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MAXIMUM MODULUS THEOREMS AND SCHWARZ LEMMATA FOR SEQUENCE SPACES, II

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1. **Introduction.** In this note, we continue the investigations of [3], proving another analogue of the maximum modulus theorem, this time for the sequence space *bv*, and we investigate maximal functions for such theorems. As in [3], we use the notation $f \in MM$ if f is analytic in the disk $|z|<1$, continuous for $|z| \le 1$ and satisfies $|f(z)| \le 1$ on $|z|=1$. We also write $f \in SL$ if $f \in MM$ and $f(0)=0$. Whenever $x=\{x_k\}$ is a sequence of complex numbers, we write $f(x)=\{f(x_k)\}$.

In [3], we proved analogues of the maximum modulus theorem for the sequence spaces s, m and c, and analogues of the Schwarz Lemma for the sequence spaces c_0, l_p and bv_0. We begin this note with the sequence space bv.

2. **The sequence space *bv*.** We write $x \in bv$, the space of sequences of bounded variation, if $x \in c$ and $\|x\|_{bv} = |\lim_{k\to\infty} x_k| + \sum_{k=1}^{\infty} |x_k - x_{k+1}|$ is finite. Note that the usual norm associated with bv is $|x_1| + \sum_{k=1}^{\infty} |x_k - x_{k+1}|$ ([1], p. 239). However, the norm used here is readily shown to be equivalent to the usual norm.

LEMMA 1. (*Compare the Lemma in* [3].) *If* $x \in bv$ *and* $f(z) = z^{p+1} (p \in \mathbb{N})$, *then* $f(x) \in bv$ *and* $\|f(x)\|_{bv} \le f(\|x\|_{bv})$.

Proof. Since $\sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty$, we have that $y_n = \sum_{k=n}^{\infty} |x_k - x_{k+1}| \to 0$ as $n \to \infty$. We also have that $y_n - y_{n+1} = |x_n - x_{n+1}|$ and $y_n \ge |\sum_{k=n}^{\infty} (x_k - x_{k+1})| = |x_n - \lim_{k\to\infty} x_k|$. Thus

$$\begin{aligned}
\|f(x)\|_{bv} - |\lim_{k\to\infty} f(x_k)| &= \sum_{k=1}^{\infty} |f(x_k) - f(x_{k+1})| = \sum_{k=1}^{\infty} |x_k^{p+1} - x_{k+1}^{p+1}| \\
&\le \sum_{k=1}^{\infty} (y_k - y_{k+1}) \sum_{r=0}^{p} \left(y_k + \left|\lim_{n\to\infty} x_n\right| \right)^r \left(y_{k+1} + \left|\lim_{n\to\infty} x_n\right| \right)^{p-r} \\
&= \sum_{k=1}^{\infty} \left(\left(y_k + \left|\lim_{n\to\infty} x\right| \right)^{p+1} - \left(y_{k+1} + \left|\lim_{n\to\infty} x_n\right| \right)^{p+1} \right)
\end{aligned}$$

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$$=\left(y_1+\left|\lim_{n\to\infty} x_n\right|\right)^{p+1}-\left(\left|\lim_{n\to\infty} x_n\right|\right)^{p+1}$$

$$=(\|x\|_{bv})^{p+1}-\left(\lim_{k\to\infty}|x_k|\right)^{p+1}.$$

whence

$$\|f(x)\|_{bv}\le f(\|x\|_{bv}).$$

Suppose that $f(z)=\sum_{n=0}^{\infty} b_n z^n$. If $f\in MM$, the radius of convergence of this series will be at least 1.

THEOREM 1. (*Compare Theorem* 5 *in* [3].) *If* $f\in MM$ *with* $\sum_{n=0}^{\infty}|b_n|\le 1$ *and* $x\in bv$ *with* $\|x\|_{bv}\le 1$, *then* $f(x)\in bv$ *and* $\|f(x)\|_{bv}\le 1$.

