ANGULAR DEPENDENCE OF ATMOSPHERIC TURBULENCE EFFECT IN SPECKLE INTERFEROMETRY

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ABSTRACT

The concepts of speckle interferometry as developed by Labeyrie, and of speckle imagery as formulated by Knox and Thompson are analyzed for dependence on field-of-view size. The preliminary analysis, assuming isoplanatism rederives the results of Korff, and derives the result previously inferred by Knox and Thompson from computer simulation, that allowable spatial frequency separation for difference of phase shift determination must be less than r_0/λ When the assumption of isoplanatism is dropped, results are obtained for the expected object power spectrum in speckle interferometry and for the expected bispectrum in speckle imagery, showing the dependence on angular spread for an object consisting of a pair of point sources. An angle, ϑ_0 , is defined (in terms of an integral over the strength of turbulence distribution along the propagation path), which bounds the range within which there are no significant anisoplanatism effects. It is noted that the effect of anisoplanatism is not to attenuate the information bearing signal but rather to impose incorrect information on the signal. Thus anisoplanatism can result in incorrect conclusions with no indication that there is a problem.

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1. INTRODUCTION AND SUMMARY

The work reported here had its origin in a suspicion that the speckle imagery concept, as formulated by Knox and Thompson[1] might be immune to the anisoplanatism problems of the Labeyrie[2] speckle interferometry technique. We reasoned that for a pair of point sources (stars) that are fully resolved and in separate isoplanatic patches, the Labeyrie technique has anisoplanatism problems because, in generating the correlation function of the composite source it suppresses position information. We know that by more conventional techniques, at least initially paying attention to the position of each part of the fully resolved image, we could generate the correlation function with no difficulty. We thought that, perhaps, since the Knox-Thompson method retains position information it might have the inherent ability to get around anisoplanatism problems.

This line of reasoning was inspired by a report[3] of an isoplanatic field-of-view for the Knox-Thompson algorithim that was much larger than we had estimated should be expected, based on existing models of the vertical distribution of turbulence.[4,5] Our estimate of the expected isoplanatic patch size was based on analysis for an adaptive optics system[6,7] and on a very cursory analysis of anisoplanatism effects for the Labeyrie speckle interferometry technique.

Responding to these considerations we undertook a very detailed analysis of atmospheric turbulence effects for speckle interferometry, (i.e., for the Labeyrie technique), and for speckle imagery, (i.e., for the Knox-Thompson algorithim) — which work we shall review here, in a once over lightly manner. We analyzed speckle interferometry initially for a single point source, obtaining results equivalent to Korff's showing that the high spatial frequencies do survive the recording and data processing at a non-trivial level. We then repeated this analysis for a pair of point sources assumed to be in the same isoplanatic patch. This not only showed the same, unaltered "survival" of the high spatial frequencies, but significantly allowed us to complete the basic set of mathematical tools which we needed for the anisoplanatism effects analysis. We then repeated the analysis of the speckle interferometry technique, but this time with a pair of point sources that were not in the same isoplanatic patch. The results of this analysis showed that the isoplanatic patch size for speckle interferometry is essentially the same as that for adaptive optics. But, somewhat surprisingly, the results indicated that anisoplanatism does not "wash out" the high spatial frequency content of the correlation function — rather, it has the effect of garbling the information contained in the correlation function. Thus the correlation function for a pair of equal intensity stars will, when there is an anisoplanatism problem, appear to represent a binary with unequal intensity stars!

We then switched to the analysis of speckle imagery. We started with the analysis of a pair of point sources, assumed to be close enough together that there were no anisoplanatism effects. In this analysis we were able to show exactly why the Knox-Thompson algorithim works, and were able to demonstrate analytically that the allowable spatial frequency separation between adjacent frequencies should not be greater than $0.427\ r_0/\lambda$. We then repeated our analysis of this problem, but without the assumption that the two point sources were in the same isoplanatic patch. This analysis showed that there was an effect due to anisoplanatism and that the isoplanatic patch size is essentially the same as for adaptive optics and for speckle imagery. But here again, we noticed that the effect of anisoplanatism was not to eliminate the contribution of the high spatial frequency content of the object from the image — but rather what anisoplanatism's effect was, was to impose spurious modulation on the recovered data. Thus, the recovered image would have fine details, but much of that detail might be spurious, i.e., have little or nothing to do with the object.

