Proceedings of the Royal Society of Edinburgh, **153**, 344–368, *2023*
DOI:10.1017/prm.2022.4

On finite groups with exactly one vanishing conjugacy class size

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(Received 16 May 2021; accepted 11 January 2022)

Let G be a finite group. An element $g \in G$ is called a vanishing element in G if there exists an irreducible character χ of G such that $\chi(g) = 0$. The size of a conjugacy class of G containing a vanishing element is called a vanishing conjugacy class size of G. In this paper, we give an affirmative answer to the problem raised by Bianchi, Camina, Lewis and Pacifici about the solvability of finite groups with exactly one vanishing conjugacy class size.

Keywords: Conjugacy class sizes; irreducible characters; vanishing elements

2020 *Mathematics subject classification:* 20E45; 20C15; 20D05

1. Introduction

Throughout this paper, G is a finite group, $Z(G)$ is the centre of G and $\mathrm{Fit}(G)$ is the fitting subgroup of G. For $x, y \in G$, $x^y = y^{-1}xy$. For $a \in G$, $o(a)$ is the order of a, $cl_G(a)$ is the conjugacy class in G containing a and $C_G(a)$ denotes the centralizer of a in G. Let $\mathrm{Irr}(G)$ denote the set of the irreducible characters of G. For a normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$, let $I_G(\theta)$ denote the inertia group of θ in G and let $\mathrm{Irr}(G|\theta)$ be the set of the irreducible constituents of the induced character θ^G. An element $g \in G$ is called vanishing in G if there is a character $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$, otherwise, g is non-vanishing in G. We denote by $\mathrm{Van}(G)$ the set of the vanishing elements of G. The size of a conjugacy class of G containing a vanishing element is called a vanishing conjugacy class size of G.

For a prime p, the set of Sylow p-subgroups of G is denoted by $\mathrm{Syl}_p(G)$. Let $\pi(G)$ be the set of prime divisors of $|G|$. For a prime r and natural numbers a and b, $|a|_r$ is the r-part of a, $|a|_{r'} = a/|a|_r$ and, $\gcd(a, b)$ and $\mathrm{lcm}(a, b)$ are the greatest common divisor and the lowest common multiple of a and b, respectively. For integers a and n with $|a| > 1$ and $n \geqslant 1$, the primitive prime divisor of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ and $l \nmid (a^i - 1)$, for $1 \leqslant i < n$. Put $Z_n(a) = \{l : l \text{ is a primitive prime divisor of } a^n - 1\} \cup \{2^m\}$, where if either $n = 1$ and $a \equiv 1 \pmod 4$ or $n = 2$ and $a \equiv -1 \pmod 4$, then $m = 1$. Otherwise, $m = 0$. Note that $Z_n(a) \neq \{1\}$, unless $(a, n) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}$, by [8].

In [2], Bianchi, Camina, Lewis and Pacifici classify the finite super-solvable groups with one vanishing conjugacy class size and put forward a problem on the solvability of the groups with one vanishing conjugacy class size. In this paper, we prove that:

THEOREM 1.1. *If G is a finite group with exactly one vanishing conjugacy class size, then G is solvable.*

In this paper, we say that G satisfies $(*)$ when all vanishing conjugacy classes of G have equal sizes.

2. Some useful lemmas and propositions

For convenience, this section is organized in the following four subsections.

2.1. On the order of elements and Hall subgroups of some finite groups

LEMMA 2.1. *Let $l \geqslant 7$ be an integer and S be a finite non-abelian simple group.*

(i) *Then, there are at least two prime numbers r and t such that $l/2 \leqslant r < t \leqslant l$.*

(ii) *If r and t are as in (i), then Alt_l and Sym_l contains no element of order tr.*

(iii) *For the triple (S, t, r) given in* tables I *and* II, *S contains no element of order tr.*

(iv) *S contains no nilpotent Hall $2'$-subgroup. Moreover, if S is isomorphic to $\mathrm{Suz}, \mathrm{Co}_3$ or Alt_l, then S contains no nilpotent Hall $(\pi(S) - \{2, 3\})$-subgroup.*

Proof. (i) and (iii) follow from [15, lemma 1] and [17], respectively and (ii) is straightforward. Working towards a contradiction, let H be a nilpotent 2-complement of S. Let $t \in \pi(S) - \{2\}$, $T \in \mathrm{Syl}_t(H)$ and $1 \neq x \in Z(T)$. Then, $|cl_S(x)|$ is a power of 2, contradicting Burnside's theorem [11, 15.2]. Obviously, Alt_l contains no nilpotent Hall $(\pi(S) - \{2, 3\})$-subgroup and also, by [4], Suz and Co_3 contain no nilpotent Hall $(\pi(S) - \{2, 3\})$-subgroup. So, (iv) follows. □

LEMMA 2.2 [16, 8.2.8]. *For a prime p, let P be a p-group and Q be a p'-group. If $P \times Q$ acts on a p-group G such that $C_G(P) \leqslant C_G(Q)$, then Q acts trivially on G.*

2.2. The conjugacy classes and centralizers of elements

LEMMA 2.3. *Let N be a normal subgroup of G, $t \in \pi(G)$ and $x, y \in G$.*

(i) *If $gcd(o(x), o(y)) = 1$ and $xy = yx$, then $C_G(xy) = C_G(x) \cap C_G(y) \leqslant C_G(x)$.*

(ii) *$|C_G(x)|_t$ divides $|N|_t |C_{G/N}(xN)|_t$ and $|cl_{G/N}(xN)|$ divides $|cl_G(x)|$.*

https://doi.org/10.1017/prm.2022.4 Published online by Cambridge University Press

https://doi.org/10.1017/prm.2022.4 Published online by Cambridge University Press

Table 1. *Orders of some vanishing elements in finite simple groups of lie type* ($q = p^k$)

$\lvert\mathrm{Out}(S)\rvert$	S	r	m	t
$\gcd(n,\, q-1)\cdot k\cdot 2$	$PSL_n(q)$ $n \geqslant 3$ $(n,\, q) \neq (3,\, 2)$ $(3,\, 4)$, $(6,\, 2)$	p	$p(q^{n-2}-1)/\gcd(n,\, q-1)$	$t \in Z_{nk}(p)$
$\gcd(2,\, q-1)\cdot k$	$PSL_2(q)$ $4 \mid (q-1)$ $q \neq 5$	$r \in Z_{1k}(p)$	$(q-1)/2$	$t \in Z_{2k}(p)$
$\gcd(2,\, q-1)\cdot k$	$PSL_2(q)$ $4 \mid (q+1)$	$r \in Z_{2k}(p)$	$(q+1)/2$	$t \in Z_{1k}(p)$
k	$PSL_2(2^k)$ 2^k-1 is not prime	$r \in Z_{1k}(2)$	2^k-1	$t \in Z_{2k}(2)$
k	$PSL_2(2^k)$ 2^k-1 is prime and $k \neq 2$	$r \in Z_{2k}(2)$	2^k+1	$t \in Z_{1k}(2)$
2	$PSL_6(2)$	2	$2(2^4-1)$	31
$3\cdot 2\cdot 2$	$PSL_3(4)$	2	4	7
$\gcd(n,\, q+1)\cdot 2k$	$PSU_n(q)$ $n \geqslant 3$ $(n,\, q) \neq (3,\, 2)$	p	$p(q^{n-2}-(-1)^{n-2})/\gcd(n,\, q+1)$	$t \in Z_{nk}(-p)$
$n \geqslant 3$ or $2 \nmid q$: $\gcd(2,\, q-1)\cdot k$ otherwise: 2k	$B_n(q)$, $C_n(q)$ $n \geqslant 2$ $(n,\, q) \neq (3,\, 2)$ $(2,\, 2)$	p	$p(q^{n-1}+1)/\gcd(2,\, q-1)$	$t \in Z_{2nk}(p)$

Table 1. *Continued*

$\lvert\mathrm{Out}(S)\rvert$	S	r	m	t
1	$B_3(2) \cong C_3(2)$	2	$2(2^2+1)$	7
$n=4$: $\gcd(2,\, q-1)^2 \cdot k \cdot 6$ $n>4$, even: $\gcd(2,\, q-1)^2 \cdot k \cdot 2$ $n>4$, odd: $\gcd(4,\, q^n-1) \cdot k \cdot 2$	$D_n(q)$ $n \geqslant 4$ $(n,\, q) \neq (4,\, 2)$	p	$p(q^{n-2}+1)/\gcd(4,\, q^n-1)$	$t \in Z_{2(n-1)k}(p)$
6	$D_4(2)$	2	$2(2^2+1)$	7
$\gcd(4,\, q^n+1) \cdot k \cdot 2$	${}^2D_n(q)$ $n \geqslant 4$	p	$p(q^{n-2}+1)/\gcd(4,\, q^n+1)$	$t \in Z_{2nk}(p)$
$p \neq 3$: k $p=3$: $2k$	$G_2(q)$ $q \neq 2$	p	$p(q+1)$	$t \in Z_{6k}(p)$
$\gcd(2,\, q) \cdot k$	$F_4(q)$	p	$p(q+1)$	$t \in Z_{12k}(p)$
$\gcd(3,\, q-1) \cdot k \cdot 2$	$E_6(q)$	p	$p(q+1)$	$t \in Z_{12k}(p)$
$\gcd(3,\, q+1) \cdot k \cdot 2$	${}^2E_6(q)$	p	$p(q+1)$	$t \in Z_{18k}(p)$
$\gcd(2,\, q-1) \cdot k$	$E_7(q)$	p	$p(q+1)$	$t \in Z_{18k}(p)$
k	$E_8(q)$	p	$p(q+1)$	$t \in Z_{30k}(p)$
$3k$	${}^3D_4(q)$	p	$p(q+1)$	$t \in Z_{12k}(p)$
$2n+1$	${}^2B_2(2^{2n+1})$ $n \geqslant 1$	2	4	$t \in Z_{2(2n+1)}(2)$
$2n+1$	${}^2G_2(3^{2n+1})$ $n \geqslant 1$	3	6	$t \in Z_{6(2n+1)}(3)$
$2n+1$	${}^2F_4(2^{2n+1})$ $n \geqslant 1$	2	$2(2^{2n+1}-1)$	$t \in Z_{12(2n+1)}(2)$
2	${}^2F_4(2)'$	2	4	13

Table 2. *Orders of some vanishing elements in some finite simple groups*

$\lvert\mathrm{Out}(S)\rvert$	S	r	m	t	$\lvert\mathrm{Out}(S)\rvert$	S	r	m	t
1	M_{11}	2	6	11	2	M_{12}	3	6	11
1	J_1	2	6	19	2	M_{22}	3	6	11
1	M_{23}	2	6	23	2	J_2	5	15	7
2	J_3	2	6	19	2	HS	5	15	11
1	McL	2	6	11	1	Ru	5	15	29
2	He	2	6	17	1	Co_1	5	15	23
2	O′N	2	6	31	1	M_{24}	5	15	23
1	Co_2	2	6	23	1	BM	5	15	47
2	Fi_{22}	2	6	13	2	Suz	7	21	13
2	HN	2	6	19	1	Co_3	7	21	23
1	Ly	2	6	67					
1	Th	2	6	31					
1	Fi_{23}	2	6	23					
1	J_4	2	6	43					
2	Fi'_{24}	2	6	29					
1	$M = F_1$	2	6	71					

(iii) *If* $\gcd(o(xN), o(yN)) = 1$ *and* $N \neq yN \in C_{G/N}(xN)$, *then there exist* $x_1, y_1 \in G$ *such that* $xN = x_1N$, $o(xN) = o(x_1)$, $yN = y_1N$, $o(yN) = o(y_1)$ *and* $y_1 \in C_G(x_1)$.

(iv) *Let* $\emptyset \neq \pi \subseteq \pi(N)$. *If* $2 \notin \pi$ *and for every* π*-element* $x \in N - Z(N)$, $|cl_G(x)| = m$, *for some integer* m, *then* N *has nilpotent Hall* π*-subgroups.*

(v) *Let* A *and* M *be subgroups of* G. *If* $N \leqslant M$ *and* $\gcd(|A|, |N|) = 1$, *then* $C_{M/N}(AN/N) = C_M(A)N/N$ *and* $|C_M(A)| = |C_N(A)||C_{M/N}(AN/N)|$.

(vi) *Let* A *be a* t'*-group of automorphisms of an abelian* t*-group* T. *Then,* $T = C_T(A) \times [T, A]$.

Proof. The proof of (i) is straightforward. For proving (ii), let $T_1 \in \mathrm{Syl}_t(C_G(x))$. Then, $T_1N/N \leqslant C_G(x)N/N \leqslant C_{G/N}(xN)$. Thus, $|T_1/(T_1 \cap N)| \mid |C_{G/N}(xN)|$. So, $|C_G(x)|_t = |T_1|$ divides $|N|_t|C_{G/N}(xN)|_t$. The remaining claim of (ii) is straightforward. Also, (iii) and (iv) are taken from [**1**, lemma 2.5(iv) and theorem 1.1]. Finally, (v) and (vi) follow from [**3**, lemma 2.7] and [**9**, lemma 5.2.3]. □

The following corollary follows immediately from lemma 2.3(v).

Corollary 2.4. *Let* $\{1\} = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_t = L$ *be a chief series of a finite group* L *such that* L_{t-1} *is a* p*-group, for some prime* p. *If* B *is a subgroup of* L *such that* $p \nmid |B|$, *then* $|C_L(B)| = |C_{L_1}(B)||C_{L_2/L_1}(BL_1/L_1)| \cdots |C_{L_t/L_{t-1}}(BL_{t-1}/L_{t-1})|$.

Lemma 2.5. *Let* $p \in \pi(G)$ *and let* N *be a Hall* p'*-subgroup of* $\mathrm{Fit}(G)$. *If for every* $x \in G - \mathrm{Fit}(G)$, $|cl_G(x)| = m$, *for some positive integer* m, *then for every* $x \in G - \mathrm{Fit}(G)$, $|cl_{G/N}(xN)|_p = |m|_p$.

Proof. Let $x \in G - \mathrm{Fit}(G)$ and P be a p-subgroup of G such that $PN/N \in \mathrm{Syl}_p(C_{G/N}(xN))$. Then, $xN \in C_{G/N}(PN/N)$. By lemma 2.3(v), $C_{G/N}(PN/N) = C_G(P)N/N$. So, $x = yn$, for some $y \in C_G(P)$ and $n \in N$. As, $N \leqslant \mathrm{Fit}(G)$ and $x \notin \mathrm{Fit}(G)$, $y \notin \mathrm{Fit}(G)$. Hence, $|cl_G(y)| = m$. Since $P \leqslant C_G(y)$, $|m|_p = |cl_G(y)|_p \leqslant |G|_p/|P|$. So, $|cl_{G/N}(xN)|_p = |G/N|_p/|C_{G/N}(xN)|_p = |G/N|_p/|PN/N| = |G|_p/|P| \geqslant |m|_p$. By lemma 2.3(ii), $|cl_{G/N}(xN)|_p \leqslant |m|_p$. Hence, $|cl_{G/N}(xN)|_p = |m|_p$. □

LEMMA 2.6. *Suppose that $N \trianglelefteq G$ is a p-group, for some prime p and G/N is a non-abelian simple group such that the order of the Schur multiplier of G/N is not divisible by p. Let $\{1\} = M_0 \leqslant M_1 \leqslant \cdots \leqslant M_t = N \leqslant G$ be a chief series of G. If $M_i/M_{i-1} \leqslant Z(G/M_{i-1})$, for every $i \in \{1, \ldots, t\}$, then $G = N \times L$, for some subgroup L of G.*

Proof. Let i be the smallest number such that $0 \leqslant i \leqslant t$ and $G/M_i = N/M_i \times M/M_i$, for some subgroup $M_i \leqslant M \trianglelefteq G$. Then, $M/M_i \cong G/N$. Working towards a contradiction, let $i > 0$. Then, $(M/M_{i-1})' \nleqslant M_i/M_{i-1}$, because M/M_i is non-abelian. So, $\{M_i/M_{i-1}\} \neq \frac{(M/M_{i-1})' M_i/M_{i-1}}{M_i/M_{i-1}} \trianglelefteq \frac{M/M_{i-1}}{M_i/M_{i-1}} \cong M/M_i \cong G/N$. Thus, $\frac{(M/M_{i-1})' M_i/M_{i-1}}{M_i/M_{i-1}} = \frac{M/M_{i-1}}{M_i/M_{i-1}}$. Since M_i/M_{i-1} is a minimal normal subgroup of G/M_{i-1}, $(M/M_{i-1})' \cap M_i/M_{i-1} = M_i/M_{i-1}$ or $\{M_{i-1}\}$. Therefore, either $(M/M_{i-1})' = M/M_{i-1}$ or $M/M_{i-1} = (M/M_{i-1})' \times M_i/M_{i-1}$. In the former case, since $M_i/M_{i-1} \leqslant Z(M/M_{i-1})$ and $M/M_i \cong G/N$, we get that $|M_i/M_{i-1}|$ divides the order of the Schur multiplier of G/N, a contradiction, because M_i/M_{i-1} is a p-group. In the latter case, regarding $M/M_{i-1} \cap N/M_{i-1} = M_i/M_{i-1}$, we have $(M/M_{i-1})' \cap N/M_{i-1} = (M/M_{i-1})' \cap M_i/M_{i-1} = \{M_{i-1}\}$. Also, $(M/M_{i-1})' \cong G/N$, hence $|G/M_{i-1}| = |G/N||N/M_{i-1}| = |(M/M_{i-1})'||N/M_{i-1}|$. Consequently, $G/M_{i-1} = (M/M_{i-1})' \times N/M_{i-1}$, a contradiction with minimality of i. Therefore, $i = 0$. Now, the lemma follows. □

