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A generalisation of von Staudt's theorem on cross-ratios

YATIR HALEVI† AND ITAY KAPLAN‡
Einstein Institute of Mathematics, Edmond J. Safra Campus, The Hebrew University of Jerusalem, Givat Ram 91904, Jerusalem, Israel.
e-mails: yatir.halevi@mail.huji.ac.il, kaplan@math.huji.ac.il

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Abstract

A generalisation of von Staudt's theorem that every permutation of the projective line that preserves harmonic quadruples is a projective semilinear map is given. It is then concluded that any proper supergroup of permutations of the projective semilinear group over an algebraically closed field of transcendence degree at least 1 is 4-transitive.

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1. *Introduction*

In his book *Geometrie der Lage* (see [**vS47**]), first appearing in 1847, Karl Georg Christian von Staudt, wanting to establish (real) projective geometry on an axiomatic approach, defined a projectivity to be a permutation of the projective line $\mathbb{P}(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ preserving *harmonic quadruples*, i.e. quadruples of distinct elements having cross-ratio -1 (which can be defined strictly geometrically), where the cross-ratio of a quadruple of distinct elements is

$$[a, b; c, d] = \frac{c-a}{c-b} \cdot \frac{d-b}{d-a}.$$

He proved that a projectivity is a composition of a finite number of perspectivities (which are basic geometric maps). It was noticed later that there is a small gap in von Staudt's reasoning, see [**Coo34**, **Voe08**] for a detailed historical background.

Given a field F, a *projectivity* (also known as a homography or a fractional linear transformation) of the projective line $\mathbb{P}(F) = F \cup \{\infty\}$ is an element of the group

$$PGL_2(F) = \left\{ \frac{ax+b}{cx+d} : a, b, c, d \in F,\ ad - bc \neq 0 \right\},$$

where the usual conventions when dealing with ∞, 0 and fractions apply here. It is easy to see that projectivities preserve cross-ratios.

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It was Schreier and Sperner who first proved in [SS35, page 191] that every permutation of $F \cup \{\infty\}$, where F is a field of characteristic $\neq 2$, preserving harmonic quadruples, i.e.

$$[a, b; c, d] = -1 \Longrightarrow [f(a), f(b); f(c), f(d)] = -1,$$

is an element of

$$P\Gamma L_2(F) = \left\{ \frac{ax^\sigma + b}{cx^\sigma + d} : a, b, c, d \in F,\ ad - bc \neq 0,\ \sigma \in Aut(F) \right\}.$$

Any result in this spirit is now called a *von Staudt theorem*.

Over the years this theorem was generalised by relaxing the assumptions on F, for instance for F a skew-field or a ring with some additional assumptions, see the introduction in [Hav15] for a survey of results in that direction.

In this paper, we follow a generalisation in a different direction. Hoffman eliminated the restriction on the characteristic of the field and replaced -1 with any field element which is fixed by $Aut(F)$, see [Hof51]. Our main result is corollary 2·16 from the text:

THEOREM. *Let F be a field, k its prime field and $\emptyset \neq O \subseteq F \setminus \{0, 1\}$ which is $Aut(F)$-invariant. If*

(i) *$k(O) \subsetneq F$ and*
(ii) *if $\mathrm{char}(F) = 2$ then F is perfect and there exists an $Aut(F)$-invariant subfield $k \subsetneq L \subsetneq F$,*

then the subgroup of permutations f of $F \cup \{\infty\}$ satisfying

$$[a, b; c, d] \in O \Longleftrightarrow [f(a), f(b); f(c), f(d)] \in O$$

is exactly $P\Gamma L_2(F)$.

The motivation for seeking such a generalisation came from infinite symmetric groups, and some model theory. It is well known that for a cardinal κ, the closed subgroups (in the product topology) of the infinite symmetric group S_κ, for a cardinal κ, in the product topology correspond exactly to automorphisms groups of first-order structures. Thus, finding closed supergroups of such groups sheds light on the the first order theory of such structures.

In [KS16], the second author and Pierre Simon proved that the affine groups $AGL_n(\mathbb{Q})$ (for $n \geq 2$) and the projective linear groups $PGL_n(\mathbb{Q})$ (for $n \geq 3$) are maximal closed in S_ω. They ask whether it is true that $P\Gamma L_2(F)$ is maximal closed, for an algebraically closed field F of transcendence degree greater that 1. The aim of this paper is to step towards answering this question.

