A NORM PROPERTY FOR SPACES OF COMPLETELY BOUNDED MAPS BETWEEN C*-ALGEBRAS

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(Received 26 November, 1985)

Let M_n be the C*-algebra of $n \times n$ complex matrices. If A is a C*-algebra, let $M_n(A)$ denote the C*-algebra of $n \times n$ matrices $a = [a_{ij}]$ with entries in A. For a linear map $\phi: A \to B$ between C*-algebras, we define the multiplicity map $\phi_n: M_n(A) \to M_n(B)$ by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. A linear map ϕ is said to be *completely bounded* if $\sup_n \|\phi_n\| < \infty$. Let $B(A, B)$, $\mathrm{CB}(A, B)$ denote the Banach space of bounded linear maps, the set of completely bounded maps from A to B, respectively.

A C*-algebra A is said to be *subhomogeneous* if all irreducible representations of A are finite dimensional with bounded dimension. Let M be the von Neumann algebra consisting of all operators which are of the form $a = \sum_{n=1}^{\infty} \oplus a_n$, where $a_n \in M_n$ and $\sup_n \|a_n\| < \infty$. Smith proved in [**5**] that if A is an infinite dimensional C*-algebra and B is a C*-algebra containing M then the closure of $\mathrm{CB}(A, B)$ in $B(A, B)$, with respect to the operator norm topology, is nowhere dense. He also asked whether this result remains true if B is replaced by a non-subhomogeneous algebra. In this note we shall settle this in the affirmative if A is separable.

A linear map $\phi: A \to B$ between C*-algebras is said to be *completely positive* if each multiplicity map $\phi_n: M_n(A) \to M_n(B)$ is positive. If $\phi \in \mathrm{CB}(A, B)$, we put the norm $\|\phi\|_{\mathrm{cb}} = \sup_n \|\phi_n\|$. If ϕ is completely positive then $\|\phi\|_{\mathrm{cb}} = \|\phi\|$ by Stinespring's theorem [**6**, IV, Theorem 3.6(ii)].

LEMMA 1. *If a C*-algebra B is not subhomogeneous then there exists a sequence $\{B_n\}$ of C*-subalgebras of B satisfying the following conditions.*

(1) *If $n \neq m$ then $a_n a_m = 0$ for any a_n in B_n and a_m in B_m.*

(2) *Given any positive number $r > 0$, there exist completely positive contractions $\psi_n: M_n \to B_n$ and $\phi_n: B \to M_n$ such that $\|\phi_n \circ \psi_n - \mathrm{id}\| < r$ on M_n for each n.*

Proof. For a representation π of a C*-algebra, let $\dim \pi$ denote the dimension of π. If a C*-subalgebra B_0 of B admits an irreducible representation π with $\dim \pi \geqq m$, there exist completely positive contractions $\psi_0: M_m \to B_0$ and $\phi_0': B_0 \to M_m$ such that $\|\phi_0' \circ \psi_0 - \mathrm{id}\| < r$ on M_m by [**5**, Lemma 2.7] or [**2**, Lemma 2], according as B_0 is unital or nonunital. Since M_m is an injective C*-algebra, the map ϕ_0' has a completely positive, norm preserving extension $\phi_0: B \to M_m$. Hence we show that there exists a sequence $\{B_n\}$ with condition (1) such that each B_n admits an irreducible representation π_n with $\dim \pi_n \geqq n$.

(i) We assume first that B has an irreducible representation π on an infinite dimensional Hilbert space H. Since $\pi(B)$ is infinite dimensional, there exists, by a standard argument, a self-adjoint element a in B such that the spectrum Y of $\pi(a)$ is infinite (see, for example, the proof of [**5**, Lemma 2.3]). Then Y is a closed subset of the spectrum X of a by [**6**, I, Corollary 8.4]. There then exists a sequence $\{f_n\}$ of positive continuous functions on $X \cup \{0\}$ such that $\|f_n\| = 1$, $f_n(0) = 0$, $f_n(Y)$ is an infinite set and $f_n f_m = 0$ for $n \neq m$.