LEMMA 2.7. *Let N be a normal 3-subgroup of G such that $G/N \cong \mathrm{Alt}_5$. If $P \in \mathrm{Syl}_2(G)$ and M is a minimal normal subgroup of G such that $M \leqslant N$, then:*

(i) *$P = \{1, x, y, xy\}$ such that $o(x) = o(y) = o(xy) = 2$;*

(ii) *$M \leqslant Z(N)$ and M is an elementary abelian 3-group. Also, either $M \leqslant Z(G)$ or $M = C_M(P) \times C_T(x) \times C_T(y) \times C_T(xy)$, where $T = [P, M]$;*

(iii) *$N_G(P)$ contains a 3-element σ such that $\sigma \notin N$, $x^\sigma = y$, $y^\sigma = xy$ and $(xy)^\sigma = x$. In particular, $C_T(x)^\sigma = C_T(y)$, $C_T(y)^\sigma = C_T(xy)$ and $C_T(xy)^\sigma = C_T(x)$;*

(iv) *for every $t \in P - \{1\}$, $u \in P - \{1, t\}$ and $1 \neq n \in C_T(t)$, we have $n^u = n^2$.*

Proof. (i) follows immediately from the facts that $P = P/(P \cap N) \cong PN/N \in \mathrm{Syl}_2(G/N)$ and $G/N \cong \mathrm{Alt}_5$. Since N is a 3-group and $M \trianglelefteq N$, $Z(N) \cap M \neq \{1\}$. Hence, we get from minimality of M that $M \cap Z(N) = M$. Consequently, $M \leqslant Z(N)$. Now, let $M \nleqslant Z(G)$. Since $M \trianglelefteq G$, P acts on M. By lemma 2.3(vi), $M = C_M(P) \times T$, where $T = [M, P]$. If $T = \{1\}$, then $M = C_M(P)$. So $N, P \leqslant C_G(M)$. Therefore, $\{N\} \neq PN/N \leqslant C_G(M)/N \trianglelefteq G/N \cong \mathrm{Alt}_5$. By simplicity of G/N, $C_G(M) = G$, a contradiction with $M \nleqslant Z(G)$. This guarantees that $T \neq \{1\}$.

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We observe that P acts on T by conjugation and $C_T(x) \times C_T(y) \times C_T(xy) \leqslant T$. Taking into account the fact that $\gcd(|P|, |T|) = 1$, Maschke's theorem yields the existence of a P-invariant subgroup T_1 of T such that $T = C_T(x) \times C_T(y) \times C_T(xy) \times T_1$. If $T_1 \neq \{1\}$, then since $(C_T(x) \times C_T(y) \times C_T(xy)) \cap T_1 = \{1\}$, we get that P acts fixed point freely on T_1. Hence, P is cyclic, a contradiction. This shows that $T_1 = \{1\}$ and $M = C_M(P) \times C_T(x) \times C_T(y) \times C_T(xy)$, as needed in (ii).

Since $G/N \cong \mathrm{Alt}_5$ and $PN/N \in \mathrm{Syl}_2(G/N)$, we get that $N_G(P)N/N = N_{G/N}(PN/N)$ is a non-abelian group of order 12. Thus, $N_G(P)$ contains a 3-element σ such that $\sigma \notin N \cup C_G(P)$. Hence, σ permutes the elements of $P - \{1\}$. Without loss of generality, we can assume that $x^\sigma = y$, $y^\sigma = xy$ and $(xy)^\sigma = x$. As $\sigma \in N_G(P)$, we can see $T^\sigma = T$. Hence, (iii) follows.

Finally, suppose that $t \in P - \{1\}$ and $u \in P - \{1, t\}$. Let $1 \neq n \in C_T(t)$. Note that $n^u \in C_{T^u}(t^u) = C_T(t)$ and regarding $o(u) = 2$, $(n^u n)^u = n^u n$. Thus, $n^u n \in C_T(t) \cap C_T(u) = C_T(P) = C_M(P) \cap T = \{1\}$. This gives $n^u n = 1$. Since $T \leqslant M$, $o(n) = 3$. It follows that $n^u = n^2$, as desired in (iv). □

Proposition 2.8. *Suppose that N is a normal subgroup of G which is a 3-group and $G/N \cong \mathrm{Alt}_5$. Let $P \in \mathrm{Syl}_2(G)$, $x_5 \in G - N$ be of order 5 and let $\{1\} = M_0 \leqslant M_1 \leqslant \cdots \leqslant M_t = N \leqslant G$ be a chief series of G. If for every $y \in G - N$, $|cl_G(y)|_3 = 3^e$, for some positive integer e, then for every $1 \neq x \in P$, there is an $1 \leqslant i \leqslant t$ such that $M_i/M_{i-1} \nleqslant Z(G/M_{i-1})$ and $|C_{M_i/M_{i-1}}(x_5M_{i-1})| \geqslant |C_{M_i/M_{i-1}}(xM_{i-1})|$.*

Proof. Set $\mathfrak{A} = \{1 \leqslant i \leqslant t : |C_{M_i/M_{i-1}}(x_5M_{i-1})| \geqslant |C_{M_i/M_{i-1}}(x_2M_{i-1})|\}$, for some $1 \neq x_2 \in P$. Since $G/N \cong \mathrm{Alt}_5$, $|C_{G/N}(x_5N)|_3 = |C_{G/N}(x_2N)|_3 = 1$. So, corollary 2.4 yields that $\mathfrak{A} \neq \emptyset$. Working towards a contradiction, let for every $i \in \mathfrak{A}$, $M_i/M_{i-1} \leqslant Z(G/M_{i-1})$, which gives that $C_{M_i/M_{i-1}}(x_5M_{i-1}) = M_i/M_{i-1} = C_{M_i/M_{i-1}}(x_2M_{i-1})$. If there exists an integer $i \in \{1, \ldots, t\} - \mathfrak{A}$, then $|C_{M_i/M_{i-1}}(x_5M_{i-1})| < |C_{M_i/M_{i-1}}(x_2M_{i-1})|$. Hence, corollary 2.4 forces $|C_G(x_5)|_3 < |C_G(x_2)|_3$, a contradiction. Therefore, $\mathfrak{A} = \{1, \ldots, t\}$. So, for every $i \in \{1, \ldots, t\}$, $M_i/M_{i-1} \leqslant Z(G/M_{i-1})$. By lemma 2.6, $G = M \times N$, where $M \cong \mathrm{Alt}_5$. Let $x_3 \in M$ be of order 3. Then, $x_3 \in G - N$ and $3^e = |cl_G(x_3)|_3 = 1$. Hence, $|cl_G(x_5)|_3 = 1$. By lemma 2.3(v), $3 \mid |C_{G/N}(x_5N)|$, a contradiction, because $G/N \cong \mathrm{Alt}_5$. Thus, there is an $i \in \mathfrak{A}$ such that $M_i/M_{i-1} \nleqslant Z(G/M_{i-1})$, as wanted. □

2.3. The conjugacy class sizes of elements outside a normal subgroup

Let $N \lhd G$ and $G = N \cup (\cup_i H_i)$, where $H_i < G$ are subgroups satisfying $H_i \cap H_j \subseteq N$ when $i \neq j$. Then, G is said to be partitioned relative to N (see [**13**, definition 1]).

Lemma 2.9 [**13**, proposition 4]. *Suppose that $N \lhd G$, G is partitioned relative to N and G/N is abelian. Let p be a prime divisor of $[G : N]$ and a Sylow p-subgroup of G be normal in G. Then, G/N is an elementary abelian p-group.*

Now, we prove proposition 2.10 which is a key tool in the proof of theorem A.

Proposition 2.10. *For an integer $m > 1$, let $G_m = \{g \in G : |cl_G(g)| = m\}$. Let N be a normal subgroup of G and $\overline{G} = G/N$. Suppose that $\overline{x}$ is the image of an element*

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x of G in $\overline{G}$ and r is a divisor of $o(\bar{x})$ such that $o(\bar{x})/r$ is not prime. If for every $\bar{y} \in \langle \bar{x} \rangle$ with $o(\bar{y}) \nmid r$, $y \in G_m$, then for every prime divisor p of $o(\bar{x})/r$, we have:

(i) *the Sylow p-subgroups of N are abelian and $C_G(x)$ contains a Sylow p-subgroup of N;*

(ii) $|N|_p |o(\bar{x})|_p \mid |C_G(x)|$.

Proof. Let p be a prime divisor of $o(\bar{x})/r$ and $P \in \mathrm{Syl}_p(N)$. By the Frattini argument, $G = NN_G(P)$. Thus, $x = nx'$, for some $n \in N$ and $x' \in N_G(P)$. First suppose that $n = 1$. Then, $x \in N_G(P)$. Set $T = \langle P, x \rangle$. For every $y \in T - \langle T \cap N, x^{o(\bar{x})/r} \rangle$, let $C_y = C_T(C_G(y))$. If $z \in T - \langle T \cap N, x^{o(\bar{x})/r} \rangle$, then there exist an element $n \in T \cap N$ and an integer α such that $1 \leqslant \alpha < o(\bar{x})$ and $z = nx^{\alpha}$. Taking into account the facts that every element of $\langle \bar{x} \rangle$ whose order divides r lies in $\langle \bar{x}^{o(\bar{x})/r} \rangle$ and $\bar{x}^{\alpha} = \bar{z} \notin \langle \bar{x}^{o(\bar{x})/r} \rangle$, we get that $o(\bar{x}^{\alpha}) \nmid r$ and the assumption yields that $z \in G_m$. Hence, $T - \langle T \cap N, x^{o(\bar{x})/r} \rangle \subseteq G_m$. Thus, for every $u, v \in T - \langle T \cap N, x^{o(\bar{x})/r} \rangle$, $|C_G(u)| = |C_G(v)|$. If there exists an element $w \in (C_u \cap C_v) - (\langle T \cap N, x^{o(\bar{x})/r} \rangle)$, then $C_G(u), C_G(v) \leqslant C_G(w)$. On the other hand, $w \in T - \langle T \cap N, x^{o(\bar{x})/r} \rangle$. Hence, $w \in G_m$. So, $|C_G(w)| = |C_G(u)| = |C_G(v)|$. Therefore, $C_G(u) = C_G(w) = C_G(v)$. Consequently, $C_u = C_v$. Now, we claim that T is abelian. If not, then for every $y \in T - \langle T \cap N, x^{o(\bar{x})/r} \rangle$, $C_y \neq T$, because $C_y \leqslant Z(C_G(y))$ is abelian. This yields that T is partitioned relative to $\langle T \cap N, x^{o(\bar{x})/r} \rangle$. Since $P \leqslant T \cap N$, $T/\langle T \cap N, x^{o(\bar{x})/r} \rangle \leqslant \langle x \langle T \cap N, x^{o(\bar{x})/r} \rangle \rangle$, which is abelian. Consequently, $T/\langle T \cap N, x^{o(\bar{x})/r} \rangle$ is abelian. Also, $x \in N_G(P)$ and hence, $T \leqslant N_G(P)$. Thus, a Sylow p-subgroup of T is normal in T. By lemma 2.9, $T/\langle T \cap N, x^{o(\bar{x})/r} \rangle$ is an elementary abelian p-group. This forces $o(\bar{x})/r$ to be prime, a contradiction. Therefore, T is abelian. Thus, P is abelian and $x \in C_G(P)$, as desired in (i). Now let $n \neq 1$. Set $T_1 = \langle P, x' \rangle$ and $H = \langle x', N \rangle$. Since $\bar{x} = \bar{x}'$, substituting x with x' in the above argument shows that T_1 is abelian. Also, T_1 contains a Sylow p-subgroup of H. Hence, the Sylow p-subgroups of H are abelian. On the other hand, $x = x'n$ for some $n \in N$. Thus, $H = \langle x, N \rangle$. Note that $p \mid o(\bar{x})$. Let x_p be the p-part of x. Then, $C_G(x_p)$ contains a Sylow p-subgroup of N. By our assumption, $x, x_p \in G_m$ and since, $C_G(x) \leqslant C_G(x_p)$, we have $C_G(x) = C_G(x_p)$. Therefore, $C_G(x)$ contains a Sylow p-subgroup of N. So, (i) follows. By (i), there exists a $P \in \mathrm{Syl}_p(N)$ such that $T = \langle P, x \rangle$ is abelian. Hence, $T \leqslant C_G(x)$. Therefore, $|T|_p = |N|_p |o(\bar{x})|_p \mid |C_G(x)|$, as desired in (ii). □

2.4. Vanishing and non-vanishing elements

A $\chi \in \mathrm{Irr}(G)$ is said to have q-defect zero for some prime q, if $q \nmid |G|/\chi(1)$.

Lemma 2.11 [**10**, corollary 2]. *Let G be non-abelian simple and $q \in \pi(G)$. Then, G has an irreducible character of q-defect zero unless one of the following holds*:

(a) *the prime q is 2 and G is isomorphic to* M_{12}, M_{22}, M_{24}, J_2, HS, Ru, Co_1 *or* BM.

(b) $q \in \{2, 3\}$ *and G is isomorphic to* Suz, Co_3 *or* Alt_n *for some* $n \geqslant 7$.

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LEMMA 2.12. *Let N be a normal subgroup of G and $p \in \pi(N)$. Suppose that $N \cong S_1 \times \cdots \times S_l$, where every S_i is isomorphic to the non-abelian simple group S.*

(a) *If S has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element of G.*

(b) *Let the triple (S, r, t) be as in* lemma 2.1(i), *and* tables I *and* II. *If $u \in \{t, r\}$, then for every $x \in N$ with $u \mid o(x)$, $x \in \mathrm{Van}(G)$.*

Proof. By our assumption, every S_i has an irreducible character θ_i of p-defect zero, because $S_i \cong S$. Thus, $\theta_1 \times \cdots \times \theta_l \in \mathrm{Irr}(N)$ is of p-defect zero. So, (a) follows from [**7**, lemma 2.7]. Also, (b) can be concluded from lemma 2.11 and (a). □

In lemma 2.13, we have brought some known results:

LEMMA 2.13. *Let M and N be normal subgroups of G, $g \in G$ and let p be a prime.*

(i) [**5**] *If $g \notin \mathrm{Fit}(G)$ is non-vanishing in G, then* $\gcd(6, o(g\mathrm{Fit}(G))) \neq 1$.

(ii) [**7**, lemma 2.1] *If $gN \in \mathrm{Van}(G/N)$, then $gN \subseteq \mathrm{Van}(G)$.*

(iii) [**7**, proposition 2.5] *If $M \cap N = \{1\}$, then $(\mathrm{Van}(G) \cap M)N \subseteq \mathrm{Van}(G)$.*

(iv) [**7**, lemma 2.4] *Let $M \leqslant N \leqslant G$ such that $\gcd(|M|, |N/M|) = 1$ and M is nilpotent. If $C_N(M) \leqslant M$ and N/M is abelian, then $N - M \subseteq \mathrm{Van}(G)$.*

(v) [**7**, lemma 5.1] *If $N \neq \{1\}$ is a p-group and $G/N \cong \mathrm{Alt}_7$, then there are distinct primes $q_1, q_2 \in \pi(\mathrm{Alt}_7) - \{p\}$ and q_i-elements $x_iN \in \mathrm{Van}(G/N)$ such that $C_N(x_1), C_N(x_2) \neq \{1\}$.*

(vi) [**14**, lemma 2.3] *Let x be a non-vanishing element in G. Then, x fixes some member of each orbit of the action of G on $\mathrm{Irr}(N)$.*

Now, we follow the ideas in the proof of [**5**, theorem A] to prove a new fact about non-vanishing elements of a group lying in its normal solvable subgroup:

PROPOSITION 2.14. *Let $N \neq \{1\}$ be a normal solvable subgroup of G and, let $x \in N$ be non-vanishing in G. If $o(x\mathrm{Fit}(G))$ is odd in $G/\mathrm{Fit}(G)$, then $x \in \mathrm{Fit}(G)$.*

Proof. The proof is by induction on $|G|$. For every non-trivial normal subgroup M of G, NM/M is normal in G/M and since $NM/M \cong N/(N \cap M)$, NM/M is solvable. By induction, for every $\{1\} \neq M \trianglelefteq G$, we get that $xM \in \mathrm{Fit}(G/M)$. Now as mentioned in the proof of [**5**, theorem A], one of the following cases occurs:

Case 1. Let $M_1 \neq M_2$ be minimal normal subgroups of G. Then, the function $\phi : G \to \hat{G} = G/M_1 \times G/M_2$, defined by $\phi(g) = (gM_1, gM_2)$ for $g \in G$, is an injective homomorphism. By induction, $\phi(x) \in \mathrm{Fit}(G/M_1) \times \mathrm{Fit}(G/M_2) = \mathrm{Fit}(\hat{G})$. So, $\phi(x) \in \phi(G) \cap \mathrm{Fit}(\hat{G}) \leqslant \mathrm{Fit}(\phi(G))$. Since ϕ induces an isomorphism between G and $\phi(G)$, we get that $x \in \mathrm{Fit}(G)$, as wanted.

