13.3	(0)	1	1	4					

Table 5. Fixed loci of purely nonsymplectic automorphisms of order 3.

K3 id	$j = 3$				K3 id	$j = 3$				K3 id	$j = 3$				K3 id	$j = 3$			
	(a_i)	k	l	g		(a_i)	k	l	g		(a_i)	k	l	g		(a_i)	k	l	g
0.3.0.1	(9)	6	0	1	0.3.3.2	(6)	3	0	1	0.3.5.3	(4)	1	1	1	0.3.7.1	(2)	1	1	3
0.3.1.1	(8)	5	1	1	0.3.4.1	(5)	3	1	2	0.3.5.4	(4)	1	0	1	0.3.7.2	(2)	0	1	2
0.3.1.2	(8)	5	0	1	0.3.4.2	(5)	2	1	1	0.3.6.1	(3)	2	1	3	0.3.8.1	(1)	1	1	4
0.3.2.1	(7)	4	1	1	0.3.4.3	(5)	2	0	1	0.3.6.2	(3)	1	1	2	0.3.8.2	(1)	0	1	3
0.3.2.2	(7)	4	0	1	0.3.5.1	(4)	3	1	3	0.3.6.3	(3)	0	1	1	0.3.9.1	(0)	1	1	5
0.3.3.1	(6)	3	1	1	0.3.5.2	(4)	2	1	2	0.3.6.4	(3)	0	0	1	0.3.9.2	(0)	0	1	4

Table 6. Fixed loci of purely nonsymplectic automorphisms of order 4.

K3 id	$j = 4$				$j = 2$				$j = 4$				$j = 2$				$j = 4$				$j = 2$								
	(a_i)	k	l	g	(a_i)	k	l	g	K3 id	(a_i)	k	l	g	(a_i)	k	l	g	K3 id	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l
0.4.0.1	(4)	0	0	1	(0)	10	0	1	0.4.1.4	(4)	0	0	1	(0)	8	1	1	0.4.1.10	(6)	1	0	1	(0)	8	0	1			
0.4.0.2	(6)	1	0	1	(0)	10	0	1	0.4.1.5	(4)	0	0	1	(0)	8	1	1	0.4.1.11	(8)	2	0	1	(0)	8	0	1			
0.4.0.3	(12)	4	0	1	(0)	10	0	1	0.4.1.6	(4)	0	1	1	(0)	8	1	1	0.4.1.12	(10)	3	0	1	(0)	8	0	1			
0.4.1.1	(4)	0	0	1	(0)	9	1	2	0.4.1.7	(12)	4	0	1	(0)	8	1	1	0.4.2.1	(4)	0	0	1	(0)	7	1	2			
0.4.1.2	(12)	4	0	1	(0)	9	1	2	0.4.1.8	(4)	0	0	1	(0)	8	0	1	0.4.2.2	(10)	3	0	1	(0)	7	1	2			
0.4.1.3	(4)	0	0	1	(0)	8	1	1	0.4.1.9	(4)	0	0	1	(0)	8	0	1	0.4.2.3	(4)	0	1	1	(0)	6	1	1			
0.4.2.4	(4)	0	0	1	(0)	6	1	1	0.4.4.2	(8)	2	0	1	(0)	4	1	3	0.4.5.12	(4)	0	0	1	(0)	0	2	1			
0.4.2.5	(4)	0	0	1	(0)	6	1	1	0.4.4.3	(2)	0	1	2	(0)	3	1	2	0.4.5.13	(4)	0	0	1	(0)	1	1	2			
0.4.2.6	(10)	3	0	1	(0)	6	1	1	0.4.4.4	(4)	0	0	1	(0)	3	1	2	0.4.5.14	(4)	0	1	1	(0)	0	2	1			
0.4.2.7	(4)	0	0	1	(0)	6	0	1	0.4.4.5	(6)	1	0	1	(0)	3	1	2	0.4.5.15	(6)	1	0	1	(0)	1	1	2			
0.4.2.8	(6)	1	0	1	(0)	6	0	1	0.4.4.6	(8)	2	0	1	(0)	3	1	2	0.4.5.16	(4)	0	0	1	(0)	0	1	1			
0.4.2.9	(8)	2	0	1	(0)	6	0	1	0.4.4.7	(4)	0	1	1	(0)	2	1	1	(0)	(6)	1	0	1	(0)	3	1	6			
0.4.3.1	(10)	3	0	1	(0)	6	1	3	0.4.4.8	(4)	0	0	1	(0)	2	1	1	0.4.6.2	(6)	1	0	1	(0)	2	1	5			
0.4.3.2	(2)	0	1	2	(0)	5	1	2	0.4.4.9	(4)	0	0	1	(0)	2	1	1	0.4.6.3	(4)	0	0	1	(0)	1	1	4			
0.4.3.3	(4)	0	0	1	(0)	5	1	2	0.4.4.10	(6)	1	0	1	(0)	2	1	1	0.4.6.4	(6)	1	0	1	(0)	1	1	4			
0.4.3.4	(8)	2	0	1	(0)	5	1	2	0.4.4.11	(4)	0	0	1	(0)	2	0	1	0.4.6.5	(0)	1	0	1	(0)	0	1	3			
0.4.3.5	(10)	3	0	1	(0)	5	1	2	0.4.5.1	(8)	2	0	1	(0)	5	1	6	0.4.6.6	(4)	0	0	1	(0)	0	1	3			
0.4.3.6	(4)	0	0	1	(0)	4	1	1	0.4.5.2	(8)	2	0	1	(0)	4	1	5	0.4.7.1	(6)	1	0	1	(0)	2	1	7			
0.4.3.7	(4)	0	0	1	(0)	4	1	1	0.4.5.3	(4)	0	0	1	(0)	3	1	4	0.4.7.2	(4)	0	0	1	(0)	1	1	6			
0.4.3.8	(4)	0	1	1	(0)	4	1	1	0.4.5.4	(6)	1	0	1	(0)	3	1	4	0.4.7.3	(6)	1	0	1	(0)	1	1	6			
0.4.3.9	(6)	1	0	1	(0)	4	1	1	0.4.5.5	(8)	2	0	1	(0)	3	1	4	0.4.7.4	(4)	0	0	1	(0)	0	1	5			
0.4.3.10	(6)	1	0	1	(0)	4	1	1	0.4.5.6	(0)	0	1	3	(0)	2	1	3	0.4.8.1	(4)	0	0	1	(0)	1	1	8			
0.4.3.11	(6)	1	1	1	(0)	4	1	1	0.4.5.7	(4)	0	0	1	(0)	2	1	3	0.4.8.2	(4)	0	0	1	(0)	0	1	7			
0.4.3.12	(8)	2	0	1	(0)	4	1	1	0.4.5.8	(4)	0	0	1	(0)	2	1	3	0.4.9.1	(4)	0	0	1	(0)	1	1	10			
0.4.3.13	(4)	0	0	1	(0)	4	0	1	0.4.5.9	(6)	1	0	1	(0)	2	1	3	0.4.9.2	(4)	0	0	1	(0)	0	1	9			
0.4.3.14	(6)	1	0	1	(0)	4	0	1	0.4.5.10	(8)	2	0	1	(0)	2	1	3												
0.4.4.1	(8)	2	0	1	(0)	5	1	4	0.4.5.11	(2)	0	1	2	(0)	1	1	2												

Table 7. Fixed loci of purely nonsymplectic automorphisms of order 5.

K3 id	$j = 5$				K3 id	$j = 5$				K3 id	$j = 5$			
	(a_i)	k	l	g		(a_i)	k	l	g		(a_i)	k	l	g
0.5.0.1	(9 4)	3	0	1	0.5.2.2	(5 2)	1	0	1	0.5.4.1	(10)	0	1	2
0.5.1.1	(7 3)	2	0	1	0.5.3.1	(3 1)	0	1	1					
0.5.2.1	(5 2)	1	1	1	0.5.3.2	(3 1)	0	0	1					

Table 8. Fixed loci of purely nonsymplectic automorphisms of order 6.