To return now to our starting point, what can we make of the fact that the isoplanatic patch size for speckle imagery appears to be as large as it is, while our model of the vertical distribution of the optical strength of turbulence predicts a smaller isoplanatic patch size. We know from our analysis that the problem is not due to applying inappropriate theory. The theoretical results for the isoplanatic patch size are essentially the same for adaptive optics, for speckle interferometry, and for speckle imagery. We see two possible explanations. First, our models for the vertical distribution of turbulence may be in error. There may be significantly less turbulence at high altitudes than we now think. Alternatively, our interpretation of the speckle images recovered by use of the Knox-Thompson algorithim may be incorrect. The fine details presented in the images may be at least partially spurious and the image size may not be a valid indication of the isoplanatic patch size. We suspect that the problem lies with our turbulence model, but are far from certain of this.

With this introduction and summary we are now ready to proceed with our analysis. We start with the Labeyrie speckle interferometry technique for a single point source.

2. LABEYRIE SPECKLE INTERFEROMETRY: SINGLE POINT SOURCE

We consider a point source of wavelength λ , and wave number $k = 2\pi/\lambda$, located at an angular position $\vec{\theta}$. This produces a wave function $U(\vec{r})$, where

$$U(\vec{r}) = A \exp [i k \vec{\theta} \cdot \vec{r} + i \phi (\vec{\theta}; \vec{r})] \quad , \tag{1}$$

at aperture position $\vec{r}$. Here $\phi(\vec{\theta} ; \vec{r})$ is the turbulence induced phase shift at position $\vec{r}$ due to propagation along the direction defined by $\vec{\theta}$.

The corresponding focal plane wavefunction will be $u(\vec{\theta})$, where

$$u(\vec{\theta}) = \mathfrak{A} \int d\vec{r}\, W(\vec{r}) \exp(-i\, k\, \vec{\theta} \cdot \vec{r})\, U(\vec{r}) \quad , \tag{2}$$

where $W(\vec{r})$ is a function which serves to define the circular aperture of diameter D , according to the equation

$$W(\vec{r}) = \begin{cases} 1 \ , & \text{if} \quad |\vec{r}| \leq \frac{1}{2} D \\ 0 \ , & \text{if} \quad |\vec{r}| > \frac{1}{2} D \end{cases} \quad , \tag{3}$$

and $\mathfrak{A}$ is a constant of proportionality. The focal plane intensity at the angular position $\vec{\theta}$ is

$$I(\vec{\theta}) = \tfrac{1}{2} |u(\vec{\theta})|^2 \tag{4}$$

The quantities of basic interest to us here are the fourier transform of this intensity

$$S(\vec{f}) = \int d\vec{\theta} \exp(-2\pi\, i\, \vec{f} \cdot \vec{\theta})\, I(\vec{\theta}) \quad , \tag{5}$$

where $\vec{f}$ is a spatial frequency (with units of cycles per radian), and the associated power spectrum

$$\mathcal{J}(\vec{f}) = \langle S^*(\vec{f})\, S(\vec{f}) \rangle \quad . \tag{6}$$

If we combine Eq.'s (2), (4), and (5) and make a multiple integral out of the product of integrals, and then make use of the well known property of the fourier transformation that the repeated fourier integral recovers the starting function, we can obtain the result that

$$S(\vec{f}) = \tfrac{1}{2} |\mathfrak{A}|^2 A^2 \exp(-2\pi i \vec{f} \cdot \vec{\theta}) \int d\vec{r}\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})$$
$$\times \exp\{i[\phi(\vec{\theta}\,;\vec{r}) - \phi(\vec{\theta}\,;\vec{r}+\lambda\vec{f})]\} \quad , \tag{7}$$

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(where we have suppressed a factor of λ^2 in this result by lumping it in with the constant of proportionality, $|\mathfrak{A}|^2$). Now if we substitute Eq. (7) into Eq. (6), make the product of integrals into a double integral and commute the integration and ensemble averaging processes, we get

$$\tilde{\mathscr{I}}(\vec{f}) = \tfrac{1}{4}\,|\mathfrak{A}|^4 (A^2)^2 \iint d\vec{r}\; d\vec{r}'\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f}) \times W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}) \left\langle \exp\{i[\phi(\vec{\theta}\,;\vec{r}) - \phi(\vec{\theta}\,;\vec{r}') + \phi(\vec{\theta}\,;\vec{r}'+\lambda\vec{f}) - \phi(\vec{\theta}\,;\vec{r}+\lambda\vec{f})]\}\right\rangle \quad . \tag{8}$$