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Case 2. Assume that G has the unique minimal normal subgroup M. By our assumption on N, $M \leqslant N$. Hence, $M \leqslant \mathrm{Fit}(G)$. Let $\Phi(G)$ denote the Frattini subgroup of G. If $\Phi(G) \neq \{1\}$, then $x\Phi(G) \in \mathrm{Fit}(G/\Phi(G))$, by induction. However, $\mathrm{Fit}(G/\Phi(G)) = \mathrm{Fit}(G)/\Phi(G)$. So, $x \in \mathrm{Fit}(G)$, as wanted. Now let $\Phi(G) = \{1\}$ and $x \notin M$. By [**12**, III, lemma 4.4], M has a complement as H in G, because $M \trianglelefteq G$ is abelian. Since $C_H(M) \trianglelefteq G$, the uniqueness of M forces $C_H(M) = \{1\}$. So, $C_G(M) = M$. Let V be the group of the irreducible characters of M. We can check that V is a faithful and irreducible G/M-module. On the other hand, by lemma 2.13(vi), xM fixes some element of each orbit of G/M on V and by induction, $xM \in \mathrm{Fit}(G/M)$. So, [**14**, theorem 4.2] forces $x^2 \in M$. Since $M \leqslant \mathrm{Fit}(G)$, $x^2 \in \mathrm{Fit}(G)$. However, $o(x\mathrm{Fit}(G))$ is odd. Hence, $x \in \mathrm{Fit(G)}$, as desired. □

Lemma 2.15. *Let $N \trianglelefteq G$ be a p-group, for some prime p and G/N be non-abelian simple. If M is a minimal normal subgroup of G such that $M \leqslant N$ and $\chi \in \mathrm{Irr}(M) - \{1_M\}$, then (i) $M \leqslant Z(N)$ and (ii) if $I_G(\chi) = G$, then $M \leqslant Z(G)$.*

Proof. Since $M \trianglelefteq N$ and N is a p-group, $\{1\} \neq M \cap Z(N) \trianglelefteq G$. As M is a minimal normal subgroup of G, $M \cap Z(N) = M$. So, (i) follows. If $I_G(\chi) = G$, then it is easy to see that M is a cyclic group of order p. By (i), $M \leqslant Z(N)$. Therefore, $\frac{G/N}{C_G(M)/N} \cong G/C_G(M) = N_G(M)/C_G(M) \lesssim \mathrm{Aut}(M)$ is cyclic. Hence, $G/N = C_G(M)/N$. Consequently, $C_G(M) = G$, so $M \leqslant Z(G)$, as wanted in (ii). □

Proposition 2.16. *Let N be a normal subgroup of G which is a 7-group and $G/N \cong \mathrm{Alt}_7$. If M is a minimal normal subgroup of G such that $M \leqslant N$ and $M \nleqslant Z(G)$, then for every $x \in G - N$ of order 3 or 6, $x \in \mathrm{Van}(G)$.*

Proof. Let $P \in \mathrm{Syl}_7(G)$ and $1_M = \lambda_1, \ldots, \lambda_t$ be the representatives of the action of P on $\mathrm{Irr}(M)$. If $\mathcal{O}_i$ is the P-orbit of λ_i, then $1 + \Sigma_{i=2}^t |\mathcal{O}_i|\lambda_i(1)^2 = \Sigma_{\lambda \in \mathrm{Irr}(M)}\lambda(1)^2 = |M| \equiv_7 0$. Thus, there exists an $i > 1$ such that $7 \nmid |\mathcal{O}_i| = [P : I_P(\lambda_i)]$. Therefore, $|\mathcal{O}_i| = 1$ and hence $P \leqslant I_G(\lambda_i)$. On the other hand, $M \leqslant Z(N)$, by lemma 2.15(i). So, $N \leqslant I_G(\lambda_i)$. This yields that $\{N\} \neq PN/N \leqslant I_G(\lambda_i)/N \leqslant G/N \cong \mathrm{Alt}_7$. Lemma 2.15(ii) shows that $I_G(\lambda_i)/N < G/N \cong \mathrm{Alt}_7$. Note that the only maximal subgroup of Alt_7 whose order is divisible by 7 is isomorphic to $PSL_2(7)$. This signifies that

$$I_G(\lambda_i)/N \text{ is isomorphic to a subgroup of } PSL_2(7). \tag{2.1}$$

So, $I_G(\lambda_i)/N$ does not contain any element of order 6 and neither does $I_G(\lambda_i)$. It follows from lemma 2.13(vi) that every element of G of order 6 is vanishing in G.

Now, let ϕ be a group isomorphism from G/N to Alt_7. It is known that Alt_7 contains exactly two conjugacy classes containing 3-elements. Let $\phi(x_3N)$ and $\phi(y_3N)$ be the representatives of these classes, for some x_3N, $y_3N \in G/N$ of orders 3. Since N is a 7-group, we can assume that $o(x_3) = o(y_3) = 3$. By [**4**], we can assume that $\phi(x_3N) \in \mathrm{Van}(\mathrm{Alt}_7)$, $\phi(x_3N)$ normalizes some Sylow 7-subgroup of Alt_7 and $\phi(y_3N)$ does not normalize any Sylow 7-subgroup of Alt_7. So,

$$x_3N \in \mathrm{Van}(G/N), \tag{2.2}$$

x_3N normalizes some Sylow 7-subgroup of G/N and y_3N does not normalize any Sylow 7-subgroup of G/N. By (2.1), $I_G(\lambda_i)/N$ is isomorphic to a subgroup of

$PSL_2(7)$. However, $PSL_2(7)$ has only one conjugacy class containing 3-elements and every element of this class normalizes some Sylow 7-subgroup of $PSL_2(7)$. Note that a Sylow 7-subgroup of $I_G(\lambda_i)/N$ is a Sylow 7-subgroup of G/N. This shows that no conjugate of y_3N lies in $I_G(\lambda_i)/N$ and if $I_G(\lambda_i)/N$ contains a 3-element uN, then $uN \in cl_{G/N}(x_3N)$. This yields that no conjugate of y_3 lies in $I_G(\lambda_i)$. Thus, for every 3-element $w \in G$, either $wN \in cl_{G/N}(x_3N)$ or no conjugate of w lies in $I_G(\lambda_i)$. In the former case, (2.2) and lemma 2.13(ii) show that $w \in \mathrm{Van}(G)$. In the latter case, $w \in \mathrm{Van}(G)$, by lemma 2.13(vi). Now, the proposition follows. □

PROPOSITION 2.17. *Suppose that N is a normal 3-subgroup of G such that $G/N \cong \mathrm{Alt}_5$. Let $Q \in \mathrm{Syl}_5(G)$ and M be a minimal normal subgroup of G such that $M \leqslant N$. If $M \nleqslant Z(G)$, then one of the following holds*:

(i) *there exist an element $1 \neq n \in C_M(Q)$ and a character $\psi \in \mathrm{Irr}(G)$ such that $\psi(n) = 0$;*

(ii) *$N_G(Q)$ contains a non-trivial 2-element x such that $|C_M(Q)| < |C_M(x)|$.*

Proof. Since $|Q| = 5$, there is an element $x_5 \in G - N$ such that $o(x_5) = 5$ and $Q = \langle x_5\rangle$. Also, regarding $G/N \cong \mathrm{Alt}_5$, we get that $N_{G/N}(QN/N) = N_G(Q)N/N$ is a dihedral group of order 10. Thus, $N_G(Q)$ contains an element x such that $o(x) = 2$ and $x \notin N \cup C_G(Q)$. Let $P \in \mathrm{Syl}_2(G)$ such that $x \in P$. By lemma 2.7 (i,iii), $P = \{1, x, y, xy\}$ such that $o(y) = o(xy) = 2$ and there is a 3-element $\sigma \in N_G(P) - N$ such that $x^\sigma = y,\ \ y^\sigma = xy,\ \ (xy)^\sigma = x$. Put $\bar{G} = G/N$ and for every $H \leqslant G$ and $g \in G$, let $\bar{H} = HN/N$ and $\bar{g}$ denote the image of g in $\bar{G}$. As $\bar{G} \cong \mathrm{Alt}_5$, we observe that

$$\bar{G} = \langle \bar{x}_5\rangle N_{\bar{G}}(\bar{P}) = \langle \bar{x}_5\rangle \bar{P}\langle\bar{\sigma}\rangle. \tag{2.3}$$

Since $\bar{G} \cong \mathrm{Alt}_5$, $(2\ 5)(3\ 4) \in N_{\mathrm{Alt}_5}(\langle(1\ 2\ 3\ 4\ 5)\rangle)$, $U = \langle(2\ 5)(3\ 4), (2\ 3)(4\ 5)\rangle \in \mathrm{Syl}_2(\mathrm{Alt}_5)$ and $(2\ 3\ 4) \in N_{\mathrm{Alt}_5}(U)$, there exists a group isomorphism ϕ from $\bar{G}$ to Alt_5 which sends $\bar{x}_5$ to $(1\ 2\ 3\ 4\ 5)$, $\bar{x}$ to $(2\ 5)(3\ 4)$, $\bar{y}$ to $(2\ 3)(4\ 5)$ and $\bar{\sigma}$ to $(2\ 3\ 4)$. Let $\bar{u} \in \langle\bar{x}_5\rangle\bar{P}$. If $o(\bar{u}) = t \in \{2, 3, 5\}$, then $o(\phi(\bar{u})) = t$. So, considering the t-elements of Alt_5 lying in $\phi(\langle\bar{x}_5\rangle\bar{P})$ shows that if $t = 2$, then $\phi(\bar{u}) \in N_{\mathrm{Alt}_5}(\langle(1\ 2\ 3\ 4\ 5)\rangle) - \langle(1\ 2\ 3\ 4\ 5)\rangle = N_{\phi(\bar{G})}(\langle\phi(\bar{x}_5)\rangle) - \langle\phi(\bar{x}_5)\rangle$ or $\phi(\bar{u}) \in \{(2\ 3)(4\ 5), (2\ 4)(3\ 5)\} = \{\phi(\bar{y}), \phi(\bar{x}\bar{y})\}$, if $t = 3$, then $\phi(\bar{u}) \in \{(1\,2\,4), (1\,5\,3), (1\,3\,2), (1\,4\,5)\}$ and if $t = 5$, then $\phi(\bar{u}) \in \langle(1\,2\,3\,4\,5)\rangle = \langle\phi(\bar{x}_5)\rangle$ or $\phi(\bar{u}) \in \{(1\,3\,4\,2\,5), (1\,4\,3\,5\,2), (1\,2\,5\,4\,3), (1\,5\,2\,3\,4)\}$. Thus, if $t = 3$, then $\phi(\bar{u}^{-1}) \in \{(1\ 4\ 2) = \phi(\bar{x}_5^3\bar{x}\bar{\sigma}^2), (1\ 3\ 5) = \phi(\bar{x}_5^2\bar{y}\bar{\sigma}^2), (1\ 2\ 3) = \phi(\bar{x}_5\bar{y}\bar{\sigma}), (1\ 5\ 4) = \phi(\bar{x}_5^4\bar{\sigma})\}$. Hence, $\bar{u}^{-1} \in \{\bar{x}_5^3\bar{x}\bar{\sigma}^2, \bar{x}_5^2\bar{y}\bar{\sigma}^2, \bar{x}_5\bar{y}\bar{\sigma}, \bar{x}_5^4\bar{\sigma}\}$. Consequently, $\bar{u}^{-1} \notin \langle\bar{x}_5\rangle\bar{P}$. Similarly, if $t = 5$, then either $\bar{u}^{-1} \notin \langle\bar{x}_5\rangle\bar{P}$ or $\bar{u} \in \langle\bar{x}_5\rangle$. In addition, we get that

$$\begin{aligned} &\text{if } \ \bar{u}, \bar{u}^{-1} \in \langle\bar{x}_5\rangle\bar{P}, \text{ then either } o(\bar{u}) = 5 \text{ and } \bar{u} \in \langle\bar{x}_5\rangle \\ &\text{or } o(\bar{u}) = 2 \text{ and } \bar{u} \in \{\bar{x}_5^i\bar{x}, \bar{y}, \bar{x}\bar{y} : 1 \leqslant i \leqslant 5\}. \end{aligned} \tag{2.4}$$

On the other hand, by lemma 2.7(ii,iii),

$$M = C_M(P) \times C_T(x) \times C_T(y) \times C_T(xy), \tag{2.5}$$

$$C_T(x)^\sigma = C_T(y), \quad C_T(y)^\sigma = C_T(xy),\ C_T(xy)^\sigma = C_T(x), \tag{2.6}$$

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where $T=[M, P]$. If $C_T(x)=\{1\}$, then (2.6) shows that $C_T(y)=C_T(xy)=\{1\}$. Thus, $M=C_M(P)$. So, $P\leqslant C_G(M)$. However, $N\leqslant C_G(M)$, by lemma 2.15(i). Hence, $\{\bar{1}\}\neq\bar{P}\leqslant C_G(M)/N\trianglelefteq G/N=\phi^{-1}(\mathrm{Alt}_5)$. Therefore, $C_G(M)=G$, because G/N is simple. Consequently, $M\leqslant Z(G)$, a contradiction. Thus, $C_T(x)\neq\{1\}$. By our assumption, M is an elementary abelian 3-group, so is $C_T(x)$. Hence, there exist subgroups $A_1,\ldots,A_t$ of $C_T(x)$ such that $|A_1|=\cdots=|A_t|=3$ and

$$C_T(x)=A_1\times\cdots\times A_t,\quad C_T(y)=B_1\times\cdots\times B_t,\ C_T(xy)=C_1\times\cdots\times C_t,\tag{2.7}$$

where $B_i=A_i^{\sigma}$ and $C_i=A_i^{\sigma^2}$, for every $1\leqslant i\leqslant t$, by (2.6). Let $n\in M$. By (2.5),

$$n=n_1n_2n_3n_4,\tag{2.8}$$

where $n_1\in C_M(P)$, $n_2\in C_T(x)$, $n_3\in C_T(y)$ and $n_4\in C_T(xy)$. Also, by (2.7),

$$\text{for every } j\in\{2,3,4\},\quad n_j=n_{j1}\ldots n_{jt},\tag{2.9}$$

where for every $1\leqslant i\leqslant t$, $n_{2i}\in A_i$, $n_{3i}\in B_i$ and $n_{4i}\in C_i$.

If $C_M(x_5)=\{1\}$, then $|C_M(x)|\geqslant|C_T(x)|>1=|C_M(x_5)|$. So, (ii) follows. Next, let $1\neq n\in C_M(x_5)$. For every $1\leqslant i\leqslant 5$, (2.8) and lemma 2.7(iii,iv) yield that

$$n^{x_5^i}=n;\ n^{x_5^ix}=n_1n_2n_3^2n_4^2;\ n^{x_5^iy}=n_1n_2^2n_3n_4^2;\ n^{x_5^ixy}=n_1n_2^2n_3^2n_4;\tag{2.10}$$

$$n^{x_5^i\sigma}=n_1^{\sigma}n_4^{\sigma}n_2^{\sigma}n_3^{\sigma};\ n^{x_5^ix\sigma}=n_1^{\sigma}(n_4^2)^{\sigma}n_2^{\sigma}(n_3^2)^{\sigma};\ n^{x_5^iy\sigma}=n_1^{\sigma}(n_4^2)^{\sigma}(n_2^2)^{\sigma}(n_3)^{\sigma};$$

$$n^{x_5^ixy\sigma}=n_1^{\sigma}n_4^{\sigma}(n_2^2)^{\sigma}(n_3^2)^{\sigma};\ n^{x_5^i\sigma^2}=n_1^{\sigma^2}n_3^{\sigma^2}n_4^{\sigma^2}n_2^{\sigma^2};\ n^{x_5^ix\sigma^2}=n_1(n_3^2)^{\sigma^2}(n_4^2)^{\sigma^2}(n_2)^{\sigma^2};$$

$$n^{x_5^iy\sigma^2}=n_1(n_3)^{\sigma^2}(n_4^2)^{\sigma^2}(n_2^2)^{\sigma^2};\ n^{x_5^ixy\sigma^2}=n_1(n_3^2)^{\sigma^2}(n_4)^{\sigma^2}(n_2^2)^{\sigma^2}.$$

Let check one of the above equalities in details. For instance, $n^{x_5^ixy\sigma}=(x_5^ixy\sigma)^{-1}n(x_5^ixy\sigma)=((xy)^{-1}((x_5^i)^{-1}nx_5^i)xy)^{\sigma}=((xy)^{-1}nxy)^{\sigma}=(n_1n_2^2n_3^2n_4)^{\sigma}$ $=n_1^{\sigma}n_4^{\sigma}(n_2^2)^{\sigma}(n_3^2)^{\sigma}$. Similarly, we can check the other ones. Note that

$$\text{for every } i\in\{2,3,4\},\quad n_i^{\sigma}=n_{i1}^{\sigma}n_{i2}^{\sigma}\ldots n_{it}^{\sigma},\tag{2.11}$$

by (2.7). We continue the proof in the following cases:

Case 1. Assume that $n_2=n_3=n_4=1$. Then, $n=n_1\in C_M(P)$. Regarding the facts that $n\in C_M(x_5)$ and $M\leqslant Z(N)$, we have $P,\langle x_5\rangle, N\leqslant G_G(n)$. Therefore, $\bar{P},\langle\bar{x}_5\rangle\leqslant C_G(n)/N\leqslant G/N=\phi^{-1}(\mathrm{Alt}_5)$. Since the only subgroup of Alt_5 whose order is divisible by 20 is Alt_5, we get that $C_G(n)/N=G/N$. Thus, $C_G(n)=G$. Consequently, $n\in Z(G)$. Therefore, $M=\langle n\rangle\leqslant Z(G)$, a contradiction.