Bogomolov and Rovinsky proved that $P\Gamma L_n(F)$ is maximal closed for $n \geq 3$ and any field F, see [BR13]. The reason for the distinction between $n = 2$ and $n \geq 3$ is that by the fundamental theorem of projective geometry, $P\Gamma L_n(F)$ (for $n \geq 3$) is exactly the collineation group of $\mathbb{P}^{n-1}(F)$. On the other hand, for $n = 2$, since $\mathbb{P}^1(F)$ is the projective line, all the points are collinear.

If $P\Gamma L_2(F)$ were not maximal closed, a proper supergroup of it must preserve one out of a known family of relations, two of them being quaternary relations (see [KS16] for details). The aim is to show that it can not preserve any member of this family of relations. In this paper, using the above theorem, we conclude that if F is an algebraically closed

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field of transcendence degree at least 1, then any group of permutation of $F \cup \{\infty\}$ properly containing $P\Gamma L_2(F)$ is 4-transitive and hence does not preserve any proper quaternary relation.

2. *Proofs*

Definition 2·1. Let f be a permutation of $F \cup \{\infty\}$ and $\emptyset \neq O \subseteq F \setminus \{0, 1\}$. We say that f is *O-preserving* if

$$[a, b; c, d] \in O \Longleftrightarrow [f(a), f(b); f(c), f(d)] \in O,$$

where a, b, c, d are distinct elements from $F \cup \{\infty\}$.

Remark 2·2. In this paper the cross-ratio is only taken for distinct points so that it takes values in $F \setminus \{0, 1\}$ (one can expand the definition to allow repetitions, but this will not be used).

Throughout we will implicitly use the following property of the cross-ratio:

> For any two quadruples of distinct elements of $F \cup \{\infty\}$, $\{A, B, C, D\}$ and $\{A, B, C, X\}$, the following holds:
> $$[A, B; C, D] = [A, B; C, X] \Longleftrightarrow D = X.$$

PROPOSITION 2·3. *For every $x \in F \setminus \{0, 1\}$ there exists a unique function*

$$g_x : X \longrightarrow F \cup \{\infty\},$$

where $X \subseteq (F \cup \{\infty\})^3$ is the set of triples of distinct elements, such that for every distinct $a, b, c \in F \cup \{\infty\}$, $\left[a, b; c, g_x(a, b, c)\right] = x$. The function g_x satisfies:

(i) *$g_x(a, b, c) \neq a, b, c$ and*
(ii) *the map $x \mapsto g_x(a, b, c)$ is injective.*

Furthermore, if f is an O-preserving permutation of $F \cup \{\infty\}$, for some $\emptyset \neq O \subseteq F \setminus \{0, 1\}$, then for every distinct $a, b, c \in F \cup \{\infty\}$ there exists a permutation $\alpha : O \to O$, such for every $x \in O$

$$f\left(g_x(a, b, c)\right) = g_{\alpha(x)}\left(f(a), f(b), f(c)\right).$$

Proof. The function g_x is uniquely determined since, by the definition of the cross-ratio, we get the following formula:

$$g_x(a, b, c) = \frac{b(c-a) - ax(c-b)}{(c-a) - x(c-b)}.$$

Properties (i) and (ii) follow easily.

As for the furthermore, for $x \in O$, define

$$\alpha(x) := \left[f(a), f(b); f(c), f(g_x(a, b, c))\right] \in O$$

and likewise

$$\alpha^{-1}(x) := \left[a, b; c, f^{-1}(g_x(f(a), f(b), f(c)))\right] \in O.$$

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They are both elements of O since f is O-preserving, and note that by uniqueness, the definition of α gives that

$$f(g_x(a,b,c)) = g_{\alpha(x)}(f(a), f(b), f(c)).$$

Claim. For every $a, b, c, y \in F \cup \{\infty\}$ distinct, if $[a,b;c,y] \in O$ then

$$\alpha([a,b;c,y]) = [f(a), f(b); f(c), f(y)]$$

and if $[f(a), f(b); f(c), y] \in O$ then

$$\alpha^{-1}([f(a), f(b); f(c), y]) = [a,b;c,f^{-1}(y)].$$

Proof. Assume that $[a,b;c,y] = x \in O$. By uniqueness necessarily

$$y = g_x(a,b,c).$$

It now follows that

$$\alpha([a,b;c,y]) = \alpha(x) = [f(a), f(b); f(c), f(g_x(a,b,c))] = [f(a), f(b); f(c), f(y)],$$

as required.