This is the quantity of basic interest in the Labeyrie speckle interferometry technique. It is the quantity measured, (after all the image processing), and hopefully contains a significant amount of information about the high spatial frequency in the source. Since we are considering a point source in this case, we would hope that $\tilde{\mathscr{I}}(\vec{f})$ has a substantial value, (unity would be ideal), for all spatial frequencies, at least up to D/λ . The value of $\tilde{\mathscr{I}}(\vec{f})$ that we shall calculate here may be considered, when properly normalized, to be the square of the turbulence limited modulation transfer function for speckle interferometry.

Using the fact that the statistics of the turbulence induced phase fluctuations, $\phi(\vec{\theta}\,;\vec{r})$, are gaussian, we can reduce Eq. (8) to the form

$$\tilde{\mathscr{I}}(\vec{f}) = \tfrac{1}{4}\,|\mathfrak{A}|^4 (A^2)^2 \iint d\vec{r}\; d\vec{r}'\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f}) \times W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}) \exp\{-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}') - \mathscr{D}(\vec{r}-\vec{r}'-\lambda\vec{f}) + \mathscr{D}(\lambda\vec{f}) + \mathscr{D}(\lambda\vec{f}) - \mathscr{D}(\vec{r}-\vec{r}'+\lambda\vec{f}) + \mathscr{D}(\vec{r}-\vec{r}')]\} \quad , \tag{9}$$

where $\mathscr{D}(\vec{\rho})$ is the wave-structure function, defined by the equation

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$$\mathcal{D}(\vec{\rho}) = \langle [\phi(\vec{\theta};\vec{r}+\vec{\rho}) - \phi(\vec{\theta};\vec{r})]^2 \rangle \quad . \tag{10}$$

Since, quite obviously $\mathcal{D}(0) = 0$, we can show that

$$\tilde{\mathcal{J}}(0) = \left(\frac{\pi}{8} |\mathfrak{A}|^2 A^2 D^2\right)^2 \quad . \tag{11}$$

This will serve as a normalization factor allowing us to obtain the modulation transfer function at spatial frequency $\vec{f}$, from $\tilde{\mathcal{J}}(\vec{f})$.

To proceed with our evaluation of $\tilde{\mathcal{J}}(\vec{f})$ we note that the wave-structure function can be written as[9]

$$\mathcal{D}(\vec{\rho}) = 6.88 (\rho/r_0)^{5/3} \quad , \tag{12}$$

where r_0 is a length, which we may call the wavefront distortion coherence diameter. For convenience we write the exponent in Eq. (9) as

$$E = -\tfrac{1}{2}[\mathcal{D}(\vec{r}-\vec{r}') - \mathcal{D}(\vec{r}-\vec{r}'-\lambda\vec{f}) + \mathcal{D}(\lambda\vec{f}) + \mathcal{D}(\lambda\vec{f}) - \mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f}) + \mathcal{D}(\vec{r}-\vec{r}')] \quad . \tag{13}$$

Making use of Eq. (12) and forming power-series expansions as appropriate, it can be shown that

$$E \approx -6.88 (\lambda|\vec{f}|/r_0)^{5/3} \quad , \quad \text{if} \quad \lambda|\vec{f}| \ll |\vec{r}-\vec{r}'| \quad , \tag{14}$$

and

$$E \approx -6.88 (|\vec{r}-\vec{r}'|/r_0)^{5/3} \quad , \quad \text{if} \quad |\vec{r}-\vec{r}'| \ll \lambda|\vec{f}| \quad . \tag{15}$$

Making use of these approximations and noting that we are interested in $\tilde{\mathcal{J}}(\vec{f})$ for values of $|\vec{f}|$ which are much larger than r_0/λ, we can approximate Eq. (9) with the relationship

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$$\tilde{\mathscr{I}}(\vec{f}) \approx \frac{1}{4} |\mathfrak{A}|^4 (A^2)^2 \iint d\vec{r}\, d\vec{r}\,' \, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}\,')\, W(\vec{r}\,'+\lambda\vec{f})$$

$$\times \exp\left[-6.88\left(|\vec{r}-\vec{r}\,'|/r_0\right)^{5/3}\right] \qquad . \qquad (16)$$