K3 id	$j = 6$				$j = 3$				$j = 2$				K3 id	$j = 6$				$j = 3$				$j = 2$					
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g		(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k
0.6.0.1	(3 0)	0	0	1	(3)	0	0	1	(0)	9	1	1	0.6.1.6	(0 6)	0	0	1	(8)	5	0	1	(0)	0	2	1		
0.6.0.2	(3 0)	0	0	1	(9)	6	0	1	(0)	0	1	1	0.6.1.7	(4 4)	1	0	1	(8)	5	0	1	(0)	3	1	2		
0.6.0.3	(1 4)	0	0	1	(9)	6	0	1	(0)	1	1	1	0.6.1.8	(6 6)	2	0	1	(8)	5	0	1	(0)	6	1	2		
0.6.0.4	(3 6)	1	0	1	(9)	6	0	1	(0)	4	1	1	0.6.1.9	(2 2)	0	0	1	(2)	0	1	2	(0)	8	0	1		
0.6.0.5	(9 6)	3	0	1	(9)	6	0	1	(0)	9	1	1	0.6.1.10	(4 4)	1	0	1	(4)	1	1	1	(0)	8	0	1		
0.6.0.6	(2 2)	0	0	1	(4)	1	0	1	(0)	8	0	1	0.6.1.11	(6 6)	2	0	1	(6)	3	1	1	(0)	8	0	1		
0.6.0.7	(2 2)	0	0	1	(8)	5	0	1	(0)	2	0	1	0.6.1.12	(3 0)	0	0	1	(3)	0	0	1	(0)	6	1	1		
0.6.0.8	(6 6)	2	0	1	(8)	5	0	1	(0)	8	0	1	0.6.1.13	(3 0)	0	0	1	(7)	4	0	1	(0)	0	1	1		
0.6.1.1	(0 6)	0	0	1	(8)	5	1	1	(0)	0	2	1	0.6.1.14	(1 4)	0	0	1	(7)	4	0	1	(0)	1	1	1		
0.6.1.2	(8 8)	3	0	1	(8)	5	1	1	(0)	9	1	2	0.6.1.15	(3 6)	1	0	1	(7)	4	0	1	(0)	4	1	1		
0.6.1.3	(2 2)	0	0	1	(8)	5	0	1	(0)	1	1	3	0.6.1.16	(7 4)	2	0	1	(7)	4	0	1	(0)	6	1	1		
0.6.1.4	(2 2)	0	0	1	(8)	5	0	1	(0)	0	1	2	0.6.1.17	(2 2)	0	0	1	(4)	1	0	1	(0)	5	0	1		
0.6.1.5	(2 2)	0	0	1	(8)	5	0	1	(0)	0	1	2	0.6.1.18	(2 2)	0	0	1	(6)	3	0	1	(0)	2	0	1		
0.6.1.19	(4 4)	1	0	1	(6)	3	0	1	(0)	5	0	1	0.6.3.19	(0 6)	0	0	1	(4)	1	1	1	(0)	0	2	1		

(Continues)

Table 8. (Continued)