The proof for α^{-1} is similar.

We may now compute

$$\begin{aligned}(\alpha \circ \alpha^{-1})(x) &= \alpha\left([a,b;c,f^{-1}(g_x(f(a), f(b), f(c)))]\right)\\ &= [f(a), f(b); f(c), g_x(f(a), f(b), f(c))] = x.\end{aligned}$$

Similarly we get that $(\alpha^{-1} \circ \alpha)(x) = x$, as needed.

Remark 2·4. The previous proposition is obviously also true if we permute the coordinates of the cross-ratio, e.g. consider a function h_x which guarantees that

$$[a,b;h_x(a,b,c),c] = x.$$

In Corollary 2·6, Lemma 2·7, Lemma 2·8 and Proposition 2·9 we work under the following assumption and notation:

ASSUMPTION 2·5. Let O be a non-empty subset of $F \setminus \{0,1\}$, f an O-preserving permutation of $F \cup \{\infty\}$ which fixes $\{0,1,\infty\}$ pointwise.

Furthermore, denote by $K := k(O)$ the field generated by the elements of O, where k is the prime field.

COROLLARY 2·6. (Assumption 2·5) *For all $a \neq b \in F$ there exist permutations $\tau_{a,b}$, $\rho_{a,b}$, $\chi_{a,b}$, $\alpha_{a,b}$, $\beta_{a,b} : O \to O$, such that for every $x \in O$:*

$$f(ax + b(1-x)) = f(a)\tau_{a,b}(x) + f(b)(1 - \tau_{a,b}(x)),$$

$$f\left(\frac{a-(1-x)b}{x}\right) = \frac{f(a) - (1-\rho_{a,b}(x))f(b)}{\rho_{a,b}(x)},$$

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$$f\left(\frac{a-xb}{1-x}\right)=\frac{f(a)-\chi_{a,b}(x)f(b)}{1-\chi_{a,b}(x)},$$

$$f\left(\frac{abx-bx-ab+a}{ax-x-b+1}\right)=\frac{f(a)f(b)\alpha_{a,b}(x)-f(b)\alpha_{a,b}(x)-f(a)f(b)+f(a)}{f(a)\alpha_{a,b}(x)-\alpha_{a,b}(x)-f(b)+1},$$

$$f\left(\frac{a-b-abx+bx}{a-b-ax+x}\right)=\frac{f(a)-f(b)-f(a)f(b)\beta_{a,b}(x)+f(b)\beta_{a,b}(x)}{f(a)-f(b)-f(a)\beta_{a,b}(x)+\beta_{a,b}(x)}.$$

(For $\alpha_{a,b}$ and $\beta_{a,b}$ we also require that $a, b\neq 1$.)

Moreover, $f\restriction O$ is a permutation of O.

Proof. We apply Proposition 2·3. Let $a\neq b\in F$. For $\tau_{a,b}$ use the identity

$$[ax+b(1-x), a; b, \infty]=x.$$

For $\rho_{a,b}$ use the identity

$$\left[a, \frac{a-(1-x)b}{x}; b, \infty\right]=x.$$

For $\chi_{a,b}$ use the identity

$$\left[a, b; \frac{a-xb}{1-x}, \infty\right]=x.$$

For $\alpha_{a,b}$ use the identity

$$\left[b, a; 1, \frac{abx-bx-ab+a}{ax-x-b+1}\right]=x.$$

For $\beta_{a,b}$ use the identity

$$\left[b, 1; a, \frac{a-b-abx+bx}{a-b-ax+x}\right]=x.$$

In order to show that $f\restriction O$ is a permutation of O, note that $[a, 1, 0, \infty]=a$ for every $a\in F\setminus\{0, 1\}$.

LEMMA 2·7. (Assumption 2·5) *For every $a, b\in K$ and $x\in O$*

$$f(a)+(1-x)f(b)\in f(K) \textit{ and}$$

$$xf(a)+f(b)\in f(K).$$

Proof. We start with the first assertion, so let $a, b\in K$ and $x\in O$. If $b=0$ there is nothing to show. If $a=0$ and $b\neq 0$, then since $\tau_{0,b}$ is a permutation, by Corollary 2·6,

$$(1-x)f(b)=(1-(\tau_{a,b}\circ\tau_{a,b}^{-1})(x))f(b)=f((1-\tau_{a,b}^{-1}(x))b)\in f(K).$$