Case 2. Assume that $n_{2i}\neq 1$, for some $1\leqslant i\leqslant t$. Without loss of generality, let $i=1$. Since $x\in N_G(\langle x_5\rangle)$ and $n\in C_M(x_5)$, we have $n^x\in C_M(\langle x_5\rangle)^x=C_M(\langle x_5\rangle)=C_M(x_5)$. By lemma 2.7(iv), $n_3^x=n_3^2$ and $n_4^x=n_4^2$. Thus, $(n^x)_{31}=n_{31}^2$ and $(n^x)_{41}=n_{41}^2$. Also, $n_2^x=n_2$, because $n_2\in C_T(x)$. Hence, $(nn^x)_{21}=(n_{21})^2\neq 1$, $(nn^x)_{31}=n_{31}(n_{31})^2=1$ and $(nn^x)_{41}=n_{41}(n_{41})^2=1$, because M is an elementary abelian 3-group and $n_{21}, n_{31}, n_{41}\in M$. As $1\neq nn^x\in C_M(x_5)$, by substituting n with nn^x, we can assume that $n_{31}=n_{41}=1$. Set $\chi=1_{C_M(P)}\times 1_{C_T(x)}\times(\theta_3\times 1_{B_2}\times\cdots\times 1_{B_t})\times(\theta_4\times 1_{C_2}\times\cdots\times 1_{C_t})$, where $\theta_3\in\mathrm{Irr}(B_1)-\{1_{B_1}\}$, $\theta_4\in\mathrm{Irr}(C_1)-\{1_{C_1}\}$

and $\theta_4(m^\sigma) = \theta_3(m^2)$, for every $m \in B_1$. Then, $1_M \neq \chi \in \mathrm{Irr}(M)$. As $M \leqslant Z(N)$, $N \leqslant I_G(\chi)$. Let $u \in I_G(\chi) - N$. Then $\bar{u} \in \bar{G}$. By (2.3), $\bar{u} = \bar{x}_5^i \bar{x}^j \bar{y}^k \bar{\sigma}^l$, for some non-negative integers i, j, k and l. Working towards a contradiction, let $l \neq 0$. Since $o(\phi(\bar{\sigma})) = 3$, $o(\bar{\sigma}) = 3$ and hence, we can assume that $l \in \{1, 2\}$. Also, $j, k \in \{0, 1\}$. Regarding $u \in I_G(\chi)$, $u^{-1} \in I_G(\chi)$. Therefore, $\chi^{u^{-1}}(n) = \chi(n)$. Also, $n \in M \leqslant Z(N)$. By (2.10), $\chi^{u^{-1}}(n) = \chi(u^{-1}nu) = \chi(\sigma^{-l}(n_1 n_2^{2^k} n_3^{2^j} n_4^{2^{|j-k|}})\sigma^l)$, so

$$\chi^{u^{-1}}(n) = \begin{cases} \chi(n_1^\sigma (n_4^{2^{|j-k|}})^\sigma (n_2^{2^k})^\sigma (n_3^{2^j})^\sigma), & \text{if } l = 1 \\ \chi(n_1^{\sigma^2} (n_3^{2^j})^{\sigma^2} (n_4^{2^{|j-k|}})^{\sigma^2} (n_2^{2^k})^{\sigma^2}), & \text{if } l = 2 \end{cases}. \tag{2.12}$$

It follows that either $l = 1$ and $\chi^{u^{-1}}(n) = \theta_3((n_{21}^{2^k})^\sigma)\theta_4((n_{31}^{2^j})^\sigma)$ or $l = 2$ and $\chi^{u^{-1}}(n) = \theta_3((n_{41}^{2^{|j-k|}})^{\sigma^2})\theta_4((n_{21}^{2^k})^{\sigma^2})$. Since $n_{31} = n_{41} = 1$, $n_{21} \neq 1$ and $\theta_4((n_{21}^{2^k})^{\sigma^2}) = \theta_3(((n_{21}^{2^k})^\sigma)^2)$, we have $\chi^{u^{-1}}(n) = \theta_3((n_{21}^{2^k})^\sigma)^l \neq 1$. However, $\chi(n) = \theta_3(n_{31})\theta_4(n_{41}) = 1$, a contradiction. This forces $\bar{u} \in \langle \bar{x}_5 \rangle \bar{P}$. Consequently,

$$I_G(\chi)/N \subseteq \langle \bar{x}_5 \rangle \bar{P}. \tag{2.13}$$

Also, for $1 \neq \gamma \in B_1$ and $1 \neq \beta \in C_1$, $\chi^{xy}(\gamma) = \chi^x(\gamma) = \chi(\gamma^2) = \theta_3(\gamma^2) \neq \theta_3(\gamma) = \chi(\gamma)$ and $\chi^y(\beta) = \chi(\beta^2) = \theta_4(\beta^2) \neq \theta_4(\beta) = \chi(\beta)$. Therefore,

$$x, y, xy \notin I_G(\chi). \tag{2.14}$$

Now, assume that $u \in I_G(\chi)$. Then, $\bar{u}, \bar{u}^{-1} \in I_G(\chi)/N$. So, in view of (2.4), (2.13) and (2.14), one of the following sub-cases holds:

Sub-case a. Assume that $o(\bar{u}) = 2$. If 4 or $5 \mid |I_G(\chi)/N|$, then taking the elements mentioned in (2.4) into account, we conclude that $I_G(\chi)/N = \bar{P}$ or $I_G(\chi)/N = N_{\bar{G}}(\langle \bar{x}_5 \rangle) = \langle \bar{x}_5 \rangle \langle \bar{x} \rangle$, contradicting (2.14). Thus, $|I_G(\chi)/N| = o(\bar{u}) = 2$ and $\bar{u} \in \{\bar{x}_5^i \bar{x} : 1 \leqslant i \leqslant 5\}$. So, $\mathcal{B} = \{(x_5^j y^l \sigma^k)^{-1} : 1 \leqslant j \leqslant 5, 0 \leqslant l \leqslant 1 \text{ and } 0 \leqslant k \leqslant 2\}$ is a transversal set of $I_G(\chi)$ in G. Hence, for every $\psi \in \mathrm{Irr}(G|\chi)$, $\psi(n) = e\Sigma_{g \in \mathcal{B}} \chi(n^{g^{-1}}) = e\Sigma_{i=1}^5 [\chi(n^{(x_5^i)}) + \chi(n^{(x_5^i \sigma)}) + \chi(n^{(x_5^i \sigma^2)})] + e\Sigma_{i=1}^5 [\chi(n^{(x_5^i y)}) + \chi(n^{(x_5^i y \sigma)}) + \chi(n^{(x_5^i y \sigma^2)})]$, for some positive integer e. By (2.10), $\psi(n) = e\Sigma_{i=1}^5 [\chi(n) + \chi(n_1^\sigma n_4^\sigma n_2^\sigma n_3^\sigma) + \chi(n_1^{\sigma^2} n_3^{\sigma^2} n_4^{\sigma^2} n_2^{\sigma^2})] + e\Sigma_{i=1}^5 [\chi(n_1 n_2^2 n_3 n_4^2) + \chi(n_1 (n_4^2)^\sigma (n_2^2)^\sigma n_3^\sigma) + \chi(n_1 n_3^{\sigma^2} (n_4^2)^{\sigma^2} (n_2^2)^{\sigma^2})]$. Thus, $\psi(n) = e\Sigma_{i=1}^5 [1 + \theta_3((n_{21})^\sigma)\theta_4((n_{31})^\sigma) + \theta_3((n_{41})^{\sigma^2})\theta_4((n_{21})^{\sigma^2})] + e\Sigma_{i=1}^5 [\theta_3(n_{31})\theta_4(n_{41}^2) + \theta_3((n_{21}^2)^\sigma)\theta_4((n_{31})^\sigma) + \theta_3((n_{41}^2)^{\sigma^2})\theta_4((n_{21}^2)^{\sigma^2})] = 5e[2(1 + \theta_3((n_{21})^\sigma) + \theta_3(((n_{21})^\sigma)^2))]$. Note that $n_{21} \neq 1$. Therefore, $\theta_3((n_{21})^\sigma) \neq 1$ is a primitive 3rd root of unitary. Hence,

$$\theta_3((n_{21})^\sigma)^2 + \theta_3((n_{21})^\sigma) + 1 = 0 \tag{2.15}$$

It follows that $\psi(n) = 0$, as wanted in (i).

Sub-case b. Assume that $I_G(\chi)/N \leqslant \langle \bar{x}_5 \rangle$. So, either $I_G(\chi)/N = \langle \bar{x}_5 \rangle$ or $I_G(\chi) = N$. If $I_G(\chi)/N = \langle \bar{x}_5 \rangle$, let $b = 1$ and if $I_G(\chi) = N$, let $b = 5$. So, $\mathcal{B} = \{(x_5^i x^j y^k \sigma^l)^{-1} : 0 \leqslant i \leqslant b-1,\ j, k \in \{0, 1\},\ l \in \{0, 1, 2\}\}$ is a transversal set of $I_G(\chi)$ in G. Hence, for every $\psi \in \mathrm{Irr}(G|\chi)$, $\psi(n) = e\Sigma_{g \in \mathcal{B}} \chi^g(n) = e\Sigma_{i=0}^{b-1} [\chi(n^{x_5^i}) + \chi(n^{x_5^i \sigma}) + \chi(n^{x_5^i \sigma^2})] + e\Sigma_{i=0}^{b-1} [\chi(n^{x_5^i x}) + \chi(n^{x_5^i x \sigma}) + \chi(n^{x_5^i x \sigma^2})] + e\Sigma_{i=0}^{b-1} [\chi(n^{x_5^i y}) +$

$\chi(n^{x_5^i y\sigma}) + \chi(n^{x_5^i y\sigma^2})] + e\Sigma_{i=0}^{b-1}[\chi(n^{x_5^i xy}) + \chi(n^{x_5^i xy\sigma}) + \chi(n^{x_5^i xy\sigma^2})]$, for some positive integer e. By (2.10) and (2.15), we can check at once that $\psi(n) = 4be[1 + \theta_3(n_{21}^\sigma) + \theta_3(n_{21}^\sigma)^2] = 0$, as wanted in (i).

Case 3. Assume that $n_2 = 1$ and there exists an $1 \leqslant i \leqslant t$ such that $n_{3i} \neq 1$ and $n_{4i} = 1$. Without loss of generality, let $i = 1$ and set $\chi = 1_{C_M(P)} \times (\theta_2 \times 1_{A_2} \times \cdots \times 1_{A_t}) \times 1_{C_T(y)} \times (\theta_4 \times 1_{C_2} \times \cdots \times 1_{C_t})$, where $\theta_2 \in \mathrm{Irr}(A_1) - \{1_{A_1}\}$, $\theta_4 \in \mathrm{Irr}(C_1) - \{1_{C_1}\}$ and $\theta_4(m^{\sigma^2}) = \theta_2(m^2)$, for every $m \in A_1$. Also, if $n_2 = 1$ and there exists an $1 \leqslant i \leqslant t$ such that $n_{4i} \neq 1$ and $n_{3i} = 1$, then without loss of generality, let $i = 1$ and set $\chi = 1_{C_M(P)} \times (\theta_2 \times 1_{A_2} \times \cdots \times 1_{A_t}) \times (\theta_3 \times 1_{B_2} \times \cdots \times 1_{B_t}) \times 1_{C_T(xy)}$, where $\theta_2 \in \mathrm{Irr}(A_1) - \{1_{A_1}\}$, $\theta_3 \in \mathrm{Irr}(B_1) - \{1_{B_1}\}$ and $\theta_3(m^\sigma) = \theta_2(m^2)$, for every $m \in A_1$. Then, $1_M \neq \chi \in \mathrm{Irr}(M)$ and arguing by analogy as Case 2 shows that for every $\psi \in \mathrm{Irr}(G|\chi)$, $\psi(n) = 0$, as wanted in (i).

Case 4. Assume that $C_M(x_5)$ does not contain any element satisfying cases 1–3. Let α, $\beta \in C_M(x_5)$. By (2.8), $\alpha = \alpha_1\alpha_2\alpha_3\alpha_4$ and $\beta = \beta_1\beta_2\beta_3\beta_4$, where α_1, $\beta_1 \in C_M(P)$, α_2, $\beta_2 \in C_T(x)$, α_3, $\beta_3 \in C_T(y)$ and α_4, $\beta_4 \in C_T(xy)$ are uniquely determined. (2.9) shows that for every $j \in \{2, 3, 4\}$, $\alpha_j = \alpha_{j1} \ldots \alpha_{jt}$ and $\beta_j = \beta_{j1} \ldots \beta_{jt}$, where for every $1 \leqslant i \leqslant t$, α_{2i}, $\beta_{2i} \in A_i$, α_{3i}, $\beta_{3i} \in B_i$ and α_{4i}, $\beta_{4i} \in C_i$. By our assumption, $\alpha_2 = \beta_2 = 1$, α_3, β_3, α_4, $\beta_4 \neq 1$ and for every $1 \leqslant i \leqslant t$, $\alpha_{3i} \neq 1$ if and only if $\alpha_{4i} \neq 1$. Also, $\beta_{3i} \neq 1$ if and only if $\beta_{4i} \neq 1$. If $\alpha_{3i} = \beta_{3i} \neq 1$ and $\alpha_{4i} = \beta_{4i}^2 \neq 1$, for some $1 \leqslant i \leqslant t$, then $(\alpha\beta)_{3i} = \alpha_{3i}\beta_{3i} = \alpha_{3i}^2 \neq 1$ and $(\alpha\beta)_{4i} = \alpha_{4i}\beta_{4i} = \alpha_{4i}^3 = 1$. However, $\alpha\beta \in C_M(x_5)$. So, $\alpha\beta$ satisfies the assumption of case 3, a contradiction. This shows that for an element $\alpha \in C_M(x_5)$ and an integer $1 \leqslant i \leqslant t$, if $\alpha_{3i} \neq 1$, then $\alpha_{4i} \neq 1$ and

$$\text{for every } \beta \in C_M(x_5), \quad (\beta_{3i}, \beta_{4i}) \in \{(1, 1), (\alpha_{3i}, \alpha_{4i}), (\alpha_{3i}^2, \alpha_{4i}^2)\}. \tag{2.16}$$

Now, working towards a contradiction, let $\alpha_1 \neq 1$. Then, since $x \in N_G(\langle x_5\rangle)$, $\alpha^x \in C_M(x_5)^x = C_M(x_5)$. By lemma 2.7(iv), $\alpha^x = \alpha_1\alpha_2\alpha_3^2\alpha_4^2$. Thus, $\alpha\alpha^x = \alpha_1^2\alpha_2^2\alpha_3^3\alpha_4^3$. Note that $\alpha_2 = 1$ and $o(\alpha_3) = o(\alpha_4) = 3$. Therefore, $1 \neq \alpha\alpha^x = \alpha_1^2 \in C_M(P)$. On the other hand, α, $\alpha^x \in C_M(x_5)$. So, $1 \neq \alpha^x\alpha = \alpha_1^2 \in C_M(x_5) \cap C_M(P)$, which is a contradiction with case 1. This shows that for every $\alpha \in C_M(x_5)$, $\alpha_1 = 1$. It follows from (2.16) that $|C_M(x_5)| \leqslant |C_T(y)|$. If $C_M(P) \neq \{1\}$, then we get that $|C_M(x_5)| < |C_M(P)||C_T(y)| = |C_M(P)||C_T(x)| = |C_M(x)|$, so (ii) follows. Next, let $C_M(P) = \{1\}$. Then, $T = M = C_M(x) \times C_M(y) \times C_M(xy)$ and $|C_M(x_5)| \leqslant |C_T(y)| = |C_M(y)| = |C_M(x)|$, by (2.16). If $|C_M(x_5)| < |C_M(x)|$, then (ii) follows. Otherwise, $|C_M(x_5)| = |C_M(x)|$. If $|C_M(x)| = 3$, then $|M| = 27$ and $|[M, \langle x_5\rangle]| = 9$. However, $\langle x_5\rangle$ acts fixed point freely on $[M, \langle x_5\rangle]$. So, $5 \mid |[M, \langle x_5\rangle]| - 1 = 8$, which is impossible. This forces $|C_M(x)| \geqslant 9$. Consequently, $t \geqslant 2$ (t was fixed in (2.7)). Since $|C_M(x_5)| = |C_M(x)| = |C_M(y)|$, we get from (2.16) that for every $1 \leqslant i \leqslant t$ and $m \in B_i$, there exists an element $\alpha \in C_M(x_5)$ such that $\alpha_{3i} = m$. So, for $1 \neq n \in C_M(x_5)$, we can assume that n_{31}, $n_{32} \neq 1$. Consequently, n_{41}, $n_{42} \neq 1$. As was mentioned above, $n_2 = 1$. Set $\chi = 1_{C_M(x)} \times (\theta_3 \times \theta_3' \times 1_{B_3} \times \cdots \times 1_{B_t}) \times (\theta_4 \times \theta_4' \times 1_{C_3} \times \cdots \times 1_{C_t})$, where $\theta_3 \in \mathrm{Irr}(B_1) - \{1_{B_1}\}$, $\theta_3' \in \mathrm{Irr}(B_2) - \{1_{B_2}\}$, $\theta_4 \in \mathrm{Irr}(C_1) - \{1_{C_1}\}$ and $\theta_4' \in \mathrm{Irr}(C_2) - \{1_{C_2}\}$ such that $\theta_3'(n_{32}) = \theta_3(n_{31})^2$ and $\theta_4'(n_{42}) = \theta_4(n_{41})^2$. Moreover, suppose that $\theta_4(n_{41}) = \theta_3(n_{31})$. Then, $1_M \neq \chi \in \mathrm{Irr}(M)$. Note that n_{31} and n_{32} are generators of B_1 and B_2, respectively. Also, n_{41} and n_{42} are generators of C_1 and C_2, respectively.