K3 id	$j = 6$				$j = 3$				$j = 2$				K3 id	$j = 6$				$j = 3$				$j = 2$					
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g		(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k
0.6.2.1	(14)	0	0	1	(7)	4	1	1	(0)	0	1	3	0.6.3.20	(44)	1	0	1	(4)	1	1	1	(0)	3	1	2		
0.6.2.2	(58)	2	0	1	(7)	4	1	1	(0)	6	1	3	0.6.3.21	(30)	0	0	1	(5)	2	0	1	(0)	0	1	4		
0.6.2.3	(06)	0	0	1	(6)	3	1	1	(0)	0	2	1	0.6.3.22	(14)	0	0	1	(5)	2	0	1	(0)	0	1	3		
0.6.2.4	(66)	2	0	1	(6)	3	1	1	(0)	6	1	2	0.6.3.23	(14)	0	0	1	(5)	2	0	1	(0)	1	1	4		
0.6.2.5	(30)	0	0	1	(7)	4	0	1	(0)	0	1	4	0.6.3.24	(52)	1	0	1	(5)	2	0	1	(0)	2	1	3		
0.6.2.6	(14)	0	0	1	(7)	4	0	1	(0)	0	1	3	0.6.3.25	(52)	1	0	1	(5)	2	0	1	(0)	3	1	4		
0.6.2.7	(14)	0	0	1	(7)	4	0	1	(0)	0	1	3	0.6.3.26	(14)	0	0	1	(3)	1	1	2	(0)	1	1	1		
0.6.2.8	(14)	0	0	1	(7)	4	0	1	(0)	1	1	4	0.6.3.27	(14)	0	0	1	(3)	0	1	1	(0)	1	1	1		
0.6.2.9	(52)	1	0	1	(7)	4	0	1	(0)	2	1	3	0.6.3.28	(22)	0	0	1	(4)	1	0	1	(0)	2	1	4		
0.6.2.10	(36)	1	0	1	(7)	4	0	1	(0)	3	1	3	0.6.3.29	(22)	0	0	1	(4)	1	0	1	(0)	1	1	3		
0.6.2.11	(74)	2	0	1	(7)	4	0	1	(0)	5	1	3	0.6.3.30	(22)	0	0	1	(4)	1	0	1	(0)	0	1	2		
0.6.2.12	(14)	0	0	1	(3)	0	1	1	(0)	4	1	1	0.6.3.31	(22)	0	0	1	(2)	0	1	2	(0)	2	0	1		
0.6.2.13	(36)	1	0	1	(5)	2	1	1	(0)	4	1	1	0.6.3.32	(22)	0	0	1	(2)	1	1	3	(0)	2	0	1		
0.6.2.14	(22)	0	0	1	(6)	3	0	1	(0)	1	1	3	0.6.3.33	(30)	0	0	1	(3)	0	0	1	(0)	0	1	1		
0.6.2.15	(22)	0	0	1	(6)	3	0	1	(0)	0	1	2	0.6.3.34	(30)	0	0	1	(3)	0	0	1	(0)	0	2	1		
0.6.2.16	(06)	0	0	1	(6)	3	0	1	(0)	0	2	1	0.6.3.35	(30)	0	0	1	(3)	0	1	1	(0)	0	1	1		
0.6.2.17	(44)	1	0	1	(6)	3	0	1	(0)	3	1	2	0.6.3.36	(30)	0	1	1	(3)	0	1	1	(0)	0	2	1		
0.6.2.18	(44)	1	0	1	(6)	3	0	1	(0)	4	1	3	0.6.3.37	(00)	0	1	2	(2)	0	1	2	(0)	0	1	2		
0.6.2.19	(22)	0	0	1	(2)	0	1	2	(0)	5	0	1	0.6.4.1	(58)	2	0	1	(5)	3	1	2	(0)	5	1	5		
0.6.2.20	(44)	1	0	1	(4)	2	1	2	(0)	5	0	1	0.6.4.2	(28)	1	0	1	(4)	2	1	2	(0)	3	1	4		
0.6.2.21	(44)	1	0	1	(4)	1	1	1	(0)	5	0	1	0.6.4.3	(14)	0	0	1	(5)	2	1	1	(0)	0	1	6		
0.6.2.22	(30)	0	0	1	(3)	0	0	1	(0)	3	1	1	0.6.4.4	(36)	1	0	1	(5)	2	1	1	(0)	2	1	5		
0.6.2.23	(30)	0	0	1	(5)	2	0	1	(0)	0	1	1	0.6.4.5	(36)	1	0	1	(5)	2	1	1	(0)	3	1	6		
0.6.2.24	(14)	0	0	1	(5)	2	0	1	(0)	1	1	1	0.6.4.6	(36)	1	0	1	(3)	1	1	2	(0)	3	1	3		
0.6.2.25	(52)	1	0	1	(5)	2	0	1	(0)	3	1	1	0.6.4.7	(06)	0	0	1	(4)	1	1	1	(0)	0	1	4		
0.6.2.26	(22)	0	0	1	(4)	1	0	1	(0)	2	0	1	0.6.4.8	(44)	1	0	1	(4)	1	1	1	(0)	2	1	4		
0.6.2.27	(22)	0	0	1	(4)	1	0	1	(0)	2	1	1	0.6.4.9	(06)	0	0	1	(2)	0	1	2	(0)	0	2	1		
0.6.2.28	(22)	0	0	1	(4)	1	1	1	(0)	2	0	1	0.6.4.10	(06)	0	0	1	(2)	1	1	3	(0)	0	2	1		
0.6.2.29	(22)	0	1	1	(4)	1	1	1	(0)	2	1	1	0.6.4.11	(30)	0	0	1	(5)	2	0	1	(0)	0	1	7		
0.6.3.1	(58)	2	0	1	(5)	3	1	2	(0)	6	1	3	0.6.4.12	(30)	0	0	1	(5)	2	0	1	(0)	0	1	7		
0.6.3.2	(06)	0	0	1	(6)	3	1	1	(0)	0	1	4	0.6.4.13	(14)	0	0	1	(5)	2	0	1	(0)	1	1	7		
0.6.3.3	(28)	1	0	1	(6)	3	1	1	(0)	3	1	4	0.6.4.14	(14)	0	0	1	(5)	2	0	1	(0)	0	1	6		
0.6.3.4	(66)	2	0	1	(6)	3	1	1	(0)	5	1	4	0.6.4.15	(52)	1	0	1	(5)	2	0	1	(0)	2	1	6		
0.6.3.5	(14)	0	0	1	(5)	2	1	1	(0)	0	1	3	0.6.4.16	(52)	1	0	1	(5)	2	0	1	(0)	1	1	5		
0.6.3.6	(36)	1	0	1	(5)	2	1	1	(0)	3	1	3	0.6.4.17	(14)	0	0	1	(3)	0	1	1	(0)	0	1	3		
0.6.3.7	(14)	0	0	1	(1)	0	1	3	(0)	4	1	1	0.6.4.18	(14)	0	0	1	(3)	0	1	1	(0)	1	1	4		
0.6.3.8	(36)	1	0	1	(3)	1	1	2	(0)	4	1	1	0.6.4.19	(14)	0	0	1	(1)	1	1	4	(0)	1	1	1		
0.6.3.9	(36)	1	0	1	(3)	2	1	3	(0)	4	1	1	0.6.4.20	(14)	0	0	1	(1)	0	1	3	(0)	1	1	1		
0.6.3.10	(22)	0	0	1	(6)	3	0	1	(0)	0	1	5	0.6.4.21	(22)	0	0	1	(4)	1	0	1	(0)	0	1	5		
0.6.3.11	(22)	0	0	1	(6)	3	0	1	(0)	0	1	5	0.6.4.22	(22)	0	0	1	(4)	1	0	1	(0)	1	1	6		
0.6.3.12	(22)	0	0	1	(6)	3	0	1	(0)	1	1	6	0.6.4.23	(22)	0	0	1	(4)	1	0	1	(0)	1	1	6		
0.6.3.13	(22)	0	0	1	(6)	3	0	1	(0)	1	1	6	0.6.4.24	(22)	0	0	1	(4)	1	0	1	(0)	2	1	7		
0.6.3.14	(06)	0	0	1	(6)	3	0	1	(0)	0	1	4	0.6.4.25	(22)	0	0	1	(2)	0	1	2	(0)	0	1	2		
0.6.3.15	(60)	1	0	1	(6)	3	0	1	(0)	1	1	4	0.6.4.26	(30)	0	0	1	(3)	0	0	1	(0)	0	1	4		
0.6.3.16	(44)	1	0	1	(6)	3	0	1	(0)	4	1	6	0.6.5.1	(410)	2	0	1	(4)	3	1	3	(0)	5	1	6		
0.6.3.17	(44)	1	0	1	(6)	3	0	1	(0)	3	1	5	0.6.5.2	(28)	1	0	1	(4)	2	1	2	(0)	2	1	6		
0.6.3.18	(44)	1	0	1	(6)	3	0	1	(0)	2	1	4	0.6.5.3	(28)	1	0	1	(2)	1	1	3	(0)	3	1	4		
0.6.5.4	(36)	1	0	1	(3)	1	1	2	(0)	2	1	5	0.6.6.1	(110)	1	0	1	(3)	2	1	3	(0)	2	1	7		
0.6.5.5	(06)	0	0	1	(4)	1	1	1	(0)	0	1	7	0.6.6.2	(28)	1	0	1	(2)	1	1	3	(0)	2	1	6		
0.6.5.6	(44)	1	0	1	(4)	1	1	1	(0)	1	1	6	0.6.6.3	(36)	1	0	1	(3)	1	1	2	(0)	1	1	7		
0.6.5.7	(44)	1	0	1	(4)	1	1	1	(0)	2	1	7	0.6.6.4	(06)	0	0	1	(2)	0	1	2	(0)	0	1	7		
0.6.5.8	(06)	0	0	1	(2)	0	1	2	(0)	0	1	4	0.6.6.5	(06)	0	0	1	(0)	0	1	4	(0)	0	1	4		
0.6.5.9	(06)	0	0	1	(0)	0	1	4	(0)	0	2	1	0.6.6.6	(14)	0	0	1	(3)	0	1	1	(0)	0	1	9		
0.6.5.10	(06)	0	0	1	(0)	1	1	5	(0)	0	2	1	0.6.6.7	(14)	0	0	1	(3)	0	1	1	(0)	0	1	9		
0.6.5.11	(14)	0	0	1	(3)	0	1	1	(0)	1	1	7	0.6.6.8	(14)	0	0	1	(3)	0	1	1	(0)	1	1	10		
0.6.5.12	(14)	0	0	1	(3)	0	1	1	(0)	0	1	6	0.6.6.9	(30)	0	0	1	(3)	0	0	1	(0)	0	1	10		
0.6.5.13	(14)	0	0	1	(1)	0	1	3	(0)	0	1	3	0.6.7.1	(110)	1	0	1	(1)	1	1	4	(0)	2	1	7		
0.6.5.14	(22)	0	0	1	(4)	1	0	1	(0)	1	1	9	0.6.7.2	(28)	1	0	1	(2)	1	1	3	(0)	1	1	8		
0.6.5.15	(22)	0	0	1	(4)	1	0	1	(0)	0	1	8	0.6.7.3	(06)	0	0	1	(2)	0	1	2	(0)	0	1	10		
0.6.5.16	(22)	0	0	1	(4)	1	0	1	(0)	0	1	8	0.6.8.1	(110)	1	0	1	(1)	1	1	4	(0)	1	1	9		
0.6.5.17	(30)	0	0	1	(3)	0	0	1	(0)	0	1	7	0.6.9.1	(012)	1	0	1	(0)	1	1	5	(0)	1	1	10		

Table 9. Fixed loci of purely nonsymplectic automorphisms of order 7.

K3 id	$j = 7$				K3 id	$j = 7$			
	(a_i)	k	l	g		(a_i)	k	l	g
0.7.0.1	(6 5 2)	2	0	1	0.7.2.1	(2 1 0)	0	1	1
0.7.1.1	(4 3 1)	1	1	1	0.7.2.2	(2 1 0)	0	0	1
0.7.1.2	(4 3 1)	1	0	1					

Table 10. Fixed loci of purely nonsymplectic automorphisms of order 8.