We may thus assume that $a, b\neq 0$ and let $x_2=\rho_{a,0}^{-1}(x)$, so $f(a/x_2)=f(a)/x$. If $b=a/x_2$ then

$$f(a)+f(b)-xf\left(\frac{a}{x_2}\right)=f(b)\in f(K).$$

Now, assume that $b \neq a/x_2$ and let $x_3 = \tau^{-1}_{a/x_2,b}(x)$. Hence

$$f\left(\frac{a}{x_2}x_3 + b(1-x_3)\right) = f\left(\frac{a}{x_2}\right)\tau_{a/x_2,b}(x_3) + f(b)(1-\tau_{a/x_2,b}(x_3))$$
$$= f(a) + f(b) - xf(b).$$

Now the second assertion. If $a = 0$ there is nothing to show. If $a \neq 0$ and $b = 0$ then since $\tau_{a,b}$ is a permutation, $xf(a) \in f(K)$. We may thus assume that $a, b \neq 0$ and let $x_2 = \chi^{-1}_{b,0}(x)$. If $a = b/(1-x_2)$ then

$$xf\left(\frac{b}{1-x_2}\right) + f(b) = \frac{x}{1-x}f(b) + f(b) = f(a) \in f(K).$$

Finally, assume that $a \neq b/(1-x_2)$, and let $x_3 = \tau^{-1}_{a,b/(1-x_2)}(x)$. Hence

$$f\left(ax_3 + \frac{b}{1-x_2}(1-x_3)\right) = f(a)\tau_{a,b/(1-x_2)}(x_3) + f\left(\frac{b}{1-x_2}\right)(1-\tau_{a,b/(1-x_2)}(x_3))$$
$$= f(a)x + f(b).$$

LEMMA 2·8. (Assumption 2·5) *For every $0 \neq a \in K$ and $x \in O$,*

$$-f(a)^2x + f(a)x + f(a) \in f(K) \text{ and}$$
$$1 + x - \frac{x}{f(a)} \in f(K).$$

Proof. Let $x \in O$ and $a \in K$ with $a \neq 0$. If $a = 1$ both assertions are trivial, so assume $a \neq 1$.

We start with the first assertion. Since $\tau_{a,0}$ is a permutation, by Corollary 2·6, we may define $x_1 = \tau^{-1}_{a,0}(x)$ so, $f(x_1a) = xf(a)$. We aim to use the permutation α_{a,x_1a}. Obviously, $x_1a \neq a$ and if $x_1a = 1$ then $xf(a) = 1$ and $-f(a)^2x + f(a)x + f(a) = 0$. Thus, by Corollary 2·6, α_{a,x_1a} is a permutation. Let $x_2 := \alpha^{-1}_{a,x_1a}(x)$, so

$$f\left(\frac{a(x_1a)x_2 - (x_1a)x_2 - a(x_1a) + a}{ax_2 - x_2 - (x_1a) + 1}\right)$$
$$= \frac{f(a)f(x_1a)x - f(x_1a)x - f(a)f(x_1a) + f(a)}{f(a)x - x - f(x_1a) + 1} = -f(a)^2x + f(a)x + f(a).$$

Now for the second assertion. Since $f \upharpoonright O$ is a permutation, we may define $x_1 := f^{-1}(x)$. Note that, since $x \in O$, necessarily $x \neq 1$ and thus, since f is permutation fixing 1, $x_1 \neq 1$ as well. We aim to use the permutation β_{a,x_1}. If $a = x_1$ then the statement is obviously true, so we may assume that $a \neq x_1$ (and both not equal to 1). By Corollary 2·6, β_{a,x_1} is a permutation, so we may define $x_2 := \beta^{-1}_{a,x_1}(x)$ and so

$$f\left(\frac{a - x_1 - ax_1x_2 + x_1x_2}{a - x_1 - ax_2 + x_2}\right) = \frac{f(a) - f(x_1) - f(a)f(x_1)x + f(x_1)x}{f(a) - f(x_1) - f(a)x + x}$$
$$= 1 + x - \frac{x}{f(a)}.$$

PROPOSITION 2·9. *Using Assumption 2·5 and if when* $\mathrm{char}(F) = 2$ *we assume further that O is closed under taking square-roots, then $f(K) = K$.*

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Proof. We first show that $K \subseteq f(K)$. Note that $O \subseteq f(K)$, indeed $f \restriction O$ is a permutation by Corollary 2·6.

