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A PROBABILISTIC PROOF OF THE OPEN MAPPING THEOREM FOR ANALYTIC FUNCTIONS

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Abstract

The conformal invariance of Brownian motion is used to give a short proof of the Open Mapping theorem for analytic functions.

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An important staple of the standard complex analysis curriculum is the *Open Mapping theorem*, which is as follows.

Theorem 1. *Let f be a nonconstant analytic function on an open set $W \subseteq \mathbb{C}$. Then $f(W)$ is open.*

The standard proof, contained in virtually any complex analysis textbook, employs contour integration and the argument principle (or, equivalently, Rouché's theorem). In this note, we give a quick and intuitive proof of the theorem which eschews contour integration, utilising instead the conformal invariance of Brownian motion and some simple topology. The following is what is referred to as conformal invariance, although it might also be referred to as *analytic invariance*, since injectivity is not required.

Theorem 2. *Let f be analytic and nonconstant on a domain U, and let $a \in U$. Let B_t be a planar Brownian motion in U starting at a and stopped at a stopping time τ. Then there is a time change $t \longrightarrow C_t$ such that $\hat{B}_t = f(B_{C_t})$ is Brownian motion starting at $f(a)$ and stopped at the stopping time C_τ^{-1}, where $C_s^{-1} = \inf\{t \ge 0 : C_t \ge s\}$.*

See [2, Theorem V.2.5] or [1, Section 2.12] for a proof of this result, which is originally due to Paul Lévy. For the proof of Theorem 1, we need the following lemma.

Lemma 3. *Let $a \in \mathbb{C}$, let B_t be a Brownian motion starting at a, and for any $\delta > 0$ let $\tau_\delta = \inf\{t \ge 0 : B_t \in \{|z - a| = \delta\}\}$. Fix $r > 0$, and let V be an open set contained in $D(a, r)$, where $D(a, r) = \{|z - a| < r\}$. Then $P(B_t \in V$ for some $t \le \tau_r) > 0$.*

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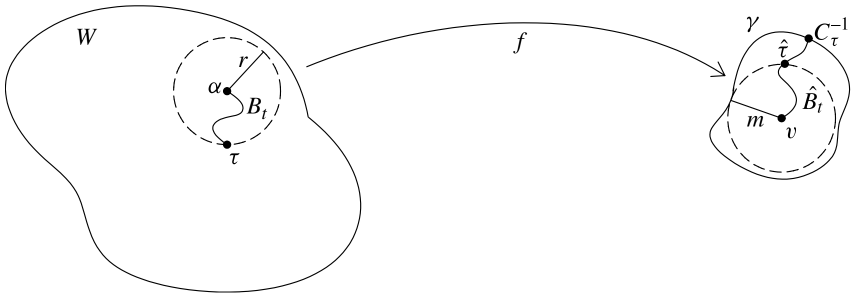

FIGURE 1. The Brownian paths in W and $f(W)$.

PROOF OF LEMMA 3. Since V is open, it contains a circular arc $C = \{a + r'e^{i\theta} : \theta_1 < t < \theta_2\}$ with $0 < r' < r$ and $0 < \theta_2 - \theta_1 \le 2\pi$. We have

$$P(B_t \in V \text{ for some } t \le \tau_r) \ge P(B_t \in C \text{ for some } t \le \tau_r) \ge P(B_{\tau_{r'}} \in C) = \frac{\theta_2 - \theta_1}{2\pi};$$

note that the last equality holds by the rotation invariance of B_t. □

PROOF OF THEOREM 1. Let $v \in f(W)$, and choose $a \in W$ such that $f(a) = v$. By the identity theorem for analytic functions (see [3, Theorem 10.18]), we can choose a small $r > 0$ such that $f(z) \ne v$ on $\{|z - a| = r\}$ and $\bar{D}(a, r) \subseteq W$, where $\bar{D}(a, r) = \{|z - a| \le r\}$. If we start a Brownian motion at a and stop it at the hitting time τ of $\{|z - a| = r\}$, then $\hat{B}_t = f(B_{C_t})$ is a Brownian motion starting at v and stopped at the stopping time C_τ^{-1}, which satisfies $\hat{B}_{C_\tau^{-1}} = f(B_\tau) \in f(\{|z - a| = r\})$. Let γ denote the curve $f(\{|z - a| = r\})$. In travelling from v to γ, $\hat{B}_t$ must cross first the circle $\{|w - v| = m\}$, where $m = \inf\{|w - v| : w \in \gamma\} > 0$; so if we let $\hat{\tau}$ be the first time $\hat{B}_t$ hits $\{|w - v| = m\}$ then $C_\tau^{-1} \ge \hat{\tau}$ almost surely. Figure 1 shows the various points and curves described here.

Lemma 3 now shows that $\hat{B}_t = f(B_{C_t})$ hits any open set in $\{|w - v| < m\}$ with positive probability before time $\hat{\tau}$, and thus before C_τ^{-1}; this shows in particular that $\{|w - v| < m\}$ is contained in the closure of $f(\bar{D}(a, r))$. However, $\bar{D}(a, r)$ is a compact set, so its image under the continuous map f is compact and thus closed, and therefore $\{|w - v| < m\} \subseteq f(\bar{D}(a, r)) \subseteq f(W)$. This proves that $f(W)$ is open. □

REMARK. In Figure 1, γ appears as a Jordan curve encircling v. Of course, if f is not injective in $\{|z - a| \le r\}$ then the curve may have self-intersections; a larger point is that the proof above does not require the fact that γ separates v from ∞, but uses only the fact that the curve does not intersect $\{|w - v| < m\}$. It is also not true in general that C_τ^{-1} must be the hitting time of γ, as the picture suggests, but again the proof above does not require this, using only the fact that $\hat{B}_{C_\tau^{-1}} \in \gamma$.

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