K3 id	$j = 8$				$j = 4$				$j = 2$				K3 id	$j = 8$				$j = 4$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g		(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.8.0.1	(2 0 0)	0	0	1	(4)	0	0	1	()	8	1	1	0.8.2.1	(4 2 2)	1	0	1	(8)	2	0	1	()	4	1	5
0.8.0.2	(2 0 0)	0	0	1	(12)	4	0	1	()	8	1	1	0.8.2.2	(1 1 2)	0	0	1	(6)	1	0	1	()	3	1	4
0.8.0.3	(0 2 4)	0	0	1	(12)	4	0	1	()	8	1	1	0.8.2.3	(4 2 2)	1	0	1	(8)	2	0	1	()	3	1	4
0.8.0.4	(6 4 4)	2	0	1	(12)	4	0	1	()	8	1	1	0.8.2.4	(1 1 2)	0	0	1	(6)	1	0	1	()	2	1	3
0.8.0.5	(1 1 2)	0	0	1	(6)	1	0	1	()	8	0	1	0.8.2.5	(2 0 0)	0	0	1	(8)	2	0	1	()	2	1	3
0.8.0.6	(1 1 2)	0	0	1	(10)	3	0	1	()	8	0	1	0.8.2.6	(5 1 0)	1	0	1	(6)	1	0	1	()	2	1	3
0.8.0.7	(3 3 4)	1	0	1	(10)	3	0	1	()	8	0	1	0.8.2.7	(0 2 4)	0	0	1	(8)	2	0	1	()	2	1	3
0.8.1.1	(7 3 2)	2	0	1	(10)	3	0	1	()	6	1	3	0.8.2.8	(2 0 0)	0	1	1	(4)	0	1	1	()	0	2	1
0.8.1.2	(1 1 2)	0	0	1	(10)	3	0	1	()	5	1	2	0.8.2.9	(1 1 2)	0	0	1	(2)	0	1	2	()	1	1	2
0.8.1.3	(3 3 4)	1	0	1	(10)	3	0	1	()	5	1	2	0.8.2.10	(2 0 0)	0	0	1	(4)	0	1	1	()	0	2	1
0.8.1.4	(2 0 0)	0	0	1	(4)	0	0	1	()	4	1	1	0.8.2.11	(2 0 0)	0	0	1	(4)	0	0	1	()	0	2	1
0.8.1.5	(2 0 0)	0	0	1	(8)	2	0	1	()	4	1	1	0.8.2.12	(0 2 4)	0	0	1	(4)	0	1	1	()	0	2	1
0.8.1.6	(1 1 2)	0	0	1	(6)	1	0	1	()	4	1	1	0.8.2.13	(1 1 2)	0	0	1	(6)	1	0	1	()	1	1	2
0.8.1.7	(1 1 2)	0	0	1	(6)	1	1	1	()	4	1	1	0.8.2.14	(2 0 0)	0	0	1	(4)	0	0	1	()	0	1	1
0.8.1.8	(1 1 2)	0	1	1	(6)	1	1	1	()	4	1	1	0.8.3.1	(5 1 0)	1	0	1	(6)	1	0	1	()	2	1	7
0.8.1.9	(0 2 4)	0	0	1	(8)	2	0	1	()	4	1	1	0.8.3.2	(1 1 2)	0	0	1	(6)	1	0	1	()	1	1	6
0.8.1.10	(4 2 2)	1	0	1	(8)	2	0	1	()	4	1	1	0.8.3.3	(5 1 0)	1	0	1	(6)	1	0	1	()	1	1	6
0.8.1.11	(3 3 4)	1	0	1	(6)	1	1	1	()	4	1	1	0.8.3.4	(2 0 0)	0	0	1	(4)	0	0	1	()	0	1	5
0.8.1.12	(1 1 2)	0	0	1	(6)	1	0	1	()	4	0	1	0.8.4.1	(2 0 0)	0	0	1	(4)	0	0	1	()	0	1	9

Table 11. Fixed loci of purely nonsymplectic automorphisms of order 9.

K3 id	$j = 9$				$j = 3$				K3 id	$j = 9$				$j = 3$			
	(a_i)	k	l	g	(a_i)	k	l	g		(a_i)	k	l	g	(a_i)	k	l	g
0.9.0.1	(6 5 2 1)	2	0	1	(7)	4	1	1	0.9.1.5	(0 1 4 1)	0	0	1	(4)	1	1	1
0.9.0.2	(1 1 1 0)	0	0	1	(7)	4	0	1	0.9.1.6	(4 3 0 0)	1	0	1	(4)	1	1	1
0.9.0.3	(3 3 3 1)	1	0	1	(7)	4	0	1	0.9.1.7	(1 1 1 0)	0	0	1	(4)	1	0	1
0.9.1.1	(3 3 3 1)	1	0	1	(4)	3	1	3	0.9.2.1	(0 1 4 1)	0	0	1	(1)	1	1	4
0.9.1.2	(3 3 3 1)	1	0	1	(4)	2	1	2	0.9.2.2	(1 1 1 0)	0	0	1	(1)	0	1	3
0.9.1.3	(1 1 1 0)	0	0	1	(4)	1	1	1	0.9.2.3	(0 1 4 1)	0	0	1	(1)	0	1	3
0.9.1.4	(1 1 1 0)	0	1	1	(4)	1	1	1									

Table 12. Fixed loci of purely nonsymplectic automorphisms of order 10.

K3 id	j = 10				j = 5				j = 2				K3 id	j = 10				j = 5				j = 2						
	(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l	g		(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l
0.10.0.1	(1 0 0 2)	0	0	1	(9 4)	3	0	1	()	0	1	2	0.10.1.13	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	1	1	1			
0.10.0.2	(0 2 1 2)	0	0	1	(9 4)	3	0	1	()	1	1	2	0.10.1.14	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	1	0	1			
0.10.0.3	(3 2 2 4)	1	0	1	(9 4)	3	0	1	()	5	1	2	0.10.2.1	(0 0 1 6)	0	0	1	(5 2)	1	1	1	()	1	1	6			
0.10.0.4	(6 4 3 2)	2	0	1	(9 4)	3	0	1	()	8	1	2	0.10.2.2	(3 2 2 4)	1	0	1	(5 2)	1	1	1	()	4	1	6			
0.10.0.5	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	6	0	1	0.10.2.3	(0 1 1 4)	0	0	1	(3 1)	0	1	1	()	1	1	4			
0.10.0.6	(1 1 0 0)	0	0	1	(7 3)	2	0	1	()	1	0	1	0.10.2.4	(1 0 0 2)	0	0	1	(5 2)	1	0	1	()	0	1	7			
0.10.0.7	(3 3 2 2)	1	0	1	(7 3)	2	0	1	()	6	0	1	0.10.2.5	(0 2 1 2)	0	0	1	(5 2)	1	0	1	()	1	1	7			
0.10.1.1	(3 2 2 4)	1	0	1	(5 2)	1	1	1	()	5	1	2	0.10.2.6	(0 2 1 2)	0	0	1	(5 2)	1	0	1	()	0	1	6			
0.10.1.2	(1 1 0 0)	0	0	1	(7 3)	2	0	1	()	0	1	5	0.10.2.7	(5 0 0 0)	1	0	1	(5 2)	1	0	1	()	1	1	6			
0.10.1.3	(0 3 1 0)	0	0	1	(7 3)	2	0	1	()	0	1	4	0.10.2.8	(4 2 1 0)	1	0	1	(5 2)	1	0	1	()	2	1	6			
0.10.1.4	(0 1 1 4)	0	0	1	(7 3)	2	0	1	()	1	1	4	0.10.2.9	(1 0 0 2)	0	0	1	(1 0)	0	1	2	()	0	1	2			
0.10.1.5	(4 1 1 2)	1	0	1	(7 3)	2	0	1	()	3	1	4	0.10.2.10	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	0	1	5			
0.10.1.6	(3 3 2 2)	1	0	1	(7 3)	2	0	1	()	5	1	5	0.10.2.11	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	1	1	6			
0.10.1.7	(3 3 2 2)	1	0	1	(7 3)	2	0	1	()	4	1	4	0.10.3.1	(0 0 1 6)	0	0	1	(1 0)	0	1	2	()	1	1	6			
0.10.1.8	(1 0 0 2)	0	0	1	(5 2)	1	0	1	()	0	1	2	0.10.3.2	(0 1 1 4)	0	0	1	(3 1)	0	1	1	()	1	1	9			
0.10.1.9	(0 2 1 2)	0	0	1	(5 2)	1	0	1	()	1	1	2	0.10.3.3	(0 1 1 4)	0	0	1	(3 1)	0	1	1	()	0	1	8			
0.10.1.10	(4 2 1 0)	1	0	1	(5 2)	1	0	1	()	3	1	2	0.10.3.4	(1 1 0 0)	0	0	1	(3 1)	0	0	1	()	0	1	10			
0.10.1.11	(1 1 0 0)	0	1	1	(3 1)	0	1	1	()	1	1	1	0.10.4.1	(0 0 1 6)	0	0	1	(1 0)	0	1	2	()	0	1	10			
0.10.1.12	(1 1 0 0)	0	0	1	(3 1)	0	1	1	()	1	0	1																

Table 13. Fixed loci of purely nonsymplectic automorphisms of order 11.

K3 id	j = 11			
	(a _i)	k	l	g
0.11.0.1	(3 2 2 3 1)	1	0	1
0.11.1.1	(1 0 0 1 0)	0	1	1
0.11.1.2	(1 0 0 1 0)	0	0	1

Table 14. Fixed loci of purely nonsymplectic automorphisms of order 12.