Proof. Using the above lemma, it follows that

$$\|f(x)\|_{bv}=\left|\lim_{k\to\infty} f(x)\right|+\sum_{k=1}^{\infty}|f(x_k)-f(x_{k+1})|$$

$$=\left|f\left(\lim_{k\to\infty} x_k\right)\right|+\sum_{k=1}^{\infty}\left|\sum_{n=0}^{\infty} b_n(x_k^n-x_{k+1}^n)\right|$$

$$\le\sum_{n=0}^{\infty}|b_n|\cdot\left(\left|\lim_{k\to\infty} x_k\right|\right)^n+\sum_{n=0}^{\infty}|b_n|\left((\|x\|_{bv})^n-\left(\left|\lim_{k\to\infty} x_k\right|\right)^n\right)$$

$$=\sum_{n=0}^{\infty}|b_n|\cdot(\|x\|_{bv})^n\le\sum_{n=0}^{\infty}|b_n|\le 1.$$

It is worth observing that the proofs of Theorem 5 in [3] and Theorem 1 above, give rise to the inequalities

$$\text{(A)}\qquad \|f(x)\|_{bv_0}\le\|x\|_{bv_0}\cdot\sum_{n=1}^{\infty}|b_n|,$$

and

$$\text{(B)}\qquad \|f(x)\|_{bv}\le\sum_{n=0}^{\infty}|b_n|.$$

Thus, we immediately obtain the following result.

THEOREM 2. (1) *If* $f\in SL$ *with* $\sum_{n=1}^{\infty}|b_n|<\infty$ *and* $x\in bv_0$ *with* $\|x\|_{bv_0}\le 1$ *then* (A) *holds. Further,*

(1.1) *if there is an* x *such that* $\|f(x)\|_{bv_0}=\|x\|_{bv_0}\ne 0$, *then* $\sum_{n=1}^{\infty}|b_n|\ge 1$;

(1.2) *if* $\sum_{n=1}^{\infty}|b_n|<1$, *then* $\|f(x)\|_{bv_0}<\|x\|_{bv_0}$ *for all* $x\in bv_0$;

(1.3) *if* $\sum_{n=1}^{\infty}|b_n|\le 1$ *and there is an* x *such that* $\|f(x)\|_{bv_0}=\|x\|_{bv_0}\ne 0$, *then* $\sum_{n=1}^{\infty}|b_n|=1$.

(2) *If* $f\in MM$ *with* $\sum_{n=0}^{\infty}|b_n|<\infty$ *and* $x\in bv$ *with* $\|x\|_{bv}\le 1$, *then* $f(x)\in bv$ *and* (B) *holds. Further*

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(2.1) *if there is an* x *such that* $\|f(x)\|_{bv}=1$, *then* $\sum_{n=0}^{\infty}|b_n|\geq 1$;
(2.2) *if* $\sum_{n=0}^{\infty}|b_n|<1$, *then* $\|f(x)\|_{bv}<1$ *for all* $x\in bv$;
(2.3) *if* $\sum_{n=0}^{\infty}|b_n|\leq 1$ *and there is an* x *such that* $\|f(x)\|_{bv}=1$, *then* $\sum_{n=0}^{\infty}|b_n|=1$.

3. **The sequence space** bv_0^λ. We write $x\in bv_0^\lambda$, the space of null sequences of bounded variation with index λ $(\lambda>0)$, if $x\in c_0$ and $\|x\|_{bv_0^\lambda}=(\sum_{k=1}^{\infty}|x_k-x_{k+1}|^\lambda)^{1/\lambda}$ is finite.

In this section, we shall make use of Jensen's inequality: $g(\lambda)=(\sum|u_k|^\lambda)^{1/\lambda}$ *is a decreasing function of* λ *for* $\lambda>0$.

We interpret this result in the wide sense in that $g(\lambda)$ may be infinite for some values of λ, but if it is finite for some value of λ, then it is finite for all larger values of λ.

Lemma 2. *If* $x\in bv_0^\lambda$ *with* $0<\lambda\leq 1$ *and if* $f(z)=z^{p+1}(p\in N)$ *then* $f(x)\in bv_0^\lambda$ *and* $\|f(x)\|_{bv_0^\lambda}\leq f(\|x\|_{bv_0^\lambda})$.