In the following, we first assume that $(n_{31}^{\sigma}, n_{32}^{\sigma}) = (n_{41}, n_{42})$ or $(n_{31}^{\sigma}, n_{32}^{\sigma}) = (n_{41}^2, n_{42}^2)$. If $(n_{31}^{\sigma}, n_{32}^{\sigma}) = (n_{41}, n_{42})$, let $u = 1$ and otherwise let $u = 2$. We note that $N \leqslant I_G(\chi)$. By (2.10), we can check that for every $g \in G$, $\chi(n^g) = 1$, for instance, $\chi(n^{x_5^i x\sigma^2}) = \chi((n_3^2)^{\sigma^2}(n_4^2)^{\sigma^2} n_2^{\sigma^2}) = [\theta_3((n_{41}^2)^{\sigma^2})\theta_3'((n_{42}^2)^{\sigma^2})][\theta_4(1)\theta_4'(1)] = \theta_3(n_{31})^{6u} = 1$. So, for every $\psi \in \mathrm{Irr}(G|\chi)$, $\psi(n) = \psi(1)$. Therefore, $1 \neq n \in \ker\psi \cap M$. Thus, $\{1\} \neq M \cap \ker\psi \trianglelefteq G$. Since M is a minimal normal subgroup of G, $M \cap \ker\psi = M$. Therefore, $\psi_M = \psi(1)1_M$, a contradiction.

Next, suppose that $(n_{31}^{\sigma}, n_{32}^{\sigma}) \in \{(n_{41}, n_{42}^2), (n_{41}^2, n_{42})\}$. If $u = x_5^i x^j y^k \sigma^l h \in I_G(\chi)$, where $h \in N$, $i \in \{1, \ldots, 5\}$, $j, k \in \{0, 1\}$ and $l \in \{1, 2\}$, then $u^{-1} \in I_G(\chi)$ and we get from (2.12) that if $l = 1$, then $\chi^{u^{-1}}(n) \in \{(\theta_4(n_{41})^2)^{2^j}, (\theta_4(n_{41}))^{2^j}\}$ and if $l = 2$, then $\chi^{u^{-1}}(n) \in \{(\theta_3(n_{31})^2)^{2^{|j-k|}}, (\theta_3(n_{31}))^{2^{|j-k|}}\}$. Hence, $\chi^{u^{-1}}(n) \neq 1$. However, $\chi(n) = 1$, a contradiction. Consequently, $\bar{u} \in \langle \bar{x}_5 \rangle \bar{P}$. Therefore, $I_G(\chi)/N \subseteq \langle \bar{x}_5 \rangle \bar{P}$. Let $\beta = n_{41} \in C_1 - \{1\}$. By lemma 2.7(iv), $\chi(\beta) = \theta_4(n_{41})$, $\chi^{x^{-1}}(\beta) = \chi(\beta^x) = \chi(\beta^2) = \theta_4(n_{41})^2$ and $\chi^{y^{-1}}(\beta) = \chi(\beta^y) = \chi(\beta^2) = \theta_4(n_{41})^2$. Since $n_{41} \neq 1$, $\theta_4(n_{41})^2 \neq \theta_4(n_{41})$. Consequently, $\chi^{x^{-1}}, \chi^{y^{-1}} \neq \chi$. Hence, $x, y \notin I_G(\chi)$. By (2.16) and since $|C_M(y)| = |C_M(x_5)|$, we can assume that there exists an element $\alpha \in C_M(x_5)$ such that $\alpha_2 = 1$, $\alpha_{31} = n_{31} \neq 1$ and for every $j \in \{2, \ldots, t\}$, $\alpha_{3j} = 1$. Then, (2.16) guarantees that $\alpha_{41} = n_{41} \neq 1$ and for every $j \in \{2, \ldots, t\}$, $\alpha_{4j} = 1$. However, $\chi(\alpha) = \theta_3(\alpha_{31})\theta_4(\alpha_{41}) = \theta_3(n_{31})\theta_4(n_{41}) = \theta_3(n_{31})^2$ and $\chi^{(x_5^i x)^{-1}}(\alpha) = \chi(\alpha^{x_5^i x}) = \chi(\alpha_2\alpha_3^2\alpha_4^2) = \theta_3(\alpha_{31}^2)\theta_4(\alpha_{41}^2) = \theta_3(n_{31})^4 = \theta_3(n_{31})$. Since $\theta_3(n_{31}) \neq 1$, $\chi^{(x_5^i x)^{-1}}(\alpha) \neq \chi(\alpha)$. This shows that $x_5^i x \notin I_G(\chi)$. Taking the elements mentioned in (2.4) into account, we conclude that $I_G(\chi)/N \leqslant \langle \bar{x}_5 \rangle$. So, either $I_G(\chi)/N = \langle \bar{x}_5 \rangle$ or $I_G(\chi) = N$. If $I_G(\chi)/N = \langle \bar{x}_5 \rangle$, let $b = 1$ and if $I_G(\chi) = N$, let $b = 5$. So, $\mathcal{B} = \{(x_5^i x^j y^k \sigma^l)^{-1} : 0 \leqslant i \leqslant b - 1, j, k \in \{0, 1\}, l \in \{0, 1, 2\}\}$ is a transversal set of $I_G(\chi)$ in G. Hence, for every $\psi \in \mathrm{Irr}(G|\chi)$, $\psi(n) = e\Sigma_{g\in\mathcal{B}}\chi^g(n) = e\Sigma_{i=0}^{b-1}[\chi(n^{x_5^i}) + \chi(n^{x_5^i\sigma}) + \chi(n^{x_5^i\sigma^2})] + e\Sigma_{i=0}^{b-1}[\chi(n^{x_5^i x}) + \chi(n^{x_5^i x\sigma}) + \chi(n^{x_5^i x\sigma^2})] + e\Sigma_{i=0}^{b-1}[\chi(n^{x_5^i y}) + \chi(n^{x_5^i y\sigma}) + \chi(n^{x_5^i y\sigma^2})] + e\Sigma_{i=0}^{b-1}[\chi(n^{x_5^i xy}) + \chi(n^{x_5^i xy\sigma}) + \chi(n^{x_5^i xy\sigma^2})]$, for some positive integer e. We note that $\chi(n_2n_3n_4) = \chi(n_2n_3^2n_4^2) = \chi(n_2^2n_3n_4^2) = \chi(n_2^2n_3^2n_4) = 1$. Thus, if $(n_{31}^{\sigma}, n_{32}^{\sigma}) = (n_{41}, n_{42}^2)$, then by (2.10), $\psi(n) = e\Sigma_{i=0}^{b-1}[1 + \theta_3(n_{31})^2 + \theta_3(n_{31})^2] + e\Sigma_{i=0}^{b-1}[1 + \theta_3(n_{31}) + \theta_3(n_{31})] + e\Sigma_{i=0}^{b-1}[1 + \theta_3(n_{31}) + \theta_3(n_{31})^2] + e\Sigma_{i=0}^{b-1}[1 + \theta_3(n_{31})^2 + \theta_3(n_{31})] = 4be(1 + \theta_3(n_{31}) + \theta_3(n_{31})^2)$. Also, if $(n_{31}^{\sigma}, n_{32}^{\sigma}) = (n_{41}^2, n_{42})$, then similarly $\psi(n) = 4be(1 + \theta_3(n_{31}) + \theta_3(n_{31})^2)$. However, $n_{31} \neq 1$. So, $o(\theta_3(n_{31})) = 3$. Therefore, $1 \neq \theta_3(n_{31})$ is a primitive third root of unitary. It follows that $(\theta_3(n_{31}))^2 + \theta_3(n_{31}) + 1 = 0$. Thus, we get that $\psi(n) = 4be(1 + \theta_3(n_{31}) + \theta_3(n_{31})^2) = 0$, as desired in (i). Now, the proof is complete. □

3. Proof of theorem a

Now, we are going to prove theorem A. Working towards a contradiction, suppose that G is non-solvable. Let M be the maximal normal solvable subgroup of G and let N/M be a minimal normal subgroup of G/M. Then, $\mathrm{Fit}(G) = \mathrm{Fit}(M)$ and $N/M = S_1/M \times \cdots \times S_t/M$ such that $S_1/M, \ldots, S_t/M$ are isomorphic to a fixed non-abelian simple group S. Let α be the size of vanishing classes of G. Set $G_\alpha = \{g \in G : |cl_G(g)| = \alpha\}$. By $(*)$, $\mathrm{Van}(G) \subseteq G_\alpha$. We are going to complete the proof in the following steps:

Step 1. $t = 1$.

Proof. Working towards a contradiction, let $t \neq 1$. Let $\pi_1 \subseteq \pi(S)$ such that if S is one of the groups mentioned in lemma 2.11(b), then $\pi_1 = \{2, 3\}$ and otherwise, $\pi_1 = \{2\}$. Fix $\pi = \pi(S) - \pi_1$. Let $p, q \in \pi$ be distinct. Suppose that $i, j \in \{1, \ldots, t\}$ and $i \neq j$. Then, for every non-trivial p-element $xM \in S_i/M$ and q-element $yM \in S_j/M$, xM, yM, $xyM \in \mathrm{Van}(G/M)$, by lemmas 2.11 and 2.12(a). Thus, xM, yM, $xyM \subseteq \mathrm{Van}(G) \subseteq G_\alpha$, by lemma 2.13(ii). So, for every $m \in M$, proposition 2.10 shows that $C_G(xym)$ contains a Sylow p-subgroup and a Sylow q-subgroup of M, which are abelian. Let $P \in \mathrm{Syl}_p(\mathrm{Fit}(G))$ and $Q \in \mathrm{Syl}_q(\mathrm{Fit}(G))$. Since $\mathrm{Fit}(G) \leqslant M$, we get that

$$P, Q \leqslant C_G(xym), \quad \text{for every } m \in M. \tag{3.1}$$

Thus, $P, Q \leqslant Z(M)$. Let F_0 be a Hall π-subgroup of $\mathrm{Fit}(G)$. Since p is an arbitrary element of π, we get that $F_0 \leqslant Z(M)$. By lemma 2.3(iii), there exist a p-element $x_1 \in N - M$, a q-element $y_1 \in N - M$ and $m_1, m_1', m'' \in M$ such that $x_1y_1 = y_1x_1$, $xm_1 = x_1$, $ym_1' = y_1$ and $xym'' = x_1y_1$. Lemma 2.3(i) and (3.1) yield that $P, Q \leqslant C_G(xym'') = C_G(x_1y_1) = C_G(x_1) \cap C_G(y_1)$. So $P, Q \leqslant C_G(x), C_G(y)$, because $P, Q \leqslant Z(M)$, $xm_1 = x_1$ and $ym_1' = y_1$. However, $p, q \in \pi$ and $i, j \in \{1, \ldots, t\}$ are arbitrary. Thus, for every $r \in \pi$ and every r-element $z \in N - M$, $F_0 \leqslant C_G(z)$. Hence, lemma 2.3(i) forces the π-elements of $N - M$ to centralize F_0. Next, let zM be a π_1-element of S_i/M. By lemma 2.3(iii), there exist a q-element $y_2 \in N - M$, a π_1-element $z_1 \in N - M$ and $u, u' \in M$ such that $zu = z_1$, $yu' = y_2$ and $y_2z_1 = z_1y_2$. By lemmas 2.11, 2.12(a) and 2.13(ii), $z_1y_2, y_2 \in \mathrm{Van}(G) \subseteq G_\alpha$. Therefore, $C_G(y_2) = C_G(y_2z_1) \leqslant C_G(z_1)$, by lemma 2.3(i). Consequently, $F_0 \leqslant C_G(z_1)$. However, $F_0 \leqslant Z(M)$ and $zu = z_1$. So, $F_0 \leqslant C_G(z)$. Since $i \in \{1, \ldots, t\}$ is arbitrary, we have that π_1-elements of $N - M$ centralize F_0. Thus, $F_0 \leqslant Z(N)$. On the other hand, every π-element $w \in N - \mathrm{Fit}(G)$ is vanishing in G, by lemma 2.12(a) and proposition 2.14. Therefore, $|cl_G(w)| = \alpha$. It follows from lemma 2.3(iv) that N contains a nilpotent Hall π-subgroup, so does N/M, contradicting lemma 2.1(iv). □

Step 2. $C_{G/M}(N/M) = \{M\}$.

Proof. Working towards a contradiction, let $C_{G/M}(N/M) \neq \{M\}$ and let C/M be a minimal normal subgroup of G/M such that $C/M \leqslant C_{G/M}(N/M)$. By step 1, N/M and C/M are isomorphic to the simple groups S_1 and S_2, respectively. Let $p \in \pi(S_1) - \pi_1$ and $p \neq q \in \pi(S_2) - \pi_2$, where for $i \in \{1, 2\}$, if S_i is one of the groups mentioned in lemma 2.11(b), then $\pi_i = \{2, 3\}$ and otherwise, $\pi_i = \{2\}$. If $\pi_1 \cup \pi_2 = \{2, 3\}$, then without loss of generality, we can assume that $\pi_1 = \{2, 3\}$. Set $\pi = \pi(N/M) - \pi_1$. Let $M \neq xM \in N/M$ be a p-element and $M \neq yM \in C/M$ be a q-element. Then, lemmas 2.11, 2.12(a) and 2.13(ii) guarantee that xM, yM, $xyM \subseteq \mathrm{Van}(G) \subseteq G_\alpha$. Thus, for every $m \in M$, proposition 2.10 shows that $C_G(xym)$ contains an abelian Sylow p-subgroup and an abelian Sylow

q-subgroup of M. Let $P \in \mathrm{Syl}_p(\mathrm{Fit}(G))$ and $Q \in \mathrm{Syl}_q(\mathrm{Fit}(G))$. Since $\mathrm{Fit}(G) \leqslant M$, we get that

$$P, Q \leqslant C_G(xym), \quad \text{for every } m \in M. \tag{3.2}$$

Thus, P, $Q \leqslant Z(M)$. Let F_0 be a Hall π-subgroup of $\mathrm{Fit}(G)$. Since p is an arbitrary element of π, we get that $F_0 \leqslant Z(M)$. By lemma 2.3(iii), there exist a p-element $x_1 \in N - M$, a q-element $y_1 \in C - M$ and m_1, m_1', $m'' \in M$ such that $x_1y_1 = y_1x_1$, $xm_1 = x_1$, $ym_1' = y_1$ and $xym'' = x_1y_1$. Hence, lemma 2.3(i) and (3.2) give that P, $Q \leqslant C_G(xym'') = C_G(x_1y_1) = C_G(x_1) \cap C_G(y_1)$. So P, $Q \leqslant C_G(y)$, $C_G(x)$, because P, $Q \leqslant Z(M)$, $xm_1 = x_1$ and $ym_1' = y_1$. Since $p \in \pi$ is arbitrary, we get that $F_0 \leqslant C_G(y)$. Consequently, $F_0 \leqslant C_G(y_1)$. However, x_1y_1, x_1, $y_1 \in \mathrm{Van}(G) \subseteq G_\alpha$. Thus, $|C_G(x_1)| = |C_G(x_1y_1)| = |C_G(y_1)|$. It follows from lemma 2.3(i) that $F_0 \leqslant C_G(y_1) = C_G(x_1y_1) = C_G(x_1)$. Since $F_0 \leqslant Z(M)$ and $xm_1 = x_1$, we get that $F_0 \leqslant C_G(x)$. Regarding the fact that $p \in \pi$ is arbitrary, we conclude that the π-elements of $N - M$ centralize F_0. Now, let $M \neq zM$ be a π'-element of N/M. Without loss of generality, we can assume that $q \notin \pi(S_1) - \pi$. By lemmas 2.11, 2.12(a) and 2.13(ii,iii), yM, $yzM \subseteq \mathrm{Van}(G)$. On the other hand, lemma 2.3(iii) forces to exist a q-element $y_2 \in C - M$, a π'-element $z_1 \in N - M$ and u, $u' \in M$ such that $y_2z_1 = z_1y_2$, $zu = z_1$ and $yu' = y_2$. Then, y_2, $y_2z_1 \in \mathrm{Van}(G) \subseteq G_\alpha$. Thus $|C_G(y_2z_1)| = |C_G(y_2)|$. So, lemma 2.3(i) guarantees that $C_G(y_2) = C_G(y_2z_1) \leqslant C_G(z_1)$. As, $F_0 \leqslant C_G(y)$, $Z(M)$ and $yu' = y_2$, we have $F_0 \leqslant C_G(y_2) \leqslant C_G(z_1)$. However, $zu = z_1$ and $F_0 \leqslant Z(M)$. Hence, $F_0 \leqslant C_G(z)$. This forces $F_0 \leqslant Z(N)$. On the other hand, every π-element $w \in N - \mathrm{Fit}(G)$ is vanishing in G, by lemmas 2.11, 2.12(a) and 2.13(ii), and proposition 2.14. Therefore, $|cl_G(w)| = \alpha$. So, lemma 2.3(iv) yields that N contains a nilpotent Hall π-subgroup, so does N/M. This is a contradiction with lemma 2.1(iv). □

Step 3. $G/M \cong \mathrm{Alt}_5$ or Alt_7.