K3 id	j = 12				j = 6				j = 4				j = 3				j = 2			
	(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l	g	(a _i)	k	l	g
0.12.0.1	(0 2 0 0 2)	0	0	1	(8 8)	3	0	1	(4)	0	0	1	(8)	5	1	1	()	9	1	2
0.12.0.2	(6 4 2 2 2)	2	0	1	(8 8)	3	0	1	(12)	4	0	1	(8)	5	1	1	()	9	1	2
0.12.0.3	(0 0 2 2 2)	0	0	1	(4 4)	1	0	1	(8)	2	0	1	(4)	1	1	1	()	8	0	1
0.12.0.4	(3 3 1 1 2)	1	0	1	(6 6)	2	0	1	(8)	2	0	1	(6)	3	1	1	()	8	0	1
0.12.0.5	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(3)	0	0	1	()	6	1	1
0.12.0.6	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(7)	4	0	1	()	0	1	1
0.12.0.7	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(10)	3	0	1	(3)	0	0	1	()	6	1	1
0.12.0.8	(1 0 0 0 0)	0	0	1	(7 4)	2	0	1	(4)	0	0	1	(7)	4	0	1	()	6	1	1
0.12.0.9	(0 1 1 1 2)	0	0	1	(3 6)	1	0	1	(6)	1	0	1	(7)	4	0	1	()	4	1	1
0.12.0.10	(0 0 0 3 4)	0	0	1	(7 4)	2	0	1	(4)	0	1	1	(7)	4	0	1	()	6	1	1
0.12.0.11	(3 3 3 0 0)	1	0	1	(3 6)	1	0	1	(6)	1	1	1	(7)	4	0	1	()	4	1	1
0.12.0.12	(3 2 2 2 2)	1	0	1	(7 4)	2	0	1	(10)	3	0	1	(7)	4	0	1	()	6	1	1
0.12.1.1	(3 2 2 2 2)	1	0	1	(5 8)	2	0	1	(10)	3	0	1	(5)	3	1	2	()	6	1	3
0.12.1.2	(0 2 0 0 2)	0	0	1	(2 8)	1	0	1	(4)	0	0	1	(6)	3	1	1	()	3	1	4
0.12.1.3	(3 3 1 1 2)	1	0	1	(6 6)	2	0	1	(8)	2	0	1	(6)	3	1	1	()	5	1	4
0.12.1.4	(0 0 4 1 0)	0	0	1	(1 4)	0	0	1	(6)	1	1	1	(1)	0	1	3	()	4	1	1
0.12.1.5	(0 1 1 1 2)	0	0	1	(3 6)	1	0	1	(6)	1	0	1	(3)	1	1	2	()	4	1	1
0.12.1.6	(0 1 1 1 2)	0	0	1	(3 6)	1	0	1	(6)	1	0	1	(3)	2	1	3	()	4	1	1
0.12.1.7	(3 3 3 0 0)	1	0	1	(3 6)	1	0	1	(6)	1	1	1	(3)	2	1	3	()	4	1	1
0.12.1.8	(0 1 3 0 0)	0	0	1	(0 6)	0	0	1	(4)	0	1	1	(4)	1	1	1	()	0	2	1
0.12.1.9	(0 0 2 2 2)	0	0	1	(4 4)	1	0	1	(2)	0	1	2	(4)	1	1	1	()	3	1	2
0.12.1.10	(0 2 0 0 2)	0	0	1	(4 4)	1	0	1	(4)	0	0	1	(4)	1	1	1	()	3	1	2

(Continues)

Table 14. (Continued)

K3 id	$j = 12$				$j = 6$				$j = 4$				$j = 3$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.12.1.11	(4 2 0 0 0)	1	0	1	(4 4)	1	0	1	(6)	1	0	1	(4)	1	1	1	()	3	1	2
0.12.1.12	(0 0 2 2 2)	0	0	1	(4 4)	1	0	1	(8)	2	0	1	(4)	1	1	1	()	3	1	2
0.12.1.13	(1 0 0 0 0)	0	0	1	(1 4)	0	0	1	(4)	0	0	1	(5)	2	0	1	()	0	1	3
0.12.1.14	(0 1 1 1 2)	0	0	1	(5 2)	1	0	1	(6)	1	0	1	(5)	2	0	1	()	2	1	3
0.12.1.15	(4 1 1 1 0)	1	0	1	(5 2)	1	0	1	(8)	2	0	1	(5)	2	0	1	()	3	1	4
0.12.1.16	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(3)	0	0	1	()	0	1	1
0.12.1.17	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(3)	0	0	1	()	0	2	1
0.12.1.18	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(3)	0	1	1	()	0	1	1
0.12.1.19	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	1	1	(3)	0	0	1	()	0	2	1
0.12.1.20	(1 0 0 0 0)	0	0	1	(3 0)	0	1	1	(4)	0	0	1	(3)	0	1	1	()	0	2	1
0.12.1.21	(1 0 0 0 0)	0	1	1	(3 0)	0	1	1	(4)	0	1	1	(3)	0	1	1	()	0	2	1
0.12.1.22	(0 0 0 3 4)	0	0	1	(3 0)	0	1	1	(4)	0	1	1	(3)	0	1	1	()	0	2	1
0.12.2.1	(3 3 1 1 2)	1	0	1	(4 10)	2	0	1	(8)	2	0	1	(4)	3	1	3	()	5	1	6
0.12.2.2	(0 2 0 0 2)	0	0	1	(2 8)	1	0	1	(4)	0	0	1	(2)	1	1	3	()	3	1	4
0.12.2.3	(0 0 2 2 2)	0	0	1	(2 8)	1	0	1	(8)	2	0	1	(2)	1	1	3	()	3	1	4
0.12.2.4	(0 1 1 1 2)	0	0	1	(3 6)	1	0	1	(6)	1	0	1	(3)	1	1	2	()	2	1	5
0.12.2.5	(0 2 0 0 2)	0	0	1	(4 4)	1	0	1	(4)	0	0	1	(4)	1	1	1	()	1	1	6
0.12.2.6	(4 2 0 0 0)	1	0	1	(4 4)	1	0	1	(6)	1	0	1	(4)	1	1	1	()	2	1	7
0.12.2.7	(0 1 3 0 0)	0	0	1	(0 6)	0	0	1	(4)	0	1	1	(0)	0	1	4	()	0	2	1
0.12.2.8	(0 1 3 0 0)	0	0	1	(0 6)	0	0	1	(4)	0	1	1	(0)	1	1	5	()	0	2	1
0.12.2.9	(1 0 0 0 0)	0	0	1	(1 4)	0	0	1	(4)	0	0	1	(1)	0	1	3	()	0	1	3
0.12.2.10	(0 0 4 1 0)	0	0	1	(1 4)	0	0	1	(0)	0	1	3	(1)	0	1	3	()	0	1	3
0.12.2.11	(1 0 0 0 0)	0	0	1	(3 0)	0	0	1	(4)	0	0	1	(3)	0	0	1	()	0	1	7
0.12.3.1	(0 1 1 1 2)	0	0	1	(1 10)	1	0	1	(6)	1	0	1	(1)	1	1	4	()	2	1	7
0.12.3.2	(0 2 0 0 2)	0	0	1	(2 8)	1	0	1	(4)	0	0	1	(2)	1	1	3	()	1	1	8
0.12.4.1	(0 2 0 0 2)	0	0	1	(0 12)	1	0	1	(4)	0	0	1	(0)	1	1	5	()	1	1	10

Table 15. Fixed loci of purely nonsymplectic automorphisms of order 14.