The fact that we are interested in spatial frequencies much greater than r_0/λ carries with it the implication that the aperture diameter, D , is much greater than the coherence diameter, r_0 . In view of the rapid fall-off of the exponential with increasing separation between $\vec{r}$ and $\vec{r}'$ in Eq. (16), except when $|\vec{r} - \vec{r}'|$ is much less than r_0 , we can make the approximation that $W(\vec{r})\, W(\vec{r}') = W(\vec{r})$ and that $W(\vec{r} + \lambda\vec{f})\, W(\vec{r}' + \lambda\vec{f}) = W(\vec{r} + \lambda\vec{f})$ in Eq. (16) . Thus we can reduce our result to the form

$$\tilde{\mathscr{I}}(\vec{f}) = \frac{1}{4} |\mathfrak{A}|^4 (A^2)^2 \left\{\int d\vec{r}\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\right\}$$

$$\times \left\{\int d\vec{x} \exp\left[-6.88\left(|\vec{x}|/r_0\right)^{5/3}\right]\right\} \qquad , \qquad (17)$$

where we have made the replacement $\vec{x} = \vec{r} - \vec{r}'$. Both the $\vec{x}$-integration and the $\vec{r}$-integration can be carried out, the latter giving rise to a quantity proportional to $\tau_{DL}(\vec{f})$, the diffraction limited optical transfer function of an aperture of diameter D , operating at wavelength λ , for spatial frequency $\vec{f}$. Thus it can be shown that

$$\frac{\tilde{\mathscr{I}}(\vec{f})}{\tilde{\mathscr{I}}(0)} = 0.435\, (r_0/D)^2\, \tau_{DL}(\vec{f}) \qquad , \quad \text{for } \lambda|\vec{f}| >> r_0 \quad . \qquad (18)$$

Here $\tilde{\mathscr{I}}(0)$ provides the normalization needed so that we can consider this ratio, $\tilde{\mathscr{I}}(\vec{f})/\tilde{\mathscr{I}}(0)$, to be the square of the modulation transfer function for speckle interferometry. This result has previously been obtained by Korff[10]. This result, of course, does not indicate any isoplanatism dependence. For that we would need a pair of point sources to provide some angular spread.

In the next section we treat the two point source version of speckle interferometer performance, but restrict attention to the case where isoplanatism is assumed. This will set the groundwork for our analysis in the subsequent section of the case where anisoplanatism effects are expected.

3. LABEYRIE SPECKLE INTERFEROMETRY: TWO POINT SOURCES: ISOPLANATISM ASSUMED

Proceeding in exactly the same manner as in the preceding section but considering a pair of point sources, (incoherent with respect to each other), located at $\vec{\theta}_j$ and $\vec{\theta}_j{}'$ and having amplitudes A_j and $A_j{}'$, respectively, it can be shown that the spatial frequency fourier transform of the focal plane intensity pattern in any (randomly selected) speckle image will be

$$S(\vec{f}) = \tfrac{1}{2}\,|\mathfrak{U}|^2 \int d\vec{r}\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f})$$
$$\times \Big\{ A_j{}^2 \exp(-2\pi i \vec{f}\cdot\vec{\theta}_j) \exp\{i[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f})]\}$$
$$+ A_{j'}^2 \exp(-2\pi i \vec{f}\cdot\vec{\theta}_{j'}) \exp\{i[\phi(\vec{\theta}_{j'};\vec{r}) - \phi(\vec{\theta}_{j'};\vec{r}+\lambda\vec{f})]\}\Big\} \quad . \qquad (19)$$

Using this expression to evaluate the power spectrum as defined by Eq. (6), we get after the appropriate mathematical manipulations, the result that

$$\mathscr{S}(\vec{f}) = \tfrac{1}{4}\,|\mathfrak{U}|^4 \iint d\vec{r}\, d\vec{r}\,'\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}\,')\, W(\vec{r}\,'+\lambda\vec{f})$$
$$\times \langle (A_j{}^2)^2 \exp\{i[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_j;\vec{r}\,'+\lambda\vec{f}) - \phi(\vec{\theta}_j;\vec{r}\,')]\}$$
$$+ (A_j{}^2)(A_{j'}^2) \exp[-2\pi i \vec{f}\cdot(\vec{\theta}_j - \vec{\theta}_{j'})]$$
$$\times \exp\{i[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_{j'};\vec{r}\,'+\lambda\vec{f}) - \phi(\vec{\theta}_{j'};\vec{r}\,')]\}$$
$$+ (A_j{}^2)(A_{j'})^2 \exp[2\pi i \vec{f}\cdot(\vec{\theta}_j - \vec{\theta}_{j'})]$$
$$\times \exp\{-i[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_j;(\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_j;\vec{r}\,'+\lambda\vec{f}) - \phi(\vec{\theta}_j;\vec{r}\,')]\}$$
$$+ (A_{j'}^2)^2 \exp\{i[\phi(\vec{\theta}_{j'};\vec{r}) - \phi(\vec{\theta}_{j'};\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_{j'};\vec{r}\,'+\lambda\vec{f}) - \phi(\vec{\theta}_{j'};\vec{r}\,')]\}\rangle . \qquad (20)$$