Proof. By steps 1 and 2, $N/M \cong S$ is non-abelian simple and $G/M \lesssim \mathrm{Aut}(N/M)$. Fix $\bar{N} = N/M$ and $\bar{G} = G/M$. For $x \in G$, let $\bar{x}$ be the image of x in $\bar{G}$.

a. Let $S \ncong \mathrm{Alt}_5$, M_{22} and let either (S, r, m, t) be as in tables I and II or $S \cong \mathrm{Alt}_l$, where $8 \leqslant l \leqslant 10$, and $(r, m, t) = (5, 15, 7)$. Then, $\bar{N}$ contains an element $\bar{x}$ of order m. By lemmas 2.11 and 2.12(a), for every $1 \leqslant i < m$, $\bar{x}^i \in \mathrm{Van}(\bar{G})$. If $S \cong \mathrm{Alt}_l$, where $8 \leqslant l \leqslant 10$, then we apply [**4**] for the previous conclusion. Consequently, $x^i \in \mathrm{Van}(G)$, by lemma 2.13(ii). Since $\mathrm{Van}(G) \subseteq G_\alpha$, m is a composite number and $r \mid m$, proposition 2.10(ii) shows that $|M|_r|o(\bar{x})|_r \mid |C_G(x)|$. So, for every $z \in \mathrm{Van}(G)$, $|M|_r|o(\bar{x})|_r \mid |C_G(z)|$. On the other hand, $\mathrm{Van}(\bar{G})$ contains an element $\bar{y}$ of order t, by lemma 2.12(b). Lemma 2.13(ii) guarantees that $y \in \mathrm{Van}(G)$. Hence, $|M|_r|o(\bar{x})|_r \mid |C_G(y)|$. Thus, lemma 2.3(ii) forces $r \mid |C_{\bar{G}}(\bar{y})|$. Therefore, $\bar{G}$ contains an element $\bar{z}$ of order rt. By lemma 2.12(b), for every $1 \leqslant i < tr$, $\bar{z}^i \in \mathrm{Van}(\bar{G})$. Consequently, $|M|_t|o(\bar{z})|_t \mid |C_G(z)|$, by lemma 2.13(ii) and proposition 2.10(ii). It follows from $(*)$ that $|M|_t|o(\bar{z})|_t \mid |C_G(x)|$. In view of lemma 2.3(ii), $t \mid |C_{\bar{G}}(\bar{x})|$. However, $t \nmid |\mathrm{Out}(S)|$. So, $t \mid |C_{\bar{N}}(\bar{x})|$, contradicting lemma 2.1(iii).

b. Assume that $S \cong \mathrm{Alt}_l$, where $l > 11$. Suppose that $m = 35$ and r and t are as in lemma 2.1(i). Then, $\bar{N}$ contains an element $\bar{x}$ of order m. By lemmas 2.11 and 2.12(a), for every $1 \leqslant i < m$, $\bar{x}^i \in \mathrm{Van}(\bar{G})$. Lemma 2.13(ii) yields $x^i \in \mathrm{Van}(G)$.

Since $\mathrm{Van}(G) \subseteq G_\alpha$, m is a composite number and $7 \mid m$, proposition 2.10(ii) shows that $|M|_7|o(\bar{x})|_7 \mid |C_G(x)|$. So, for every $z \in \mathrm{Van}(G)$, $|M|_7|o(\bar{x})|_7 \mid |C_G(z)|$. On the other hand, $\mathrm{Van}(\bar{G})$ contains an element $\bar{y}$ of order t, by lemma 2.12(b). Lemma 2.13(ii) forces $y \in \mathrm{Van}(G)$. Hence, $|M|_7|o(\bar{x})|_7 \mid |C_G(y)|$. Lemma 2.3(ii) implies that $7 \mid |C_{\bar{G}}(\bar{y})|$. Therefore, $\bar{G}$ contains an element $\bar{z}$ of order $7t$. By lemma 2.12(b), for every $1 \leqslant i < 7t$, $\bar{z}^i \in \mathrm{Van}(\bar{G})$. So, $|M|_t|o(\bar{z})|_t \mid |C_G(z)|$, by lemma 2.13(ii) and proposition 2.10(ii). Thus, $|M|_t|o(\bar{z})|_t \mid |C_G(u)|$, for every $u \in \mathrm{Van}(G)$. Also, lemma 2.12(b) forces $\mathrm{Van}(\bar{G}) \cap \bar{N}$ to contain an element $\bar{w}$ of order r. Hence, $t \mid |C_{\bar{G}}(\bar{w})|$, by lemmas 2.3(ii) and 2.13(ii). However, $t \nmid |\mathrm{Out}(S)|$. So, $t \mid |C_{\bar{N}}(\bar{w})|$, contradicting lemma 2.1(ii).

c. Let $S \cong M_{22}$ or Alt_{11}, $m = 8$ and let $t = 11$. [**4**] implies that $\bar{N}$ contains an element $\bar{x}$ of order m such that for every $1 \leqslant i < 4$, $\bar{x}^i \in \mathrm{Van}(\bar{G})$. Consequently, $x^i \in \mathrm{Van}(G)$, by lemma 2.13(ii). So, proposition 2.10(ii) shows that $2|M|_2 \mid |C_G(x)|$. Hence, $(*)$ forces $2|M|_2 \mid |C_G(z)|$, for every $z \in \mathrm{Van}(G)$. On the other hand, $\mathrm{Van}(\bar{G})$ contains an element $\bar{y}$ of order t, by lemma 2.12(b). By lemma 2.13(ii), $y \in \mathrm{Van}(G)$ and hence, $2|M|_2 \mid |C_G(y)|$. Thus, lemma 2.3(ii) implies that $2 \mid |C_{\bar{G}}(\bar{y})|$. This shows that $\mathrm{Aut}(S)$ contains an element of order $2t = 22$, which is a contradiction, by considering [**4**].

The above cases show that $N/M \cong \mathrm{Alt}_5$ or Alt_7. By step 2, $C_{G/M}(N/M) = \{M\}$. Thus, $G/M \lesssim \mathrm{Aut}(N/M)$. Working towards a contradiction, let $G/M \neq N/M$. Then, $G/M \cong \mathrm{Sym}_7$ or Sym_5. If $\bar{G} \cong \mathrm{Sym}_7$, let $(r, t, d) = (5, 7, 10)$ and if $\bar{G} \cong \mathrm{Sym}_5$, let $(r, t, d) = (3, 5, 6)$. [**4**] guarantees that $\mathrm{Van}(\bar{G})$ contains an element $\bar{x}$ of order d such that for every $1 \leqslant i < d$, $\bar{x}^i \in \mathrm{Van}(\bar{G})$. Since $\mathrm{Van}(G) \subseteq G_\alpha$, d is a composite number and $r \mid d$, proposition 2.10(ii) shows that $|M|_r|o(\bar{x})|_r \mid |C_G(x)|$. So for every $z \in \mathrm{Van}(G)$, $|M|_r|o(\bar{x})|_r \mid |C_G(z)|$. On the other hand, $\mathrm{Van}(\bar{G})$ contains an element $\bar{y}$ of order t, by lemma 2.12(b). Lemma 2.13(ii) yields that $y \in \mathrm{Van}(G)$. Hence, $|M|_r|o(\bar{x})|_r \mid |C_G(y)|$. Thus, lemma 2.3(ii) forces $r \mid |C_{\bar{G}}(\bar{y})|$. Therefore, $\bar{G}$ contains an element $\bar{z}$ of order rt, which is a contradiction, regarding the orders of elements of $\bar{G}$. This shows that $G/M = N/M$. Thus, $G/M \cong \mathrm{Alt}_5$ or Alt_7. □

Step 4. $\pi(M/\mathrm{Fit}(G)) \subseteq \{2\}$.

Proof. By step 3, $G/M \cong \mathrm{Alt}_5$ or Alt_7. It is worth mentioning that by [**4**],

$$\mathrm{Van}(\mathrm{Alt}_5) = \mathrm{Alt}_5 - \{1\} \text{ and } \{g \in \mathrm{Alt}_7 : o(g) \in \{5, 7\}\} \subseteq \mathrm{Van}(\mathrm{Alt}_7). \tag{3.3}$$

Fix $F_0 = \{1\}$ and for $1 \leqslant i \leqslant n$, let $F_i/F_{i-1} = \mathrm{Fit}(G/F_{i-1})$ such that $F_n = M$. Let $P \in \mathrm{Syl}_2(M)$. Working towards a contradiction, suppose that $\pi(M/F_1) \not\subseteq \{2\}$. Then, one of the following cases occurs:

Case 1. Let $\pi(F_n/F_{n-1}) = \{2\}$. Since by our assumption $\pi(M/F_1) \not\subseteq \{2\}$, we have $n \geqslant 3$ and obviously, $\pi(F_{n-1}/F_{n-2}) \neq \{2\}$. Set $W = (P \cap F_{n-1})F_{n-2}$. Then, $W \trianglelefteq G$, $\{W\} \neq F_{n-1}/W \trianglelefteq G/W$ is nilpotent and $\{W\} \neq Z/W \trianglelefteq G/W$, where $F_{n-1} \leqslant Z$ and $Z/F_{n-1} = Z(M/F_{n-1})$. Also, $Z/F_{n-1} \leqslant M/F_{n-1}$ is a 2-group and $\gcd(|F_{n-1}/W|, 2) = 1$. Let $C/W = C_{Z/W}(F_{n-1}/W)$. If $W \neq yW \in C/W$, then for every $W \neq xW \in F_{n-1}/W$, there is an element $w \in W$ such that

$$y^{-1}xyF_{n-2} = xwF_{n-2}. \tag{3.4}$$

We can assume that xF_{n-2} is a $2'$-element. Since W/F_{n-2} is a 2-group, $o(wF_{n-2})$ is a power of 2. Also, $wF_{n-2}, xF_{n-2} \in F_{n-1}/F_{n-2}$ and F_{n-1}/F_{n-2} is nilpotent. So, $xwF_{n-2} = wxF_{n-2}$. By (3.4), $o(xF_{n-2}) = \mathrm{lcm}(o(xF_{n-2}), o(wF_{n-2}))$. Thus, $wF_{n-2} = F_{n-2}$. Consequently, $yF_{n-2} \in C_{G/F_{n-2}}(xF_{n-2})$. So, $yF_{n-2} \in C_{G/F_{n-2}}(O_{2'}(F_{n-1}/F_{n-2}))$. However, $C/F_{n-2} \leqslant Z/F_{n-2}$. Therefore, $O_{2'}(C/F_{n-2}) \leqslant O_{2'}(F_{n-1}/F_{n-2})$ is nilpotent. Thus, $C/F_{n-2} = O_2(C/F_{n-2}) \times O_{2'}(C/F_{n-2})$ is nilpotent. Hence, $C/F_{n-2} \leqslant \mathrm{Fit}(G/F_{n-2}) = F_{n-1}/F_{n-2}$. So, $C_{Z/W}(F_{n-1}/W) = C/W \leqslant F_{n-1}/W$. By lemma 2.13(ii,iv), $Z - F_{n-1} \subseteq \mathrm{Van}(G)$. Now, let $s \in \pi(F_{n-1}/F_{n-2}) - \{2\}$. Then, $(P \cap Z)/(P \cap F_{n-1})$ acts on a Sylow s-subgroup of F_{n-1}/F_{n-2}, by conjugation and one of the following sub-cases occurs:

a. Let the action of $(P \cap Z)/(P \cap F_{n-1})$ on a Sylow s-subgroup of F_{n-1}/F_{n-2} be fixed point freely. Then, $(P \cap Z)/(P \cap F_{n-1}) \cong (P \cap Z)F_{n-1}/F_{n-1} = Z/F_{n-1}$ is a cyclic 2-group, because Z/F_{n-1} is abelian. Hence, Z/F_{n-1} contains a subgroup $\langle z_1F_{n-1}\rangle$ of order 2, which is normal in G/F_{n-1}. Obviously, $C_{G/F_{n-1}}(\langle z_1F_{n-1}\rangle) = G/F_{n-1}$. Thus, there is an element $xF_{n-1} \in C_{G/F_{n-1}}(\langle z_1F_{n-1}\rangle)$ of order 5, so $o(xz_1F_{n-1}) = 10$. By lemma 2.13(i,ii) and since $Z - F_{n-1} \subseteq \mathrm{Van}(G)$, z_1, x, $xz_1 \in \mathrm{Van}(G) \subseteq G_\alpha$. Hence, proposition 2.10 shows that $|F_{n-1}|_5 |o(xz_1F_{n-1})|_5 \mid |C_G(xz_1)|$. So, for every $h \in \mathrm{Van}(G)$, $5|F_{n-1}|_5 \mid |C_G(h)|$. If $G/M \cong \mathrm{Alt}_5$, let $p = 3$ and otherwise, let $p = 7$. Suppose that y is a p-element of $G - M$. By (3.3) and lemma 2.13(ii), $y \in \mathrm{Van}(G)$. Thus, $5|F_{n-1}|_5 \mid |C_G(y)|$. Since $\pi(M/F_{n-1}) = \{2\}$, lemma 2.3(ii) forces $5 \mid |C_{G/M}(yM)|$, which is impossible.

b. Assume that there exist a 2-element $z \in (Z \cap P) - F_{n-1}$ and an s-element $y \in F_{n-1} - F_{n-2}$ such that $yF_{n-2} \in C_{G/F_{n-2}}(zF_{n-2})$. We can assume by lemma 2.3(iii) that $z \in C_G(y)$, and by proposition 2.14, $y \in \mathrm{Van}(G)$. As stated before, $Z - F_{n-1} \subseteq \mathrm{Van}(G)$, so z, $zy \in \mathrm{Van}(G)$. Thus, zy satisfies the assumption of proposition 2.10. Let H_{n-1}/F_{n-2} be a Hall s'-subgroup of F_{n-1}/F_{n-2}. Then, $H_{n-1} \trianglelefteq G$ and proposition 2.10 shows that $|H_{n-1}|_s |o(yzH_{n-1})|_s \mid |C_G(yz)|$. It follows that for every $h \in \mathrm{Van}(G)$, $s|H_{n-1}|_s \mid |C_G(h)|$. If $G/M \cong \mathrm{Alt}_5$, let $r \in \{3, 5\} - \{s\}$ and $p \in \{3, 5\} - \{r\}$ and if $G/M \cong \mathrm{Alt}_7$, let $r \in \{5, 7\} - \{s\}$ and $p \in \{5, 7\} - \{r\}$. Let w be an r-element of $G - M$. By (3.3) and lemma 2.13(ii), $wM \subseteq \mathrm{Van}(G)$. Thus $s|H_{n-1}|_s \mid |C_G(w)|$. So, there exists an s-element $w' \in G - H_{n-1}$ such that $w'H_{n-1} \in C_{G/H_{n-1}}(wH_{n-1})$. However, $|C_{G/M}(wM)| = r$. Hence, $w' \in M - H_{n-1}$. Then, w, w', $ww' \in \mathrm{Van}(G)$, by proposition 2.14 and lemma 2.13(ii). So, proposition 2.10 shows that $r|H_{n-1}|_r \mid |C_G(ww')|$. Consequently, for every $h \in \mathrm{Van}(G)$, $r|H_{n-1}|_r \mid |C_G(h)|$. Assume that v is a p-element of $G - M$. By (3.3) and lemma 2.13(ii), $v \in \mathrm{Van}(G)$. Thus, $r|H_{n-1}|_r \mid |C_G(v)|$. Note that $\pi(M/H_{n-1}) = \{2, s\}$. By lemma 2.3(ii), $r \mid |C_{G/M}(vM)|$, which is impossible.