K3 id	$j = 14$				$j = 7$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.14.0.1	(0 1 2 0 0 0)	0	0	1	(6 5 2)	2	0	1	()	0	1	3
0.14.0.2	(0 0 0 1 2 4)	0	0	1	(6 5 2)	2	0	1	()	2	1	3
0.14.0.3	(3 2 2 1 1 2)	1	0	1	(6 5 2)	2	0	1	()	5	1	3
0.14.0.4	(0 0 1 1 2 2)	0	0	1	(4 3 1)	1	0	1	()	3	0	1
0.14.1.1	(3 2 1 1 1 4)	1	0	1	(4 3 1)	1	1	1	()	5	1	6
0.14.1.2	(0 1 1 0 0 2)	0	0	1	(4 3 1)	1	0	1	()	0	1	6
0.14.1.3	(0 0 1 1 2 2)	0	0	1	(4 3 1)	1	0	1	()	1	1	6
0.14.1.4	(0 0 1 1 2 2)	0	0	1	(4 3 1)	1	0	1	()	2	1	7
0.14.1.5	(4 1 1 0 0 0)	1	0	1	(4 3 1)	1	0	1	()	2	1	6
0.14.1.6	(0 0 0 1 2 4)	0	0	1	(2 1 0)	0	1	1	()	2	1	3
0.14.2.1	(0 1 0 0 0 4)	0	0	1	(2 1 0)	0	1	1	()	0	1	9
0.14.2.2	(0 0 0 1 2 4)	0	0	1	(2 1 0)	0	1	1	()	1	1	9

Table 16. Fixed loci of purely nonsymplectic automorphisms of order 15.

K3 id	$j = 15$				$j = 5$				$j = 3$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.15.0.1	(0 0 1 2 2 0 0)	0	0	1	(7 3)	2	0	1	(2)	0	1	2
0.15.0.2	(3 3 1 1 1 0 0)	1	0	1	(7 3)	2	0	1	(4)	2	1	2
0.15.0.3	(3 2 1 1 1 3 1)	1	0	1	(7 3)	2	0	1	(5)	3	1	2
0.15.0.4	(0 0 0 0 3 3 1)	0	0	1	(3 1)	0	1	1	(6)	3	0	1
0.15.0.5	(3 2 2 3 0 0 0)	1	0	1	(5 2)	1	1	1	(6)	3	0	1
0.15.1.1	(0 1 0 0 3 0 0)	0	0	1	(3 1)	0	1	1	(0)	0	1	4
0.15.1.2	(0 0 0 0 3 3 1)	0	0	1	(3 1)	0	1	1	(1)	1	1	4
0.15.1.3	(0 0 1 2 2 0 0)	0	0	1	(1 0)	0	1	2	(2)	0	1	2

Table 17. Fixed loci of purely nonsymplectic automorphisms of order 16.

K3 id	$j = 16$				$j = 8$				$j = 4$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.16.0.1	(3 2 1 1 1 2 2)	1	0	1	(7 3 2)	2	0	1	(10)	3	0	1	()	6	1	3
0.16.0.2	(0 1 0 0 0 1 2)	0	0	1	(3 3 4)	1	0	1	(10)	3	0	1	()	5	1	2
0.16.0.3	(3 2 2 2 1 0 0)	1	0	1	(3 3 4)	1	0	1	(10)	3	0	1	()	5	1	2
0.16.0.4	(0 0 0 2 1 1 2)	0	0	1	(3 3 4)	1	0	1	(6)	1	1	1	()	4	1	1
0.16.0.5	(3 3 2 0 0 0 0)	1	0	1	(3 3 4)	1	0	1	(6)	1	1	1	()	4	1	1
0.16.1.1	(4 1 0 0 0 1 0)	1	0	1	(5 1 0)	1	0	1	(6)	1	0	1	()	2	1	7
0.16.1.2	(0 1 0 0 0 1 2)	0	0	1	(5 1 0)	1	0	1	(6)	1	0	1	()	1	1	6

Table 18. Fixed loci of purely nonsymplectic automorphisms of order 18.

K3 id	$j = 18$				$j = 9$				$j = 6$				$j = 3$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.18.0.1	(0 1 0 1 0 0 0 0)	0	0	1	(6 5 2 1)	2	0	1	(14)	0	0	1	(7)	4	1	1	()	0	1	3
0.18.0.2	(3 2 1 1 1 1 1 2)	1	0	1	(6 5 2 1)	2	0	1	(5 8)	2	0	1	(7)	4	1	1	()	6	1	3
0.18.0.3	(0 0 0 1 1 1 1 2)	0	0	1	(3 3 3 1)	1	0	1	(5 2)	1	0	1	(7)	4	0	1	()	2	1	3
0.18.0.4	(3 2 2 1 1 1 0 0)	1	0	1	(3 3 3 1)	1	0	1	(7 4)	2	0	1	(7)	4	0	1	()	5	1	3
0.18.0.5	(0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 4 1)	0	0	1	(4 4)	1	0	1	(4)	1	1	1	()	5	0	1
0.18.0.6	(3 3 1 1 1 0 0 0 0)	1	0	1	(3 3 3 1)	1	0	1	(4 4)	1	0	1	(4)	2	1	2	()	5	0	1
0.18.0.7	(0 0 0 0 0 3 2 2)	0	0	1	(4 3 0 0)	1	0	1	(2 2)	0	1	1	(4)	1	1	1	()	2	1	1
0.18.1.1	(3 2 2 1 1 1 0 0)	1	0	1	(3 3 3 1)	1	0	1	(4 10)	2	0	1	(4)	3	1	3	()	5	1	6
0.18.1.2	(0 0 0 1 1 1 1 2)	0	0	1	(3 3 3 1)	1	0	1	(2 8)	1	0	1	(4)	2	1	2	()	2	1	6
0.18.1.3	(0 1 0 0 0 0 0 2)	0	0	1	(4 3 0 0)	1	0	1	(0 6)	0	0	1	(4)	1	1	1	()	0	1	7
0.18.1.4	(0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 4 1)	0	0	1	(4 4)	1	0	1	(4)	1	1	1	()	1	1	6
0.18.1.5	(0 0 0 0 1 1 1 4)	0	0	1	(1 1 1 0)	0	1	1	(4 4)	1	0	1	(4)	1	1	1	()	2	1	7
0.18.1.6	(0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 4 1)	0	0	1	(4 4)	1	0	1	(4)	1	1	1	()	2	1	7
0.18.1.7	(4 1 0 0 0 0 0 0)	1	0	1	(4 3 0 0)	1	0	1	(4 4)	1	0	1	(4)	1	1	1	()	2	1	7
0.18.1.8	(0 1 0 1 0 0 0 0)	0	0	1	(0 1 4 1)	0	0	1	(1 4)	0	0	1	(1)	0	1	3	()	0	1	3
0.18.2.1	(0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 4 1)	0	0	1	(1 10)	1	0	1	(1)	1	1	4	()	1	1	9

Table 19. Fixed loci of purely nonsymplectic automorphisms of order 20.

K3 id	$j = 20$				$j = 10$				$j = 5$				$j = 4$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.20.0.1	(0 0 0 2 0 0 0 1 2)	0	0	1	(3 2 2 4)	1	0	1	(5 2)	1	1	1	(2)	0	1	2	()	5	1	2
0.20.0.2	(3 2 2 2 0 0 0 0 0)	1	0	1	(3 2 2 4)	1	0	1	(5 2)	1	1	1	(10)	3	0	1	()	5	1	2
0.20.0.3	(0 0 1 1 1 0 0 0 0)	0	0	1	(0 2 1 2)	0	0	1	(5 2)	1	0	1	(6)	1	0	1	()	1	1	2
0.20.0.4	(0 0 0 1 1 1 2 0 0)	0	0	1	(0 2 1 2)	0	0	1	(5 2)	1	0	1	(2)	0	1	2	()	1	1	2
0.20.0.5	(0 0 1 0 0 0 0 2 2)	0	0	1	(4 2 1 0)	1	0	1	(5 2)	1	0	1	(2)	0	1	2	()	3	1	2
0.20.0.6	(0 0 0 0 0 1 2 2 2)	0	0	1	(4 2 1 0)	1	0	1	(5 2)	1	0	1	(8)	2	0	1	()	3	1	2
0.20.1.1	(0 0 1 1 1 0 0 0 0)	0	0	1	(0 0 1 6)	0	0	1	(1 0)	0	1	2	(6)	1	0	1	()	1	1	6

Table 20. Fixed loci of purely nonsymplectic automorphisms of order 21.