Since $f_n(a)$ is regarded as an element of B, let B_n denote the C*-subalgebra generated by the algebra $f_n(a)Bf_n(a)$. Then $\{B_n\}$ satisfies condition (1). Let Q_n denote the projection of H on the closed subspace generated by $\pi(f_n(a))(H)$. Since the spectrum of $\pi(f_n(a))$ contains $f_n(Y)$, the space $Q_n(H)$ is infinite dimensional. Let $\pi(B_n)^-$ be the weak closure of $\pi(B_n)$. Then $Q_n(H)$ is an invariant subspace under $\pi(B_n)^-$. Since $\pi(B_n)^- \mid Q_n(H) = Q_n\pi(B)^- \mid Q_n(H) = B(Q_n(H))$, we have the desired irreducible representation π_n of B_n defined by $\pi_n(a) = \pi(a) \mid Q_n(H)$.

(ii) Suppose that any irreducible representation of B is finite dimensional. Let $B^\wedge$ denote the set of primitive ideals of B with the Jacobson topology. Since B is of type 1, a primitive ideal uniquely corresponds to a unitarily equivalent class of irreducible representations. Also B contains an essential ideal J which has continuous trace by [**4**, Theorem 6.2.11]. Let $J^\wedge$ be the set of primitive ideals K of B such that $K \not\supseteq J$. Then $J^\wedge$ is a dense subset of $B^\wedge$ as J is essential. By [**4**, Proposition 4.4.10], there exists a sequence $\{\pi_n\}$ of distinct irreducible representations such that $\dim \pi_n \geqq n$ and $K_n \in J^\wedge$, where K_n denotes the kernel of π_n. Since J has continuous trace, $J^\wedge$ is a locally compact Hausdorff space by [**4**, Theorem 6.1.11]. Then there exists a sequence $\{f_n\}$ of positive continuous functions on $B^\wedge$ such that the support of f_n is contained in $J^\wedge$, $f_n(K_n) = 1$, $\|f_n\| = 1$ and $f_n f_m = 0$ for $n \neq m$. By Dauns-Hofmann's theorem [**4**, Corollary 4.4.8], each f_n is regarded as an element of the centre of the multiplier of B.

Let B_n denote the C*-subalgebra generated by f_nB. Then $\{B_n\}$ satisfies condition (1). Since $\pi_n(f_n b) = f_n(K_n)\pi_n(b) = \pi_n(b)$ for all b in B, the restriction of π_n to B_n is the desired irreducible representation.

LEMMA 2. *Let $C_0(N)$ denote the C*-algebra of sequences convergent to* 0. *If a C*-algebra B is not subhomogeneous then there exists a bounded linear map $\phi: C_0(N) \to B$ such that ψ is not completely bounded whenever $\|\psi - \phi\| < \frac{1}{2}$.*

Proof. For a positive integer n, let $l_\infty(n)$ be the C*-algebra of finite sequences $\{x_i\}_{i=1}^n$ and put

$$k(n) = \sup\{\|\psi\|_{cb} \mid \psi: l_\infty(n) \to M_2n,\ \|\psi\| \leqq 1\}.$$

Using Lanford's example, Loebl [**3**] and Tsui [**7**] showed that $k(n) \to \infty$ as $n \to \infty$; since both $l_\infty(n)$ and M_2n are finite dimensional, we choose $v_n: l_\infty(n) \to M_2n$ such that $\|v_n\|_{cb} = k(n)$ and $\|v_n\| = 1$. From Lemma 1, we have a sequence $\{B_n\}$ of B and completely positive contractions $\psi_n: M_2n \to B_n$ and $\phi_n: B \to M_2n$ such that $\|\phi_n \circ \psi_n - \mathrm{id}\| < \frac{1}{2}$ on M_2n for each n.

Each element a in $C_0(N)$ can be written as $a = \sum_{n=1}^{\infty} \oplus a_n$, where $a_n \in l_\infty(n)$ and $\lim_n \|a_n\| = 0$. If $a = \sum_{n=1}^{\infty} \oplus a_n \in C_0(N)$ then $\|\sum_{n=1}^{\infty} (\psi_n \circ v_n)(a_n)\| = \sup_n \|(\psi_n \circ v_n)(a_n)\|$ and hence $\sum_{n=1}^{\infty}(\psi_n\circ\nu_n)(a_n)$ belongs to B. We define a linear map $\phi:C_0(N)\to B$ by

$$\phi(a)=\sum_{n=1}^{\infty}(\psi_n\circ\nu_n)(a_n).$$

Then ϕ has norm at most one.