Case 2. Let $2 \neq s \in \pi(F_n/F_{n-1})$. Assume that $S/F_{n-1} \in \mathrm{Syl}_s(M/F_{n-1})$, L/F_{n-1} is a Hall s'-subgroup of M/F_{n-1} and $H/F_{n-1} = Z(S/F_{n-1})$. First, let $G/M \cong \mathrm{Alt}_5$ and $p = 2$. Then, G/M acts on H/F_{n-1}. This action is not fixed point freely, because the Sylow 2-subgroups of G/M are abelian and non-cyclic. Thus, there exists a 2-element $M \neq xM \in G/M$ such that $C_{H/F_{n-1}}(xF_{n-1}) \neq \{F_{n-1}\}$. Let $F_{n-1} \neq yF_{n-1} \in C_{H/F_{n-1}}(xF_{n-1})$. Since $s \neq 2$, (3.3), lemma 2.13(ii) and proposition 2.14 force x, y, $xy \in \mathrm{Van}(G)$. So, xy satisfies the assumption of proposition 2.10. Next, let $G/M \cong \mathrm{Alt}_7$. By lemma 2.13(v), $\mathrm{Van}(G/M)$ contains a p-element xM such that $C_{M/L}(xL) \neq \{L\}$ and $p \in \pi(G/M) - \{2, s\}$. Let $L \neq yL \in C_{M/L}(xL)$. We can assume that $o(y)$ is a power of s. Then, $x, y, xy \in \mathrm{Van}(G)$, by lemma 2.13(ii) and proposition 2.14. Consequently, xy satisfies the assumption of proposition 2.10. In both cases, proposition 2.10 shows that $|L|_p|o(xyL)|_p \mid |C_G(xy)|$. So, for every $h \in \mathrm{Van}(G)$, $p|L|_p \mid |C_G(h)|$. Let $zM \in G/M$ be of order r, where if $G/M \cong \mathrm{Alt}_5$, $r = 5$ and otherwise, $r \in \{5, 7\} - \{p\}$. By (3.3) and lemma 2.13(ii), $z \in \mathrm{Van}(G)$. Thus, $p|L|_p \mid |C_G(z)|$. As, $p \nmid |M/L|$, lemma 2.3(ii) yields $p \mid |C_{G/M}(zM)|$, which is impossible.

These contradictions show that $\pi(M/\mathrm{Fit}(G)) \subseteq \{2\}$. □

Step 5. $\pi(G/M) - \{2\} \subseteq \pi(\mathrm{Fit}(G))$.

Proof. By steps 3 and 4, $\pi(M/\mathrm{Fit}(G)) \subseteq \{2\}$ and $G/M \cong \mathrm{Alt}_5$ or Alt_7. Let $p \in \pi(G/M) - \{2\}$. Then, there are the elements $xM, yM \in \mathrm{Van}(G/M)$ such that $|G_{G/M}(xM)|_p = |G/M|_p$ and $p \nmid |C_{G/M}(yM)|$. By lemma 2.13(ii), $x, y \in \mathrm{Van}(G)$. So, $(*)$, corollary 2.4 and lemma 2.3(ii) force $p \mid |C_G(x)|$ and $|C_G(x)|_p = |C_G(y)|_p \mid |M|_p|C_{G/M}(yM)|_p = |\mathrm{Fit}(G)|_p$, because $p \neq 2$. Therefore, $p \mid |\mathrm{Fit}(G)|$, as desired. □

Step 6. $M = \mathrm{Fit}(G)$.

Proof. By steps 3 and 4, $\pi(M/\mathrm{Fit}(G)) \subseteq \{2\}$ and $G/M \cong \mathrm{Alt}_5$ or Alt_7. Working towards a contradiction, suppose that $M \neq \mathrm{Fit}(G)$. Let $P \in \mathrm{Syl}_2(M)$. Set $P_1 = P \cap \mathrm{Fit}(G)$ and assume that $Z/\mathrm{Fit}(G)$ is the maximal normal abelian subgroup of $G/\mathrm{Fit}(G)$ such that $Z \leqslant M$. Then, $P_1 \in \mathrm{Syl}_2(\mathrm{Fit}(G))$, $Z(M/\mathrm{Fit}(G)) \leqslant Z/\mathrm{Fit}(G)$ and $Z/\mathrm{Fit}(G)$ is a 2-group. By step 5, $\pi(G/M) - \{2\} \subseteq \pi(\mathrm{Fit}(G))$. Hence, $\mathrm{Fit}(G)/P_1$ is a non-trivial Hall $2'$-subgroup of M/P_1 that is nilpotent and normal in G/P_1. If $xP_1 \in C_{M/P_1}(\mathrm{Fit}(G)/P_1)$, then $[x, \mathrm{Fit}(G)] \subseteq P_1$. So, for every $2'$-element $f \in \mathrm{Fit}(G)$, there exists an element $g \in P_1$ such that $xfx^{-1}f^{-1} = g$. Hence, $xfx^{-1} = gf$. However, $\mathrm{Fit}(G)$ is nilpotent and $P_1 \leqslant \mathrm{Fit}(G)$. Therefore, $o(f) = \mathrm{lcm}(o(f), o(g))$. This forces $g = 1$. Thus, $x \in C_M(O_{2'}(\mathrm{Fit}(G)))$. Since $M/\mathrm{Fit}(G)$ is a 2-group, we have $x = x_1x_2 = x_2x_1$ such that $x_1 \in M$ is a 2-element and $x_2 \in \mathrm{Fit}(G)$ is a $2'$-element. Let $Q_1 \in \mathrm{Syl}_2(C_M(O_{2'}(\mathrm{Fit}(G)))$ such that $x_1 \in Q_1$. Then, $P_1 \leqslant Q_1$ and $Q_1\mathrm{Fit}(G) = Q_1 \times O_{2'}(\mathrm{Fit}(G))$ is a nilpotent subgroup of G. However, $Q_1\mathrm{Fit}(G)/\mathrm{Fit}(G) = C_M(O_{2'}(\mathrm{Fit}(G))\mathrm{Fit}(G)/\mathrm{Fit}(G) \trianglelefteq G/\mathrm{Fit}(G)$. Consequently, $Q_1 \times O_{2'}(\mathrm{Fit}(G)) = Q_1\mathrm{Fit}(G) \trianglelefteq G$. Therefore, $\mathrm{Fit}(G) = O_{2'}(\mathrm{Fit}(G)) \times P_1 \leqslant O_{2'}(\mathrm{Fit}(G)) \times Q_1 \leqslant \mathrm{Fit}(G)$. This yields that $O_{2'}(\mathrm{Fit}(G)) \times Q_1 = \mathrm{Fit}(G)$, so $Q_1 = P_1$. Thus, $x_1 \in P_1$. As $x = x_1x_2$ and $x_2 \in \mathrm{Fit}(G)$, we get $x \in \mathrm{Fit}(G)$. Thus,

$$C_{M/P_1}(\mathrm{Fit}(G)/P_1) \leqslant \mathrm{Fit}(G)/P_1. \tag{3.5}$$

Hence, $C_{Z/P_1}(\mathrm{Fit}(G)/P_1) \leqslant \mathrm{Fit}(G)/P_1$. By lemma 2.13(iv,ii).

$$Z - \mathrm{Fit}(G) \subseteq \mathrm{Van}(G). \tag{3.6}$$

In the following, we first assume that $G/M \cong \mathrm{Alt}_7$ and $Z = M$. We observe that G/M acts on $Z/\mathrm{Fit}(G)$, by conjugation. Hence, there are an odd prime q, a q-element $yM \in \mathrm{Van}(G/M)$ and an element $\mathrm{Fit}(G) \neq x\mathrm{Fit}(G) \in Z/\mathrm{Fit}(G)$ such that $y\mathrm{Fit}(G) \in C_{G/\mathrm{Fit}(G)}(x\mathrm{Fit}(G))$, by lemma 2.13(v). Therefore, $x, y, xy \in \mathrm{Van}(G)$. So, proposition 2.10 shows that $|\mathrm{Fit}(G)|_q|o(xy\mathrm{Fit}(G))|_q \mid |C_G(xy)|$. It follows that for every $h \in \mathrm{Van}(G)$, $q|\mathrm{Fit}(G)|_q \mid |C_G(h)|$. Let $p \in \{5, 7\} - \{q\}$. Suppose that z is a p-element of $G - M$. By lemma 2.13(i), $z \in \mathrm{Van}(G)$. Thus, $q|\mathrm{Fit}(G)|_q \mid |C_G(z)|$. Regarding $\pi(M/\mathrm{Fit}(G)) = \{2\}$, we get that $q \mid |C_{G/M}(zM)|$, which is impossible.

Now, let $G/M \cong \mathrm{Alt}_5$, $Z = M$ and a Sylow 2-subgroup of $G/\mathrm{Fit}(G)$ is abelian. Then, for every 2-element $x_2 \in G - Z$, $x_2\mathrm{Fit}(G) \in C_{G/\mathrm{Fit}(G)}(Z/\mathrm{Fit}(G))$. Set $C/\mathrm{Fit}(G) = C_{G/\mathrm{Fit}(G)}(Z/\mathrm{Fit}(G))$. So, $x_2 \in C - Z$. Therefore, $Z \neq x_2Z \in C/Z \trianglelefteq G/Z = G/M \cong \mathrm{Alt}_5$. This forces $C/Z = G/Z$, consequently, $C = G$. Thus, $C_{G/\mathrm{Fit}(G)}(Z/\mathrm{Fit}(G)) = G/\mathrm{Fit}(G)$. Hence, $G/\mathrm{Fit}(G)$ contains an element $x\mathrm{Fit}(G)$ of order 6 such that $x^2 \in G - M$ and $x^3 \in Z - \mathrm{Fit}(G)$. Since $M \neq xM$, $x^2M \in G/M \cong \mathrm{Alt}_5$, xM, $x^2M \in \mathrm{Van}(G/M)$. By lemma 2.13(ii), x^2, $x \in \mathrm{Van}(G)$. Also, (3.6) shows that $x^3 \in \mathrm{Van}(G)$. So, proposition 2.10 shows that $3|\mathrm{Fit}(G)|_3 \mid |C_G(x)|$. Regarding $[G : \mathrm{Fit}(G)]_3 = 3$, we get that that $|G|_3 \mid |C_G(x)|$. Hence, $|cl_G(x)|_3 = 1$. Since $x \in \mathrm{Van}(G)$, $(*)$ forces 3 not to divide the vanishing conjugacy class sizes of G. It follows from [**6**, theorem A] that G has a normal 3-complement, which is impossible.

Then, suppose that $M \neq Z$, $Z/\mathrm{Fit}(G)$ is an elementary abelian 2-group and $G/M \cong \mathrm{Alt}_5$ or Alt_7. Assume that L/Z is a chief factor of G such that $L \leqslant M$. If $o(u\mathrm{Fit}(G)) = 2$, for every $u\mathrm{Fit}(G) \in L/\mathrm{Fit}(G)$, then $L/\mathrm{Fit}(G)$ is abelian, contradicting our assumption on Z. So, $L/\mathrm{Fit}(G)$ contains an element $u\mathrm{Fit}(G)$ of order 4. Since $Z/\mathrm{Fit}(G)$ and L/Z are elementary abelian 2-groups, $u \in L - Z$ and $\mathrm{Fit}(G) \neq u^2\mathrm{Fit}(G) \in Z/\mathrm{Fit}(G)$. In the following, set $\bar{G} = G/P_1$ and, for every $H \leqslant G$ and $x \in G$, let $\bar{H} = HP_1/P_1$ and $\bar{x}$ be the image of x in $\bar{G}$. Since $|\bar{\mathrm{Fit}}(G)|$ is odd, we can assume that $o(\bar{u}) = 4$ and $\bar{u}^2 \in \bar{Z} - \bar{\mathrm{Fit}}(G)$. Let $\{\bar{1}\} = \bar{N}_0 \leqslant \cdots \leqslant \bar{N}_t = \bar{\mathrm{Fit}}(G) \leqslant \bar{G}$ be a normal series of $\bar{G}$ such that for every $1 \leqslant i \leqslant t$, $\bar{N}_i/\bar{N}_{i-1}$ is a chief factor of $\bar{G}$. Suppose that i is the smallest number such that $0 \leqslant i \leqslant t$ and

$$\bar{u}^2\bar{N}_i \in C_{\bar{M}/\bar{N}_i}(\bar{\mathrm{Fit}}(G)/\bar{N}_i). \tag{3.7}$$

By (3.5), $i \neq 0$. Working towards a contradiction, let

$$\bar{u}^2\bar{N}_{i-1} \in C_{\bar{M}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1}). \tag{3.8}$$

Assume that $\bar{n}\bar{N}_{i-1} \in \bar{\mathrm{Fit}}(G)/\bar{N}_{i-1}$ is arbitrary. If $\bar{n} \in \bar{N}_i$, then (3.8) forces $\bar{n}\bar{N}_{i-1} \in C_{\bar{M}/\bar{N}_{i-1}}(\bar{u}^2\bar{N}_{i-1})$. Next, let $\bar{n} \notin \bar{N}_i$. By (3.7), $[\bar{n}, \bar{u}^2] \in \bar{N}_i$. So, there is an element $\bar{m} \in \bar{N}_i$ such that $[\bar{n}, \bar{u}^2] = \bar{m}$. Thus, $\bar{n}(\bar{u}^2)\bar{n}^{-1}\bar{N}_{i-1} = \bar{m}\bar{u}^2\bar{N}_{i-1}$. By (3.8), $\bar{m}\bar{u}^2\bar{N}_{i-1} = \bar{u}^2\bar{m}\bar{N}_{i-1}$. Since $\bar{u}^2\bar{N}_{i-1}$ is a 2-element and $\bar{m}\bar{N}_{i-1} \in \bar{\mathrm{Fit}}(G)/\bar{N}_{i-1}$, which is a $2'$-group, we get that $\bar{m}\bar{N}_{i-1} = \bar{N}_{i-1}$. Consequently, $\bar{n}\bar{N}_{i-1} \in C_{\bar{M}/\bar{N}_{i-1}}(\bar{u}^2\bar{N}_{i-1})$, for every $\bar{n}\bar{N}_{i-1} \in \bar{\mathrm{Fit}}(G)/\bar{N}_{i-1}$. Therefore, $\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1} \leqslant C_{\bar{M}/\bar{N}_{i-1}}(\bar{u}^2\bar{N}_{i-1})$. Hence, $\bar{u}^2\bar{N}_{i-1} \in C_{\bar{M}/\bar{N}_{i-1}}(\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1})$, contradicting our assumption on i. So,

$$\bar{u}^2\bar{N}_{i-1} \notin C_{\bar{M}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1}). \tag{3.9}$$

Since $\{\bar{N}_{i-1}\} \neq Z(\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1}) \cap \bar{N}_i/\bar{N}_{i-1} \trianglelefteq \bar{G}/\bar{N}_{i-1}$ and $\bar{N}_i/\bar{N}_{i-1}$ is a chief factor of $\bar{G}$, $Z(\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1}) \cap \bar{N}_i/\bar{N}_{i-1} = \bar{N}_i/\bar{N}_{i-1}$. So, $\bar{N}_i/\bar{N}_{i-1} \leqslant Z(\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1})$. Therefore, $\bar{\mathrm{Fit}}(G)/\bar{N}_{i-1} \leqslant C_{\bar{Z}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1})$. As $Z/\mathrm{Fit}(G)$ is abelian, we get that $(\bar{Z}/\bar{N}_{i-1})C_{\bar{G}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1})/C_{\bar{G}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1})$ is abelian. Hence, [**7**, the proof of lemma 2.3] yields the existence of a character $\lambda \in \mathrm{Irr}(\bar{N}_i/\bar{N}_{i-1})$ such that

$$I_{\bar{Z}/\bar{N}_{i-1}}(\lambda) = C_{\bar{Z}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1}). \tag{3.10}$$