Proof. Let $y_n=\sum_{k=n}^{\infty}|x_k-x_{k+1}|^\lambda$, so that $y_n\to 0$ as $n\to\infty$. By Jensen's inequality, $(y_n)^{1/\lambda}=(\sum_{k=n}^{\infty}|x_k-x_{k+1}|^\lambda)^{1/\lambda}\geq\sum_{k=n}^{\infty}|x_k-x_{k+1}|\geq|x_n|$. Also, $|x_n-x_{n+1}|^\lambda=y_n-y_{n+1}$, so that

$$(\|x^{p+1}\|_{bv_0^\lambda})^\lambda=\sum_{k=1}^{\infty}|x_k^{p+1}-x_{k+1}^{p+1}|^\lambda\leq\sum_{k=1}^{\infty}|x_k-x_{k+1}|^\lambda\left(\sum_{r=0}^{p}|x_k|^r|x_{k+1}|^{p-r}\right)^\lambda$$

$$\leq\sum_{k=1}^{\infty}(y_k-y_{k+1})\left(\sum_{r=0}^{p}(y_k)^{r/\lambda}(y_{k+1})^{(p-r)/\lambda}\right)^\lambda\leq\sum_{k=1}^{\infty}(y_k-y_{k+1})\sum_{r=0}^{p}y_k^r y_{k+1}^{p-r}$$

by Jensen's enequality, since $1/\lambda\geq 1$

$$=\sum_{k=1}^{\infty}(y_k^{p+1}-y_{k+1}^{p+1})=y_1^{p+1}=(\|x\|_{bv_0^\lambda})^{(p+1)\lambda},$$

whence $$\|f(x)\|_{bv_0}\leq f(\|x\|_{bv_0}).$$

By using this lemma and the techniques of the proof of Theorem 5 in [3], we can readily prove the following result.

Theorem 3. *If* $x\in bv_0^\lambda$ *with* $0<\lambda\leq 1$ *and* $\|x\|_{bv_0^\lambda}\leq 1$ *and if* $f(z)\in SL$ *with* $\sum_{n=1}^{\infty}|b_n|^\lambda$ *finite, then* $f(x)\in bv_0^\lambda$ *and* $\|f(x)\|_{bv_0^\lambda}\leq(\sum_{n=1}^{\infty}|b_n|^\lambda)^{1/\lambda}\|x\|_{bv_0^\lambda}$. *Further, if* $\sum_{n=1}^{\infty}|b_n|^\lambda\leq 1$, *then* $\|f(x)\|_{bv_0}\leq\|x\|_{bv_0}$.

Other statements, similar to those in Theorem 2 above can be made as well.

For $\lambda>1$, it is not possible to obtain such a result as the following example shows: Let $x_k=(2^\lambda-1)^{1/\lambda}2^{1-k}$ so that $\|x\|_{bv_0^\lambda}=1$; however $\|x^2\|_{bv_0^\lambda}=3(2^\lambda-1)^{2/\lambda}(4^\lambda-1)^{-1/\lambda}>1$.

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4. **The sequence space bv^λ.** We write $x \in bv^\lambda$, the space of sequences of bounded variation with index $\lambda (\lambda > 0)$, if $x \in c$ and

$$\|x\|_{bv^\lambda} = \left(\left| \lim_{k\to\infty} x_k \right|^\lambda + \sum_{k=1}^{\infty} |x_k - x_{k+1}|^\lambda \right)^{1/\lambda}$$

is finite.

In a similar way to that in which Lemma 2 above adapts the proof of the lemma in [3], we can adapt the proof of Lemma 1 above, and the proof of Theorem 1 above, to obtain

LEMMA 3. *If $x \in bv^\lambda$ with $0 < \lambda \le 1$ and if $f(z) = z^{p+1} (p \in N)$ then $f(x) \in bv^\lambda$ and $\|f(x)\|_{bv^\lambda} \le f(\|x\|_{bv^\lambda})$.*

THEOREM 4. *If $x \in bv^\lambda$ with $0 < \lambda \le 1$ and $\|x\|_{bv^\lambda} \le 1$, and if $f(z) \in MM$ with $\sum_{n=0}^\infty |b_n|^\lambda$ finite, then $f(x) \in bv^\lambda$ and*

$$\|f(x)\|_{bv^\lambda} \le \left(\sum_{n=0}^{\infty} |b_n|^\lambda \right)^{1/\lambda}$$

Further, if $\sum_{n=0}^\infty |b_n|^\lambda \le 1$, then $\|f(x)\|_{bv^\lambda} \le 1$.