Suppose that there exists a completely bounded map $\psi:C_0(N)\to B$ such that $\|\psi-\phi\|<\frac{1}{2}$. Then

$$\begin{aligned}\|\psi\|_{cb}&\geqq\|\phi_n\circ\psi\mid l_\infty(n)\|_{cb}\\&\geqq\|\nu_n\|_{cb}-\|(\phi_n\circ\psi_n-\mathrm{id})\circ\nu_n\|_{cb}-\|\phi_n\circ(\psi-\phi)\mid l_\infty(n)\|_{cb}\\&\geqq(1-\tfrac{1}{2}-\|\psi-\phi\|)k(n)\\&=(\tfrac{1}{2}-\|\psi-\phi\|)k(n)\to\infty\end{aligned}$$

as $n\to\infty$. This contradicts the fact that $\|\psi\|_{cb}$ is finite, and completes the proof.

The above lemma is based on an idea of Smith [**5**, Theorem 2.5].

THEOREM 3. *Let A be a separable, infinite dimensional C^*-algebra. If a C^*-algebra B is not subhomogeneous then there exists a bounded linear map $\Phi:A\to B$ such that ψ is not completely bounded whenever $\|\psi-\Phi\|<\frac{1}{4}$.*

Proof. (i) We first show that there exist completely bounded maps $\phi_0:A\to C_0(N)$ and $\nu_0:C_0(N)\to A$ such that $\phi_0\circ\nu_0$ is the identity map on $C_0(N)$, $\|\phi_0\|_{cb}\leqq 2$ and $\|\nu_0\|_{cb}\leqq 2$.

Let αN denote the one-point compactification of the discrete set of positive integers. If A is unital, by [**1**, Lemma 5], there exist completely positive unital maps $\phi':A\to C(\alpha N)$, $\nu':C(\alpha N)\to A$ such that $\phi'\circ\nu'$ is the identity map on $C(\alpha N)$. Since $C(\alpha N)$ is regarded as $C_0(N)+\mathbb{C}I$, we define $\phi_0:A\to C_0(N)$ by $\phi_0(b)=\phi'(b)-f(\phi'(b))I$, where f denotes a state on $C(\alpha N)$ such that $f(a)=0$ for all a in $C_0(N)$, so that $\|\phi_0\|_{cb}\leqq 2$ and $\phi_0(I)=0$. Then ϕ_0 and the restriction ν_0 of ν' to $C_0(N)$ are the desired maps. We remark that $\|\nu_0\|_{cb}\leqq 1$.

If A is nonunital, by the above, there exist $\phi'':A+\mathbb{C}I\to C_0(N)$ and $\nu'':C_0(N)\to A+\mathbb{C}I$ such that $\phi''\circ\nu''$ is the identity map, $\|\phi''\|_{cb}\leqq 2$, $\phi''(I)=0$ and $\|\nu''\|_{cb}\leqq 1$. Let h be a state on $A+\mathbb{C}I$ such that $h(a)=0$ for all a in A. We define $\nu_0:C_0(N)\to A$ by $\nu_0(b)=\nu''(b)-h(\nu''(b))I$. Then $\|\nu_0\|_{cb}\leqq 2$. Since $\phi''(I)=0$, the restriction $\phi_0=\phi''\mid A$ and ν_0 are the desired maps.

(ii) Let ϕ be the map as in Lemma 2. Using ϕ_0 and ν_0 as in (i), we put $\Phi=\phi\circ\phi_0:A\to B$. Suppose that there exists a completely bounded map $\psi:A\to B$ with $\|\psi-\Phi\|<\frac{1}{4}$. Then $\psi\circ\nu_0:C_0(N)\to B$ is a completely bounded map with

$$\|\psi\circ\nu_0-\phi\|=\|\psi\circ\nu_0-\Phi\circ\nu_0\|\leqq\|\psi-\Phi\|\,\|\nu_0\|<\tfrac{1}{2}.$$

By Lemma 2, $\psi\circ\nu_0$ is not completely bounded. This is a contradiction, and completes the proof.

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