If there exists an element $\bar{g}\bar{N}_{i-1} \in \bar{G}/\bar{N}_{i-1}$ such that $\bar{u}^{\bar{g}}\bar{N}_{i-1} \in I_{\bar{G}/\bar{N}_{i-1}}(\lambda)$, then $(\bar{u}^2)^{\bar{g}}\bar{N}_{i-1} \in I_{\bar{Z}/\bar{N}_{i-1}}(\lambda)$. Thus, since by (3.10), $I_{\bar{Z}/\bar{N}_{i-1}}(\lambda) = C_{\bar{Z}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1}) \trianglelefteq \bar{G}/\bar{N}_{i-1}$, we get that $(\bar{u}^2)\bar{N}_{i-1} \in C_{\bar{Z}/\bar{N}_{i-1}}(\bar{N}_i/\bar{N}_{i-1})$, which is a contradiction with (3.9). This shows that no conjugate of $\bar{u}^2\bar{N}_{i-1}$ and no conjugate of $\bar{u}\bar{N}_{i-1}$ fix λ. So, lemma 2.13(vi) implies that $\bar{u}\bar{N}_{i-1}$, $\bar{u}^2\bar{N}_{i-1} \in \mathrm{Van}(\bar{G}/\bar{N}_{i-1})$ and by lemma 2.13(ii),

$$u, u^2 \in \mathrm{Van}(G). \tag{3.11}$$

Also, if $Z/\mathrm{Fit}(G)$ is not an elementary abelian 2-group, then there exists an element $u \in Z - \mathrm{Fit}(G)$ such that $o(u\mathrm{Fit}(G)) = 4$. So, (3.6) shows that

$$u, u^2 \in \mathrm{Van}(G). \tag{3.12}$$

Next, suppose that $G/M \cong \mathrm{Alt}_5$, $Z = M$ and a Sylow 2-subgroup of $G/\mathrm{Fit}(G)$ is not abelian. Then, there exists a 2-element $u \in G - Z = G - M$ such that $o(u\mathrm{Fit}(G)) = 4$ and $\mathrm{Fit}(G) \neq u^2\mathrm{Fit}(G) \in Z/\mathrm{Fit}(G)$. Since $uM \in G/M \cong \mathrm{Alt}_5$, we have $uM \in \mathrm{Van}(G/M)$. We conclude from lemma 2.13(ii) and (3.6) that

$$u, u^2 \in \mathrm{Van}(G). \tag{3.13}$$

Nevertheless, if $Z \neq M$ or $G/M \cong \mathrm{Alt}_5$, $Z = M$ and a Sylow 2-subgroup of $G/\mathrm{Fit}(G)$ is not abelian, then $(*)$, (3.11), (3.12) and (3.13) yield that there is an element $u \in \mathrm{Van}(G)$ such that $u^2 \in \mathrm{Van}(G)$, $o(u\mathrm{Fit}(G)) = 4$ and $|cl_G(u)| = |cl_G(u^2)|$. Hence, proposition 2.10 shows that $4|\mathrm{Fit}(G)|_2 \mid |C_G(u)|$. Consequently, $4|\mathrm{Fit}(G)|_2 \mid |C_G(w)|$, for every $w \in \mathrm{Van}(G)$. If $x_5 \in G - \mathrm{Fit}(G)$ is a 5-element, then $M \neq x_5M$ is a 5-element of G/M. However, $G/M \cong \mathrm{Alt}_5$ or Alt_7. Thus, $x_5M \in \mathrm{Van}(G/M)$. By lemma 2.13(ii), $x_5M \subseteq \mathrm{Van}(G)$. Therefore, $4|\mathrm{Fit}(G)|_2 \mid |C_G(x_5)|$. As, $|C_{G/M}(x_5M)|_2 = 1$, x_5M contains a $\{2, 5\}$-element y such that $o(y\mathrm{Fit}(G)) = 10$, $y_5 \in x_5M$ and $y_2 \in M - \mathrm{Fit}(G)$, where y_2 and y_5 are the 2-part and the 5-part of y, respectively. As mentioned above, y_5, $y \in \mathrm{Van}(G)$ and $y_2 \in M - \mathrm{Fit}(G)$. Thus, $|C_G(y)| = |C_G(y_5)|$. By lemma 2.3(i),

$$C_G(y_5) = C_G(y) \leqslant C_G(y_2). \tag{3.14}$$

On the other hand, if $P_5 \in \mathrm{Syl}_5(\mathrm{Fit}(G))$, then since $5 \in \pi(\mathrm{Fit}(G))$, $P_5 \neq \{1\}$ and $\langle y\rangle = \langle y_2\rangle \times \langle y_5\rangle$ acts on P_5, by conjugation. By (3.14), $C_{P_5}(y_5) \leqslant C_{P_5}(y_2)$. It follows from lemma 2.2 that $y_2 \in C_G(P_5)$. Fix $A := C_G(P_5)$. Then, $1 \neq y_2 \in (A - \mathrm{Fit}(G)) \cap M$. Hence, $\mathrm{Fit}(G) < (AP_5 \cap M) \trianglelefteq G$. Thus, $\{\mathrm{Fit}(G)\} \neq (AP_5 \cap M)/\mathrm{Fit}(G) \trianglelefteq G/\mathrm{Fit}(G)$. Let $B/\mathrm{Fit}(G)$ be a minimal normal subgroup of $G/\mathrm{Fit}(G)$ such that $B \leqslant (AP_5 \cap M)$. Then, $\bar{B}/\bar{\mathrm{Fit}}(G)$ is abelian. By (3.5) and lemma 2.13(iv),

$$\bar{B} - \bar{\mathrm{Fit}}(G) \subseteq \mathrm{Van}(\bar{G}). \tag{3.15}$$

Obviously, there exists an element $g \in B \cap A$ such that $g \notin \mathrm{Fit}(G)$. Since $\bar{g} \in \bar{B} - \bar{\mathrm{Fit}}(G)$, we get from (3.15) and lemma 2.13(ii) that $g \in \mathrm{Van}(G)$. However, $g \in A =$

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$C_G(P_5)$ and $[G:P_5]_5=5$. It follows that $|cl_G(g)|_5 \leqslant 5$. So, $|cl_G(w)|_5 \leqslant 5$, for every $w \in \mathrm{Van}(G)$. Let $x_2 \in G-M$ be a p-element, where if $G/M \cong \mathrm{Alt}_5$, then $p=2$ and otherwise, $p=7$. Then, $x_2M \in \mathrm{Van}(G/M)$. By lemma 2.13(ii), $x_2 \in \mathrm{Van}(G)$. So, $|P_5| \mid |C_G(x_2)|$. We have $|cl_{G/M}(x_2M)|_5=1$ and $M/\mathrm{Fit}(G)$ is a 2-group. Thus, corollary 2.4 forces $P_5 \leqslant C_G(x_2)$. Therefore, $x_2 \in C_G(P_5)$. This yields that $M \neq x_2M \in C_G(P_5)M/M \trianglelefteq G/M \cong \mathrm{Alt}_5$ or Alt_7. Since G/M is simple, $C_G(P_5)M/M = G/M$. Consequently, $C_G(P_5)M=G$. Thus, $C_G(P_5)$ contains a 5-element x_5 such that $x_5 \notin M$. So, $M \neq x_5M \in G/M$. It follows that $x_5M \in \mathrm{Van}(G/M)$. By lemma 2.13(ii), $x_5 \in \mathrm{Van}(G)$. Also, $P_5\langle x_5\rangle \leqslant C_G(x_5)$. So, $5 \nmid |cl_G(x_5)|$. Hence, $(*)$ forces 5 not to divide the vanishing conjugacy class sizes of G. It follows from [**6**, theorem A] that G has a normal 5-complement, which is impossible.

This shows that $M=\mathrm{Fit}(G)$, as wanted. □

Step 7. We get the final contradiction.

Proof. By step 6, $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_5$ or Alt_7. First, let $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_5$. Then, for every $x \in G-\mathrm{Fit}(G)$, $\mathrm{Fit}(G) \neq x\mathrm{Fit}(G) \in G/\mathrm{Fit}(G) \cong \mathrm{Alt}_5$. So, $x\mathrm{Fit}(G) \in \mathrm{Van}(G/\mathrm{Fit}(G))$. By lemma 2.13(ii), $x \in \mathrm{Van}(G)$. By step 5, $3, 5 \in \pi(\mathrm{Fit}(G))$. Let E be a Hall $3'$-subgroup of $\mathrm{Fit}(G)$. Set $\tilde{G}=G/E$ and, for every $H \leqslant G$ and $x \in G$, let $\tilde{H}=HE/E$ and $\tilde{x}$ be the image of x in $\tilde{G}$. Then, $\tilde{\mathrm{Fit}}(G)$ is a 3-group and lemma 2.5 shows that for every $\tilde{1} \neq \tilde{x} \in \tilde{G}-\tilde{\mathrm{Fit}}(G)$, $|cl_{\tilde{G}}(\tilde{x})|_3=3^e$, for some positive integer e. Let $\tilde{x}_5 \in \tilde{G}-\tilde{\mathrm{Fit}}(G)$ be of order 5 and let $\{\tilde{1}\}=\tilde{M}_0 \leqslant \tilde{M}_1 \leqslant \cdots \leqslant \tilde{M}_t = \tilde{\mathrm{Fit}}(G) \leqslant \tilde{G}$ be a chief series of $\tilde{G}$. By proposition 2.8, there is an $1 \leqslant i \leqslant t$ such that $\tilde{M}_i/\tilde{M}_{i-1} \nleqslant Z(\tilde{G}/\tilde{M}_{i-1})$ and $|C_{\tilde{M}_i/\tilde{M}_{i-1}}(\tilde{x}_5\tilde{M}_{i-1})| \geqslant |C_{\tilde{M}_i/\tilde{M}_{i-1}}(\tilde{x}\tilde{M}_{i-1})|$, for some 2-element $\tilde{M}_{i-1} \neq \tilde{x}\tilde{M}_{i-1} \in N_{\tilde{G}/\tilde{M}_{i-1}}(\langle \tilde{x}_5\tilde{M}_{i-1}\rangle)$. Hence, proposition 2.17 implies the existence of a non-trivial element $\tilde{n}\tilde{M}_{i-1} \in C_{\tilde{M}_i/\tilde{M}_{i-1}}(\tilde{x}_5\tilde{M}_{i-1})$ and a character $\psi \in \mathrm{Irr}(\tilde{G}/\tilde{M}_{i-1})$ such that $\psi(\tilde{n}\tilde{M}_{i-1})=0$. So, $\tilde{n}\tilde{M}_{i-1} \in \mathrm{Van}(\tilde{G}/\tilde{M}_{i-1})$. By lemma 2.13(ii), $n \in nM_{i-1} \subseteq \mathrm{Van}(G)$. Since $3 \nmid |E|$ and $\tilde{M}_i$ is a 3-group, we can assume that n is a 3-element. Thus, $E \leqslant C_G(n)$, because $n \in \mathrm{Fit}(G)$, $E \leqslant \mathrm{Fit}(G)$ is a $3'$-group and $\mathrm{Fit}(G)$ is nilpotent. We note that $|G/E|_5=|G/\mathrm{Fit}(G)|_5=|\mathrm{Alt}_5|_5=5$. Therefore, $|cl_G(n)|_5 \leqslant |G/E|_5=5$. Hence,

$$|cl_G(w)|_5 \leqslant 5, \text{ for every } w \in \mathrm{Van}(G). \tag{3.16}$$

Let $x_3 \in G-\mathrm{Fit}(G)$ be a 3-element. Then, $|C_{G/\mathrm{Fit}(G)}(x_3\mathrm{Fit}(G))|_5=1$. By (3.16) and corollary 2.4, $|C_{\mathrm{Fit}(G)}(x_3)|_5=|\mathrm{Fit}(G)|_5$. So, $P_5 \leqslant C_{\mathrm{Fit}(G)}(x_3)$, where $P_5 \in \mathrm{Syl}_5(\mathrm{Fit}(G))$. Thus, $x_3 \in P_5C_G(P_5)-\mathrm{Fit}(G)$. This yields that $\{\mathrm{Fit}(G)\} \neq P_5C_G(P_5)/\mathrm{Fit}(G) \trianglelefteq G/\mathrm{Fit}(G)$. However, $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_5$ is simple. Therefore, $P_5C_G(P_5)/\mathrm{Fit}(G)=G/\mathrm{Fit}(G)$. Hence, $P_5C_G(P_5)=G$. This guarantees the existence of a 5-element $y_5 \in C_G(P_5)-\mathrm{Fit}(G)$. Then, $y_5 \in \mathrm{Van}(G)$. Also, $P_5\langle y_5\rangle \leqslant C_G(y_5)$. This signifies that $5|P_5| \mid |C_G(y_5)|$. Taking into account the fact that $[G:P_5]_5=5$, we get that $|G|_5 \mid |C_G(y_5)|$. Therefore, $5 \nmid |cl_G(y_5)|$. So, $(*)$ forces 5 not to divide any vanishing conjugacy class size of G. It follows from [**6**, theorem A] that G has a normal 5-complement, which is impossible.

Next, let $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_7$. By step 5, $3, 5, 7 \in \pi(\mathrm{Fit}(G))$. Let F be a Hall $7'$-subgroup of $\mathrm{Fit}(G)$. Set $\bar{G}=G/F$ and, for every $H \leqslant G$ and $x \in G$, let $\bar{H}=HF/F$ and $\bar{x}$ be the image of x in $\bar{G}$. Then, $\bar{\mathrm{Fit}}(G) \trianglelefteq \bar{G}$ is a 7-group and

https://doi.org/10.1017/prm.2022.4 Published online by Cambridge University Press

$\bar{G}/\bar{\mathrm{Fit}}(G) \cong G/\mathrm{Fit}(G) \cong \mathrm{Alt}_7$. Suppose that $\{\bar{1}\} = \bar{N}_0 \leqslant \cdots \leqslant \bar{N}_t = \bar{\mathrm{Fit}}(G) \leqslant \bar{G}$ is a chief series of $\bar{G}$. Assume that for every $i \in \{1, \ldots, t\}$, $\bar{N}_i/\bar{N}_{i-1} \leqslant Z(\bar{G}/\bar{N}_{i-1})$. By lemma 2.6, $\bar{G} = \bar{L} \times \bar{\mathrm{Fit}}(G)$, where $\bar{L} \cong \mathrm{Alt}_7$. Let $x \in L - F$ be a 7-element. Then, $\bar{\mathrm{Fit}}(G)\langle\bar{x}\rangle \leqslant C_{\bar{G}}(\bar{x})$. However, $[G : \mathrm{Fit}(G)]_7 = 7$. Thus, $|\bar{G}|_7 \mid |C_{\bar{G}}(\bar{x})|$. Since $\gcd(|F|, 7) = 1$ and x is a 7-element, we get from lemma 2.3(v) that $C_{\bar{G}}(\bar{x}) = C_G(x)F/F \cong C_G(x)/C_F(x)$. Thus, $|G|_7 = |\bar{G}|_7 \mid |C_G(x)|$. Therefore, $7 \nmid |cl_G(x)|$. However, $\bar{1} \neq \bar{x} \in \bar{L}$ is a 7-element and $\bar{L} \cong \mathrm{Alt}_7$. Hence, $\bar{x} \in \mathrm{Van}(\bar{G})$. By lemma 2.13(ii), $x \in \mathrm{Van}(G)$. So, 7 does not divide the vanishing conjugacy class sizes of G. Hence, [**6**, theorem A] implies that G has a normal 7-complement, which is impossible. This guarantees the existence of an element $i \in \{1, \ldots, t\}$ such that $\bar{N}_i/\bar{N}_{i-1} \nleqslant Z(\bar{G}/\bar{N}_{i-1})$. Let $y \in G$ be a $\{2, 3\}$-element such that $\bar{y}\bar{\mathrm{Fit}}(G) \in \bar{G}/\bar{\mathrm{Fit}}(G)$ is of order 6. Let y_2 and y_3 be the 2-part and the 3-part of y, respectively. Then, $y_2 \notin \mathrm{Fit}(G)$ and $o(\bar{y}_3\bar{\mathrm{Fit}}(G)) = 3$. It follows from proposition 2.16 that $\bar{y}\bar{N}_{i-1}, \bar{y}_3\bar{N}_{i-1} \in \mathrm{Van}(\bar{G}/\bar{N}_{i-1})$. By lemma 2.13(ii), $y, y_3 \in \mathrm{Van}(G)$. Thus, $|C_G(y)| = |C_G(y_3)|$, by $(*)$. By lemma 2.3(i),

$$C_G(y_3) = C_G(y) \leqslant C_G(y_2). \tag{3.17}$$

Also, $3 \mid |\mathrm{Fit}(G)|$. So, $\langle y\rangle = \langle y_3\rangle \times \langle y_2\rangle$ acts on P_3, where $\{1\} \neq P_3 \in \mathrm{Syl}_3(\mathrm{Fit}(G))$. By (3.17), $C_{P_3}(y_3) \leqslant C_{P_3}(y_2)$. It follows from lemma 2.2 that $y_2 \in C_G(P_3)$. Then, $y_2 \in C_G(P_3) - \mathrm{Fit}(G)$. This yields that $\mathrm{Fit}(G) \neq y_2\mathrm{Fit}(G) \in C_G(P_3)P_3/\mathrm{Fit}(G) \trianglelefteq G/\mathrm{Fit}(G) \cong \mathrm{Alt}_7$. Since $G/\mathrm{Fit}(G)$ is simple, $C_G(P_3)P_3/\mathrm{Fit}(G) = G/\mathrm{Fit}(G)$. Consequently, $G = C_G(P_3)P_3$. Thus, $C_G(P_3)$ contains a 3-element x_3 such that $x_3 \notin \mathrm{Fit}(G)$. By proposition 2.16, $\bar{x}_3\bar{N}_{i-1} \in \mathrm{Van}(\bar{G}/\bar{N}_{i-1})$. It follows from lemma 2.13(ii) that $x_3 \in \mathrm{Van}(G)$. Also, $P_3\langle x_3\rangle \leqslant C_G(x_3)$. So, $|cl_G(x_3)|_3 \leqslant 3$. Hence, $(*)$ forces 3^2 not to divide the vanishing conjugacy class sizes of G. Let $x_5 \in G - \mathrm{Fit}(G)$ be a 5-element. Since $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_7$, $x_5\mathrm{Fit}(G) \in \mathrm{Van}(G/\mathrm{Fit}(G))$. By lemma 2.13(ii), $x_5 \in \mathrm{Van}(G)$. Thus, $3^2 \nmid |cl_G(x_5)|$. However, $[G : \mathrm{Fit}(G)]_3 = 3^2$. So, corollary 2.4 forces $3 \mid |C_{G/\mathrm{Fit}(G)}(x_5\mathrm{Fit}(G))|$, which is impossible, because $G/\mathrm{Fit}(G) \cong \mathrm{Alt}_7$.

The above steps show that G is solvable. Now, the proof of theorem A is complete. □

Acknowledgements

The author expresses her gratitude to Professor Silvio Dolfi for the helpful discussions. Also, the author thanks the referee for careful reading and several useful comments.

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