Again we cannot obtain a similar theorem for $\lambda > 1$; the same example as in §3 *suffices to show this.*

5. **Maximal elements.** We write $f \in \overline{MM}$ if f is analytic in a region containing the closed unit disk and $f \in MM$. If $f \in \overline{MM}$ and $f(0) = 0$, then we write $f \in \overline{SL}$. For $f \in \overline{MM}$, it is well known what the maximal elements are.

PROPOSITION. (See, e.g., [2], p. 129.] *If $f \in \overline{MM}$ and $|f(z)| = 1$ whenever $|z| = 1$, then $f(z) = e^{i\theta} z^\gamma \prod_{k=1}^N (\alpha_k z - \beta_k)/(\overline{\beta_k} z - \overline{\alpha_k})$ where θ is real, γ is a non-negative integer and $|\alpha_k| > |\beta_k| > 0$. (By convention, N is a non-negative integer, and empty products have value* 1.)

If $f \in \overline{MM}$ and $x \in m$ (or c or c_0) with $\|x\|_m = \sup_k |x_k| = 1$ (or $\|x\|_c = \|x\|_m$ or $\|x\|_{c_0} = \|x\|_m$) then it is easy to see that f must have the form as in the above proposition (except that in the case of c_0, where we need $f \in \overline{SL}$, the result demands that $\gamma \ge 1$). For $x \in bv_0$ or $x \in bv$, the result is more interesting.

THEOREM 5. *If $f \in \overline{SL}$ and, for every $x \in bv_0$ with $\|x\|_{bv_0} = 1$, we have $\|f(x)\|_{bv_0} = \|x\|_{bv_0}$, then $f(z) = e^{i\theta} z$ where θ is real.*

Proof. First, consider $x = \{z, 0, 0, 0, \ldots\}$ where $|z| = 1$, so that $\|x\|_{bv_0} = 1$. Thus $\|f(x)\|_{bv_0} = |f(z) - f(0)| = |f(z)| = 1$. From the proposition, we obtain that

$$f(z) = e^{i\theta} z^\gamma \prod_{k=1}^{N} (\alpha_k z - \beta_k)/(\overline{\beta_k} z - \overline{\alpha_k}).$$

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Let

$$X=\{\{0, z/2, 0, 0, 0, \ldots\}, \{z/3, 0, z/3, 0, 0, 0, \ldots\}, \{0, z/4, 0, z/4, 0, 0, 0, \ldots\}, \{z/5, 0, z/5, 0, z/5, 0, 0, 0, \ldots\}, \ldots\} \quad \text{where} \quad |z|=1,$$

Let $x=X_n$, so that $\|x\|_{bv_0}=1=\|f(x)\|_{bv_0}=n\,|f(z/n)|$. Thus

$$1=n^{1-\gamma}\prod_{k=1}^{N}|(\alpha_k z-n\beta_k)/(\overline{\beta_k}z-n\overline{\alpha_k})|.$$

Now

$$\lim_{n\to\infty}\prod_{k=1}^{N}|(\alpha_k z-n\beta_k)/(\overline{\beta_k}z-n\overline{\alpha_k})|=\prod_{k=1}^{N}|\beta_k/\alpha_k|=M.$$

Since $0<M<1$, it follows that $\gamma=1$ and $N=0$, so that $f(z)=e^{i\theta}z$.

THEOREM 6. *If $f\in\overline{MM}$, and for every $x\in bv$ with $\|x\|_{bv}=1$, we have $\|f(x)\|_{bv}=1$, then it follows that*

(a) *if* $f(0)=0$ *then* $f(z)=e^{i\theta}z$,

(b) *if* $f(0)\neq 0$ *then* $f(z)=e^{i\theta}$,

where θ is real.