Let $a, b \in K$ and $x \in O$, which exists since O is non-empty. By Lemma 2·7, $f(a) + f(b) - xf(b) \in f(K)$. By the same lemma

$$f(a) + f(b) = xf(b) + (f(a) + f(b) - xf(b)) \in f(K).$$

In order to show that if $f(c) \in f(K)$ then also $-f(c)$, first notice that by considering $\rho_{0,c}$ in Corollary 2·6 we see that

$$-\frac{1-x}{x} f(c) \in f(K).$$

By considering $\chi_{d,0}$ in the same corollary, we see that

$$\frac{f(d)}{1-x} \in f(K),$$

for any $d \in K$. In particular $-f(c)/x \in f(K)$, for any $c \in K$. Similarly, by considering $\tau_{a,0}$, we get that $xf(c) \in f(K)$ for all $c \in K$ and together $-f(c) \in f(K)$ for all such c, as needed.

As for the multiplication, by Lemma 2·8,

$$-f(a)^2 x + f(a)x + f(a) \in f(K),$$

for every $x \in O$. Using the above, and since $f(a),\ f(a)x \in f(K)$,

$$-f(a)^2 x \in f(K).$$

So $-f(a)^2 x = f(b)$ for some $b \in K$. Since $-f(b)/x \in f(K)$ as well, $f(a)^2 \in f(K)$.

Let $a \neq 0 \in K$, by Lemma 2·8, $1 + x - x/f(a) \in f(K)$ for any $x \in O$. Using the above, and since $1, x \in f(K)$,

$$\frac{x}{f(a)} \in f(K).$$

Using a similar argument to the previous paragraph, $-1/f(a) \in f(K)$, so $1/f(a) \in f(K)$.

Finally, we first assume that $\mathrm{char}(F) \neq 2$. Let $a, b \in K$. Since $(f(a) + f(b))^2 \in f(K)$ we get that

$$2f(a)f(b) \in f(K).$$

To get that $f(K)$ is a subfield, we need this final claim:

Claim. If $a \in K$ then $f(a)/2 \in f(K)$.

Proof. We may assume that $a \neq 0$. Since $f(a) \in f(K)$ then $1/f(a) \in f(K)$ and so also $2/f(a)$. Take the inverse again and $f(a)/2 \in f(K)$.

We conclude that if $\mathrm{char}(F) \neq 2$, $f(K)$ is a field and so $K \subseteq f(K)$.

Assume that $\mathrm{char}(F) = 2$. For any $n \geq 0$, let

$$(f(K))^{2^n} := \{a^{2^n} : a \in f(K)\}.$$

Note that since $f(K)$ is closed under squares, $\{(f(K))^{2^n} : n \geq 0\}$ forms a decreasing sequence under inclusion. The following is an easy observation:

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Claim. For every $n \geq 0$, $(f(K))^{2^n}$ is also closed under addition, additive and multiplicative inverses and taking square powers.

Consider $L := \bigcap_{n\geq 0}(f(K))^{2^n}$.

Claim. L is a field containing O. Therefore, $K \subseteq L \subseteq f(K)$.

Proof. Since O is closed under square-roots it is contained in L and by the previous claim L is closed under addition, additive and multiplicative inverses and taking square-powers. Let $a, b \in L$ and let $n \geq 0$. We will show that $ab \in (f(K))^{2^n}$. Since $a, b \in (f(K))^{2^{n+1}}$, by the following form of Hua's identity (first mentioned in [**Hua49**] but we use the more manageable form from [**Jac68**, page 2]):

$$a - (a^{-1} + (b^{-2} - a)^{-1})^{-1} = a^2 b^2$$

and, by the last claim, $a^2b^2 \in (f(K))^{2^{n+1}}$. Since the Frobenius map is injective, $ab \in (f(K))^{2^n}$, as required.

Either way, $K \subseteq f(K)$, but since f^{-1} is also O-preserving by definition, we actually have $K \subseteq f(K) \subseteq K$ as required.

From now on we no longer assume Assumption 2·5.

Definition 2·10. For an $Aut(F)$-invariant subfield $K \subseteq F$, a *K-chain* is an image of $K \cup \{\infty\}$ under the action of an element of $P\Gamma L_2(F)$.