K3 id	$j = 21$				$j = 7$				$j = 3$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.21.0.1	(3 2 1 1 1 3 0 0 0 0)	1	0	1	(4 3 1)	1	1	1	(4)	3	1	3
0.21.0.2	(0 0 1 0 0 1 1 1 1 0 0)	0	0	1	(4 3 1)	1	0	1	(1)	0	1	3
0.21.0.3	(0 0 0 0 0 1 1 1 3 1)	0	0	1	(4 3 1)	1	0	1	(2)	1	1	3
0.21.0.4	(3 3 1 0 0 0 0 1 0 0)	1	0	1	(4 3 1)	1	0	1	(3)	2	1	3

Table 21. Fixed loci of purely nonsymplectic automorphisms of order 22.

K3 id	$j = 22$				$j = 11$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.22.0.1	(0 0 0 1 1 0 0 0 1 2)	0	0	1	(3 2 2 3 1)	1	0	1	()	1	1	5
0.22.0.2	(3 2 2 1 1 0 0 0 0 0)	1	0	1	(3 2 2 3 1)	1	0	1	()	4	1	5
0.22.0.3	(3 2 1 1 1 2 1 0 0 0)	1	0	1	(3 2 2 3 1)	1	0	1	()	5	1	5
0.22.1.1	(0 0 0 1 0 0 0 0 1 4)	0	0	1	(1 0 0 1 0)	0	1	1	()	1	1	10

Table 22. Fixed loci of purely nonsymplectic automorphisms of order 24.

K3 id	$j = 24$				$j = 12$				$j = 8$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.24.0.1	(3 2 1 1 1 1 1 1 0 0 0)	1	0	1	(3 2 2 2 2)	1	0	1	(7 3 2)	2	0	1
0.24.0.2	(0 0 1 0 0 0 1 1 0 0 0)	0	0	1	(0 0 4 1 0)	0	0	1	(3 3 4)	1	0	1
0.24.0.3	(0 0 0 0 0 0 0 3 1 1 2)	0	0	1	(3 3 3 0 0)	1	0	1	(1 1 2)	0	1	1
0.24.0.4	(3 3 1 0 0 0 0 0 0 0 0)	1	0	1	(3 3 3 0 0)	1	0	1	(3 3 4)	1	0	1
0.24.0.5	(0 0 0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 1 1 2)	0	0	1	(5 1 0)	1	0	1
0.24.0.6	(0 0 0 0 0 0 1 1 1 2 2)	0	0	1	(4 1 1 1 0)	1	0	1	(4 2 2)	1	0	1
0.24.0.7	(0 0 0 0 0 2 3 0 0 0 0)	0	0	1	(0 0 0 3 4)	0	0	1	(2 0 0)	0	1	1
0.24.0.8	(0 0 0 1 2 0 0 0 0 0 0)	0	0	1	(0 0 0 3 4)	0	0	1	(0 2 4)	0	0	1
0.24.1.1	(0 0 0 0 1 1 1 1 1 0 0)	0	0	1	(0 1 1 1 2)	0	0	1	(5 1 0)	1	0	1

Table 23. Fixed loci of purely nonsymplectic automorphisms of order 24, part 2.

K3 id	$j = 6$				$j = 4$				$j = 3$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.24.0.1	(5 8)	2	0	1	(10)	3	0	1	(5)	3	1	2	()	6	1	3
0.24.0.2	(1 4)	0	0	1	(6)	1	1	1	(1)	0	1	3	()	4	1	1
0.24.0.3	(3 6)	1	0	1	(6)	1	1	1	(3)	2	1	3	()	4	1	1
0.24.0.4	(3 6)	1	0	1	(6)	1	1	1	(3)	2	1	3	()	4	1	1
0.24.0.5	(5 2)	1	0	1	(6)	1	0	1	(5)	2	0	1	()	2	1	3
0.24.0.6	(5 2)	1	0	1	(8)	2	0	1	(5)	2	0	1	()	3	1	4
0.24.0.7	(3 0)	0	1	1	(4)	0	1	1	(3)	0	1	1	()	0	2	1
0.24.0.8	(3 0)	0	1	1	(4)	0	1	1	(3)	0	1	1	()	0	2	1
0.24.1.1	(1 10)	1	0	1	(6)	1	0	1	(1)	1	1	4	()	2	1	7

Table 24. Fixed loci of purely nonsymplectic automorphisms of order 26.

K3 id	$j = 26$				$j = 13$				$j = 2$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.26.0.1	(0 0 0 1 0 0 0 0 0 1 1 2)	0	0	1	(3 3 2 1 0 0)	1	0	1	()	1	1	6
0.26.0.2	(0 0 0 0 0 0 0 1 2 1 1 2)	0	0	1	(3 3 2 1 0 0)	1	0	1	()	2	1	6
0.26.0.3	(3 2 2 1 0 0 0 0 0 1 0 0)	1	0	1	(3 3 2 1 0 0)	1	0	1	()	4	1	6

Table 25. Fixed loci of purely nonsymplectic automorphisms of order 27.

K3 id	$j = 27$				$j = 9$				$j = 3$			
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.27.0.1	(0 0 0 0 1 0 0 0 0 0 1 3 1)	0	0	1	(0 1 4 1)	0	0	1	(1)	1	1	4
0.27.0.2	(0 0 0 1 1 0 0 0 0 0 1 0 0)	0	0	1	(0 1 4 1)	0	0	1	(1)	0	1	3

Table 26. Fixed loci of purely nonsymplectic automorphisms of order 28.

K3 id	j = 28			j = 14			j = 7			j = 4			j = 2		
	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g
0.28.0.1	(3 2 1 1 1 2 0 0 0 0 0 0)	1	0 1	(3 2 1 1 1 4)	1	0 1	(4 3 1)	1	1 1	(8)	2	0 1	()	5	1 6
0.28.0.2	(0 0 0 1 0 0 2 0 0 0 0 0)	0	0 1	(0 0 0 1 2 4)	0	0 1	(2 1 0)	0	1 1	(0)	0	1 3	()	2	1 3
0.28.0.3	(0 0 0 0 0 0 2 2 1 0 0 0)	0	0 1	(0 0 0 1 2 4)	0	0 1	(2 1 0)	0	1 1	(8)	2	0 1	()	2	1 3

Table 27. Fixed loci of purely nonsymplectic automorphisms of order 30.

K3 id	j = 30			j = 15			j = 10		
	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g
0.30.0.1	(0 0 1 0 0 0 0 0 0 0 0 0)	0	0 1	(0 0 1 2 2 0 0)	0	0 1	(0 3 1 0)	0	0 1
0.30.0.2	(0 0 0 0 0 0 0 1 1 1 1 1 2)	0	0 1	(3 3 1 1 1 1 0 0)	1	0 1	(4 1 1 2)	1	0 1
0.30.0.3	(3 2 1 1 1 1 1 0 0 0 0 0 0)	1	0 1	(3 2 1 1 1 3 1)	1	0 1	(3 3 2 2)	1	0 1
0.30.0.4	(0 0 0 0 1 1 1 0 0 0 0 0 0)	0	0 1	(0 0 0 0 3 3 1)	0	0 1	(0 1 1 4)	0	0 1
0.30.0.5	(0 0 0 1 0 0 0 0 0 0 0 0 1 2)	0	0 1	(3 2 2 3 0 0 0)	1	0 1	(0 0 1 6)	0	0 1
0.30.0.6	(3 2 2 1 0 0 0 0 0 0 0 0 0)	1	0 1	(3 2 2 3 0 0 0)	1	0 1	(3 2 2 4)	1	0 1
0.30.0.7	(0 0 0 0 0 0 1 1 3 0 0 0 0)	0	0 1	(0 0 0 0 3 3 1)	0	0 1	(1 1 1 0 0)	0	1 1
0.30.0.8	(0 0 0 0 2 0 0 0 0 0 1 0 0 0)	0	0 1	(0 0 1 2 2 0 0)	0	0 1	(1 0 0 2)	0	0 1
0.30.1.1	(0 0 0 0 1 1 1 0 0 0 0 0 0)	0	0 1	(0 0 0 0 3 3 1)	0	0 1	(0 1 1 4)	0	0 1
0.30.1.2	(0 0 1 0 0 0 0 0 0 0 0 0 0)	0	0 1	(0 0 1 2 2 0 0)	0	0 1	(0 0 1 6)	0	0 1

Table 28. Fixed loci of purely nonsymplectic automorphisms of order 30, part 2.