Proof. (a) If $f(0)=0$, then we follow the proof of Theorem 5 to obtain that $f(z)=e^{i\theta}z$

(b) If $f(0)\neq 0$, we first consider $x=\{z, 0, 0, 0, \ldots\}$ with $|z|=1$, so that $\|x\|_{bv}=1$. Thus

$$f(x)=\{f(z), f(0), f(0), f(0), \ldots\} \quad \text{and} \quad \|f(x)\|_{bv}=|f(0)|+|f(z)-f(0)|=1.$$

If $|f(0)|=1$, then $|f(z)-f(0)|=0$ on $|z|=1$ and the minimum modulus theorem gives that $f(z)=f(0)=e^{i\theta}$.

Suppose hereafter that $0<|f(0)|<1$. Let $F(z)=(f(z)-f(0)/1-|f(0)|)$. Thus $|F(z)|=1$ on $|z|=1$, so that

$$F(z)=e^{i\theta}z^{\gamma}\prod_{k=1}^{N}(\alpha_k z-\beta_k)/(\overline{\beta_k}z-\overline{\alpha_k})$$

and, *a fortiori*, $f(z)=f(0)+\{1-|f(0)|\}e^{i\theta}z^{\gamma}$

$$\prod_{k=1}^{N}(\alpha_k z-\beta_k)/(\overline{\beta_k}z-\overline{\alpha_k}).$$

Define X as in the proof of Theorem 5 and let $x=X_n$, so that $\|x\|_{bv}=1$. Further

$$\|f(x)\|_{bv}=|f(0)|+n\,|f(z/n)-f(0)|=1$$

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so that
$$1=|f(0)|+\{1-|f(0)|\}n^{1-\gamma}\prod_{k=1}^{N}|(\alpha_k z-n\beta_k)/(\overline{\beta_k}z-n\overline{\alpha_k})|.$$

Now
$$\lim_{n\to\infty}\prod_{k=1}^{N}|(\alpha_k z-n\beta_k)/(\overline{\beta_k}z-n\overline{\alpha_k})|=\prod_{k=1}^{N}|\beta_k/\alpha_k|=M.$$

Since $0<M<1$, it follows that $\gamma=1$ and $N=0$. Thus
$$f(z)=f(0)+e^{i\theta}\{1-|f(0)|\}z=p+qz, \text{ say,}$$
where $0<|p|<1$ and $0<|q|<1$.

Now choose any $x\in bv$ with $\|x\|_{bv}=1$ and $\lim_{k\to\infty}x_k=z$ where $|z|=1$. Then
$$\begin{aligned}\|f(x)\|_{bv}&=|f(z)+\sum_{k=1}^{\infty}|q|\cdot|x_k-x_{k+1}|\\&=|p+qz|+|q|\,(\|x\|_{bv}-|z|)=|p+qz|=1.\end{aligned}$$

This is impossible unless either $p=0$, $|q|=1$ or $q=0$, $|p|=1$, both of which are excluded. Hence $f(z)=e^{i\theta}$.

These last two theorems give the answer to the question posed in [3] as to whether $\sum|b_n|\le 1$ is a necessary condition, if we insist that $\|x\|=1$. The answer is yes, but in an unexpected way.

Maximal element theorems for bv_0^λ and bv^λ can be proved in similar ways to those used in Theorems 5 and 6 using
$$Y=\{\{0,z/2^{1/\lambda},0,0,0,\ldots\},\{z/3^{1/\lambda},0,z/3^{1/\lambda},0,0,0,\ldots\}\ldots\}$$
instead of X. However, the proofs will demand, as a necessary condition for the existence of maximal elements, that $\lambda=\gamma$ and since γ is an integer, we must have $\gamma=1$. Thus we obtain Theorem 5 only for $bv_0^1=bv_0$ and Theorem 6 only for $bv^1=bv$.

References

1. Dunford, N., and Schwartz, J. T., *Linear Operators*, Part **I,** Interscience, New York, Fourth Printing, 1967.

2. Conway, J. B., *Functions of One Complex Variable*, Springer-Verlag, New York, Second Printing, 1975.

3. Shawyer, B. L. R., *Maximum Modulus Theorems and Schwarz Lemmata for Sequence Spaces*, Canad. Math. Bull., **18** (1975), 593–596.

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