Remark 2·11. The term *K-chain* is due originally to von Staudt who introduced it for any real subline of the complex projective line. Note that the usual definition of K-chain is for any subfield $K \subseteq F$ and only images under the action of $PGL_2(F)$ (see [**Her95**, definition 2·2·2]). Naturally, these definitions are equivalent for $Aut(F)$-invariant subfields. See [**Her95**] for more on this subject, and in a higher level of generality.

We recall that an action of a group G on a set X (with $|X| \geq k$) is *k-transitive* if G acts transitively on the set of k-tuples of distinct elements of X. For example, the action of the group $PGL_2(F)$ on $F \cup \{\infty\}$ is 3-transitive.

COROLLARY 2·12. *Let f be an O-preserving permutation of $F \cup \{\infty\}$, for some non-empty $O \subseteq F \setminus \{0, 1\}$ which is $Aut(F)$-invariant. If, when* $\mathrm{char}(F) = 2$, *we further assume O is closed under taking square-roots, then f sends K-chains to K-chains, where $K = k(O)$.*

Proof. Let $X = T(K \cup \{\infty\})$ be a K-chain, where $T \in P\Gamma L_2(F)$. Since O is $Aut(F)$-invariant, $f \circ T$ is also O-preserving. Thus we may assume that $X = K \cup \{\infty\}$. Since $PGL_2(F)$ is 3-transitive and preserves the cross-ratio, by composing with an element of $PGL_2(F)$ we may assume that f fixes $\{0, 1, \infty\}$ pointwise. By Proposition 2·9, $f(K \cup \{\infty\}) = K \cup \{\infty\}$ as needed.

We recall some definitions from affine geometry. Given a K-vector space V, an *affine line* is a set of the form $Ka + b$ for some $a, b \in V$ with $a \neq 0$. A map $T : V \to V$ is called

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semilinear if there exists a field automorphism $\sigma \in Aut(K)$ such that for all $v, u \in V$ and $x, y \in K$

$$T(xv + yu) = \sigma(x)T(v) + \sigma(y)T(u).$$

FACT 2·13 (The Fundamental Theorem of Affine Geometry). [**BR98**, theorem 3·5·6] Let K be any field and V a K-vector space of dimension at least 2. If f a permutation of V sending affine lines to affine lines then there exists a semilinear map $T : V \to V$ and $b \in V$ such that $f(x) = T(x) + b$ for all $x \in V$.

The following theorem is a generalisation of the main theorem in [**Hof51**].

THEOREM 2·14. *Let $k \subsetneq F$ be a field, where k is the prime field of F, and let f be a permutation of $F \cup \{\infty\}$.*

If $\mathrm{char}(F) = 2$ *we assume further that*:

(a) *F is perfect and*

(b) *there exists an $Aut(F)$-invariant subfield $k \subsetneq L \subsetneq F$.*

Then the following are equivalent:

(i) $f \in P\Gamma L_2(F)$;

(ii) *f is O-preserving for all non-empty $O \subseteq F \setminus \{0, 1\}$ which are $Aut(F)$-invariant and satisfy $k(O) \subsetneq F$*;

(iii) *f is O-preserving for some non-empty $O \subseteq F \setminus \{0, 1\}$ which is $Aut(F)$-invariant and satisfies $k(O) \subsetneq F$*;

(iv) *There exists an $Aut(F)$-invariant subfield $K \subsetneq F$ such that f sends K-chains to K-chains*;

(v) *For all $Aut(F)$-invariant subfields $K \subsetneq F$, f sends K-chains to K-chains.*

Remark 2·15. The implication (iv) $\Rightarrow$ (i) is well known for fields F with $\mathrm{char}(F) \neq 2$, see [**Her95**, theorem 9·2·5], but we provide a direct proof.

Proof. (i) $\Rightarrow$ (ii). This is by the definition of $P\Gamma L_2(F)$.

(ii) $\Rightarrow$ (iii). If $\mathrm{char}(F) \neq 2$ just take $O = k \setminus \{0, 1\}$ (which is non-empty). If $\mathrm{char}(F) = 2$, then take $O = L \setminus \{0, 1\}$, where L is the given $Aut(F)$-invariant subfield.

(iii) $\Rightarrow$ (iv). Let $K = k(O)$. Since O is $Aut(F)$-invariant, K is $Aut(F)$-invariant and if $\mathrm{char}(F) = 2$ then O is closed under taking square-roots since F is perfect and hence the inverse of the Frobenius map is an automorphism. Now apply Corollary 2·12.