K3 id	j = 6			j = 5			j = 3			j = 2		
	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g
0.30.0.1	(0 6)	0	0 1	(7 3)	2	0 1	(2)	0	1 2	()	0	1 4
0.30.0.2	(2 8)	1	0 1	(7 3)	2	0 1	(4)	2	1 2	()	3	1 4
0.30.0.3	(5 8)	2	0 1	(7 3)	2	0 1	(5)	3	1 2	()	5	1 5
0.30.0.4	(6 0)	1	0 1	(3 1)	0	1 1	(6)	3	0 1	()	1	1 4
0.30.0.5	(2 2)	0	0 1	(5 2)	1	1 1	(6)	3	0 1	()	1	1 6
0.30.0.6	(4 4)	1	0 1	(5 2)	1	1 1	(6)	3	0 1	()	4	1 6
0.30.0.7	(1 4)	0	0 1	(3 1)	0	1 1	(1)	1	1 4	()	1	1 1
0.30.0.8	(0 0)	0	1 2	(1 0)	0	1 2	(2)	0	1 2	()	0	1 2
0.30.1.1	(1 10)	1	0 1	(3 1)	0	1 1	(1)	1	1 4	()	1	1 9
0.30.1.2	(0 6)	0	0 1	(1 0)	0	1 2	(2)	0	1 2	()	0	1 10

Table 29. Fixed loci of purely nonsymplectic automorphisms of order 32.

K3 id	j = 32			j = 16		
	(a _i)	k	l g	(a _i)	k	l g
0.32.0.1	(0 0 0 0 0 1 0 0 0 0 0 0 1 2 2)	0	0 1	(4 1 0 0 0 1 0)	1	0 1
0.32.0.2	(0 0 0 0 0 1 1 1 0 0 0 0 1 0 0)	0	0 1	(0 1 0 0 0 1 2)	0	0 1

Table 30. Fixed loci of purely nonsymplectic automorphisms of order 32, part 2.

K3 id	j = 8			j = 4			j = 2		
	(a _i)	k	l g	(a _i)	k	l g	(a _i)	k	l g
0.32.0.1	(5 1 0)	1	0 1	(6)	1	0 1	()	2	1 7
0.32.0.2	(5 1 0)	1	0 1	(6)	1	0 1	()	1	1 6

Table 31. Fixed loci of purely nonsymplectic automorphisms of order 34.

K3 id	$j = 34$			$j = 17$			$j = 2$					
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.34.0.1	(000010011100000000)	0	0	1	(00001231)	0	0	1	()	0	1	8
0.34.0.2	(0000000112100000)	0	0	1	(00001231)	0	0	1	()	1	1	8

Table 32. Fixed loci of purely nonsymplectic automorphisms of order 36.

K3 id	$j = 36$			$j = 18$			$j = 12$					
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.36.0.1	(321111100000001000)	1	0	1	(322111100)	1	0	1	(33112)	1	0	1
0.36.0.2	(000000002111000000)	0	0	1	(00001114)	0	0	1	(42000)	1	0	1
0.36.0.3	(00010000000000100)	0	0	1	(01010000)	0	0	1	(00410)	0	0	1

Table 33. Fixed loci of purely nonsymplectic automorphisms of order 36, part 2.

K3 id	$j = 9$			$j = 6$			$j = 4$			$j = 3$			$j = 2$							
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g				
0.36.0.1	(3331)	1	0	1	(410)	2	0	1	(8)	2	0	1	(4)	3	1	3	()	5	1	6
0.36.0.2	(1110)	0	1	1	(44)	1	0	1	(6)	1	0	1	(4)	1	1	1	()	2	1	7
0.36.0.3	(0141)	0	0	1	(14)	0	0	1	(0)	0	1	3	(1)	0	1	3	()	0	1	3

Table 34. Fixed loci of purely nonsymplectic automorphisms of order 38.

K3 id	$j = 38$			$j = 19$			$j = 2$					
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g
0.38.0.1	(000001100000010000)	0	0	1	(000121100)	0	0	1	()	0	1	9
0.38.0.2	(000000100001210000)	0	0	1	(000121100)	0	0	1	()	1	1	9

Table 35. Fixed loci of purely nonsymplectic automorphisms of order 42.

K3 id	$j = 42$			$j = 21$				
	(a_i)	k	l	g	(a_i)	k	l	g
0.42.0.1	(32111110000000000000)	1	0	1	(3211130000)	1	0	1
0.42.0.2	(0000000001111110000000)	0	0	1	(0000011131)	0	0	1
0.42.0.3	(00000001000000001112)	0	0	1	(3310000100)	1	0	1

Table 36. Fixed loci of purely nonsymplectic automorphisms of order 42, part 2.

K3 id	$j = 14$			$j = 7$			$j = 6$			$j = 3$			$j = 2$							
	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g	(a_i)	k	l	g				
0.42.0.1	(321114)	1	0	1	(431)	1	1	1	(410)	2	0	1	(4)	3	1	3	()	5	1	6
0.42.0.2	(411000)	1	0	1	(431)	1	0	1	(28)	1	0	1	(2)	1	1	3	()	2	1	6
0.42.0.3	(001122)	0	0	1	(431)	1	0	1	(110)	1	0	1	(3)	2	1	3	()	2	1	7

Table 37. Fixed loci of purely nonsymplectic automorphisms of orders 13, 17, 19, 25, 33, 40, 44, 48, 50, 54 and 66.

K3 id	j	(a_i)	k	l	g
0.13.0.1	13	(3 3 2 1 0 0)	1	0	1
0.17.0.1	17	(0 0 0 0 1 2 3 1)	0	0	1
0.19.0.1	19	(0 0 0 1 2 1 1 0 0)	0	0	1
	25	(0 0 0 0 2 1 0 0 0 1 0 0)	0	0	1
0.25.0.1	5	(1 0)	0	1	2
	33	(0 0 0 0 0 1 0 0 0 0 3 1 0 0 0 0)	0	0	1
	11	(1 0 0 1 0)	0	1	1
0.33.0.1	3	(0)	1	1	5
	40	(0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0)	0	0	1
	20	(0 0 1 1 1 0 0 0 0)	0	0	1
	10	(0 0 1 6)	0	0	1
	8	(5 1 0)	1	0	1
	5	(1 0)	0	1	2
	4	(6)	1	0	1
0.40.0.1	2	(0)	1	1	6
	44	(0 0 0 0 0 0 0 0 1 2 0 0 0 0 0 0 1 0 0 0 0)	0	0	1
	22	(0 0 0 1 0 0 0 0 1 4)	0	0	1
	11	(1 0 0 1 0)	0	1	1
	4	(4)	0	0	1
0.44.0.1	2	(0)	1	1	10
	48	(0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0)	0	0	1
	24	(0 0 0 0 1 1 1 1 1 0 0)	0	0	1
	16	(4 1 0 0 0 1 0)	1	0	1
	12	(0 1 1 1 2)	0	0	1
	8	(5 1 0)	1	0	1
	6	(1 1 0)	1	0	1
	4	(6)	1	0	1
	3	(1)	1	1	4
0.48.0.1	2	(0)	2	1	7
	50	(0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0)	0	0	1
	25	(0 0 0 0 2 1 0 0 0 1 0 0)	0	0	1
	10	(0 0 1 6)	0	0	1
	5	(1 0)	0	1	2
0.50.0.1	2	(0)	0	1	10
	54	(0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0)	0	0	1
	27	(0 0 0 0 1 0 0 0 0 0 1 3 1)	0	0	1
	18	(0 0 1 1 1 1 0 0)	0	0	1
	9	(0 1 4 1)	0	0	1
	6	(1 1 0)	1	0	1
0.54.0.1	3	(1)	1	1	4
	2	(0)	1	1	9
	66	(0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0)	0	0	1
	33	(0 0 0 0 0 1 0 0 0 0 3 1 0 0 0 0)	0	0	1
	22	(0 0 0 1 0 0 0 0 1 4)	0	0	1
	11	(1 0 0 1 0)	0	1	1
	6	(0 1 2)	1	0	1
	3	(0)	1	1	5
0.66.0.1	2	(0)	1	1	10

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Competing interest. The authors have no competing interest to declare.

Data availability statement. The source code for HECKE is available at <https://github.com/thofma/Hecke.jl> and the database at [21].

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