(iv) $\Rightarrow$ (i). By composing with an element of $PGL_2(F)$, we may assume that f fixes $\{0, 1, \infty\}$ pointwise. We plan to use Fact 2·13, so we must show that, in F as a K-vector space, $f \restriction F$ sends affine lines to affine lines. Since f fixes $\{\infty\}$, and sends K-chains to K-chains, it is sufficient to show the following, where by a *projective affine line* we mean a union of an affine line with $\{\infty\}$,

Recall also that any K-chain is equal to some $T(K \cup \{\infty\})$, where $T(x)$ is of the form

$$\frac{ax^\sigma + b}{cx^\sigma + d},$$

for $a, b, c, d \in F$, with $ad - bc \neq 0$, and $\sigma \in Aut(F)$. Since K is $Aut(F)$-invariant we may assume that $\sigma = id$.

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Claim. A subset of $F \cup \{\infty\}$ is a K-chain which includes ∞ if and only if it is a projective affine line.

Proof. A projective affine line has the form $a(K \cup \{\infty\}) + b$, for $a, b \in F$, so it is a K-chain. For the other direction, by translation it is enough to show that any K-chain containing 0 and ∞ is a projective affine line.

Note that projective affine lines that contain 0 are just of the form aK (for $a \neq 0$), and that both families of projective affine lines containing 0 and K-chains containing 0 and ∞ are closed under scalar multiplication (by non-zero elements from F) and inverse ($x \mapsto 1/x$). So it is enough to show that after applying finitely many operations of the form above on a K-chain containing 0 and ∞ gives a projective affine line (containing 0). Assume that we are given a K-chain of the form $T(K \cup \{\infty\})$ for T as above which contains 0 and ∞.

(i) If T is of the form $ax + b$ with $a \neq 0$, then $b = 0$ and we are done.
(ii) If T is of the form $b/(cx + f)$ with $c \neq 0$, then we are done by the first bullet (after dividing by b and taking inverse).
(iii) If the T is of the form $(ax + b)/(cx + d)$ with $a, c \neq 0$, then after multiplying by c/a, we may assume that $a = c = 1$, and then since the chain contains 0, $b \in K$, and since the chain contains ∞, $d \in K$. So it is equal to $(K \cup \{\infty\})$.

As a result, f preserves the system of affine lines in the K-vector space F. Since $K \subsetneq F$, $\dim_K F \geq 2$ so by the fundamental theorem of affine geometry (Fact 2·13) and since $f(0) = 0$, f must be additive and so also $f(-a) = -f(a)$ for all $a \in F$.

The conjugation of f by the $PGL_2(F)$ map $x \mapsto 1/x$ also satisfies the above, so it is also additive. This translates to

$$\frac{f(a)f(b)}{f(a) + f(b)} = f\left(\frac{ab}{a+b}\right),$$

for all nonzero $a, b \in F$. By setting in the equation $a = 1$ and $b = t - 1$ (for $t \neq 1$) we obtain, $f(t)f(t^{-1}) = 1$, thus f commutes with inversion.

Putting in the same equation $b = 1 - a$, for $a \neq 0, 1$, we obtain $f(a)f(1-a) = f(a(1-a))$, which gives $f(a^2) = f(a)^2$.

If $\mathrm{char}(F) \neq 2$ then, since f is additive, $f(x/2) = f(x)/2$ for all $x \in F$. Set $a = x + y$ in the last equation to get $f(xy) = f(x)f(y)$ for all $x, y \in F$.

If $\mathrm{char}(F) = 2$ we once again use Hua's identity:

$$f(a^2b^2) = f\left(a - (a^{-1} + (b^{-2} - a^{-1})^{-1}\right)$$

$$= f(a) - (f(a)^{-1} + (f(b)^{-2} - f(a)^{-1})^{-1}) = f(a^2)f(b^2).$$

Hence $f(ab)^2 = (f(a)f(b))^2$, so $f(ab) = f(a)f(b)$.

Either way, we get that f is an automorphism of F, an in particular $f \in P\Gamma L_2(F)$.

(i) $\Rightarrow$ (v) is clear and (v) $\Rightarrow$ (iv) follows by taking $K = k$.

As a direct corollary of Theorem 2·14 we get the following.

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COROLLARY 2·16. *Let F be a field, k its prime field and $\emptyset \neq O \subseteq F \setminus \{0, 1\}$ which is $Aut(F)$-invariant. If*:

(i) *$k(O) \subsetneq F$ and*

(ii) *if* $\mathrm{char}(F) = 2$ *then F is perfect and there exists an $Aut(F)$-invariant subfield $k \subsetneq L \subsetneq F$,*

then the subgroup of O-preserving permutations of $F \cup \{\infty\}$ is exactly $P\Gamma L_2(F)$.

It was shown by Hoffman in [**Hof51**], that if F is a field, $a \in F \setminus \{0, 1\}$ and f is an $\{a\}$-preserving permutation of $F \cup \{\infty\}$ then $f \in P\Gamma L_2(F)$. One may ask, what about if f preserves a set of cardinality larger than 1? Theorem 2·14 only gives a partial answer. More specifically, can the proper containment of fields assumptions be dropped? Can the perfectness assumption when $\mathrm{char}(F) = 2$ be dropped? For example:

Question 2·17. Does the subgroup of permutations of $\mathbb{Q}(\sqrt{2}) \cup \{\infty\}$ which are $\{\pm\sqrt{2}\}$-preserving properly contain $P\Gamma L_2(\mathbb{Q}(\sqrt{2}))$?

3. *Every proper extension of $P\Gamma L_2(F)$ is* 4-*transitive*

Our final aim is to show that, as a corollary of Theorem 2·14, any group of permutations of $F \cup \{\infty\}$, for F algebraically closed of transcendence degree at least 1, which properly contains $P\Gamma L_2(F)$ must be 4-transitive and as a result does not preserve any non-trivial 4-relation.

LEMMA 3·1. *Let $\{O_i\}_{i\in I}$ be the orbits of $Aut(F)$ acting on $F \setminus \{0, 1\}$. Then the orbits of $P\Gamma L_2(F)$ acting on quadruples of distinct elements from $F \cup \{\infty\}$ are*

$$\{(a, b, c, d) : [a, b; c, d] \in O_i\}_{i\in I}.$$

Proof. Let $T \circ \sigma \in P\Gamma L_2(F,)$, for $T \in PGL_2(F)$ and $\sigma \in Aut(F)$. Since elements of $PGL_2(K)$ preserve the cross-ratio, and $\sigma(O_i) = O_i$ by definition, $P\Gamma L_2(F)$ preserves the orbits.

Now, let (x, y, z, w), (a, b, c, d) be quadruples of distinct elements such that

$$[x, y; z, w], [a, b; c, d] \in O_i.$$

By applying an element of $Aut(F)$ we may assume that $[x, y; z, w] = [a, b; c, d]$.

Since $PGL_2(F)$ is 3-transitive, there exists $T \in PGL_2(K)$ such that $T(x) = a$, $T(y) = b$, $T(z) = c$. So we have that

$$[a, b; c, T(w)] = [a, b; c, d].$$

As a, b, c are distinct we have that $T(w) = d$.

THEOREM 3·2. *Let F be an algebraically closed field of transcendence degree at least* 1, *and H be a group of permutations of $F \cup \{\infty\}$ properly containing $P\Gamma L_2(F)$. Then H is* 4-*transitive.*

Proof. By Lemma 3·1, the action of $P\Gamma L_2(F)$ breaks the space of quadruples of distinct elements from $F \cup \{\infty\}$ into infinitely many finite orbits (corresponding to finite Galois orbits) and one infinite orbit (corresponding to the Galois orbit of transcendentals).

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Thus it is enough to show that every orbit of the action of H on the space of quadruples of distinct elements from $F \cup \{\infty\}$ intersects the orbit corresponding to the transcendentals.

Aiming for a contradiction, assume there exists an orbit X of the action of H, which only contains orbits with algebraic cross-ratio, i.e.

$$X = \bigcup_{i \in I_0} \{(a, b, c, d) : [a, b; c, d] \in O_i\},$$

for some $I_0 \subseteq I$ and O_i finite, where I and O_i are as in Lemma 3·1. Let $O = \bigcup_{i \in I_0} O_i$ be the cross-ratios arising from quadruples from X and let $K = k(O)$, where k is the prime field. Note that $K \subsetneq F$ and that O is $Aut(F)$ invariant. By assumption every element of H is O-preserving and thus by Corollary 2·16, $H \subseteq P\Gamma L_2(F)$, contradiction.

Question 3·3. What about other fields? For instance, is it true that every group of permutations of $\mathbb{Q}(\sqrt{2}) \cup \{\infty\}$ properly containing $P\Gamma L_2(\mathbb{Q}(\sqrt{2}))$ must be 4-transitive?

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