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It is to be noted that this expression contains terms corresponding to "interference" between the two point sources — but in this case it is "interference" of the two random intensity patterns, and not interference between the electromagnetic fields.

To proceed beyond this point we again make use of the fact that the phase fluctuation statistics are gaussian. We introduce the approximation that for the evaluation of this integral the statistics of wavefront distortion can be considered to be isotropic with respect to $\vec{r}$, $\vec{r}'$ (or at least invariant under 180° reversal), without regard for the orientation of the direction of $\vec{\theta}_j - \vec{\theta}_j{}'$ as projected on the $\vec{r}$-plane. This allows us to write

$$\langle[\phi(\vec{\theta}_j ; \vec{r}) - \phi(\vec{\theta}_{j'} ; \vec{r}')]^2\rangle$$

$$\approx \tfrac{1}{2}\{\langle[\phi(\vec{\theta}_j ; \vec{r}) - \phi(\vec{\theta}_{j'} ; \vec{r}')]^2\rangle + \langle[\phi(\vec{\theta}_j ; \vec{r}') - \phi(\vec{\theta}_{j'} ; \vec{r})]^2\rangle\} , \qquad (21)$$

to be substituted into the expression we get when we take the ensemble average of the exponential terms in Eq. (20) , using the fact that the phase statistics are gaussian. Thus we can reduce Eq. (20) to the form

$$\begin{aligned} \mathcal{I}(\vec{f}) = \tfrac{1}{4}\, |\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}) \\ \times \Big\{ [(A_j{}^2)^2 + (A_{j'}{}^2)^2] \exp\Big(-\{\mathcal{D}(\vec{r}-\vec{r}') + \mathcal{D}(\lambda\vec{f}) - \tfrac{1}{2}[\mathcal{D}(\vec{r}-\vec{r}'-\lambda\vec{f}) \\ + \mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f})]\}\Big) + 2(A_j{}^2)(A_{j'}{}^2) \cos[2\pi\vec{f}\cdot(\vec{\theta}_j - \vec{\theta}_{j'})] \exp\{-\mathcal{D}(\lambda\vec{f}) \\ - \tfrac{1}{2}\langle[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_{j'};\vec{r}')]^2\rangle - \tfrac{1}{2}\langle[\phi(\vec{\theta}_j;\vec{r}') - \phi(\vec{\theta}_{j'};\vec{r})]^2\rangle \\ + \tfrac{1}{4}\langle[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_{j'};\vec{r}'+\lambda\vec{f})]^2\rangle + \tfrac{1}{4}\langle[\phi(\vec{\theta}_j;\vec{r}') - \phi(\vec{\theta}_{j'};\vec{r}+\lambda\vec{f})]^2\rangle \\ + \tfrac{1}{4}\langle[\phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f}) - \phi(\vec{\theta}_{j'};\vec{r}')]^2\rangle + \tfrac{1}{4}\langle[\phi(\vec{\theta}_j;\vec{r}'+\lambda\vec{f}) - \phi(\vec{\theta}_{j'};\vec{r})]^2\rangle\}\Big\} \end{aligned} \qquad (22)$$

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Now restricting our attention to the case where isoplanatism is assumed so that

$$\mathcal{D}(\vec{\rho}) \approx \langle [\phi(\vec{\theta}_j;\vec{r}+\vec{\rho}) - \phi(\vec{\theta}_{j'};\vec{r})]^2 \rangle \qquad , \tag{23}$$

and making use of the fact that the wave-structure function is isotropic, [so that $\mathcal{D}(\vec{\rho}) = \mathcal{D}(-\vec{\rho})$], we can reduce Eq. (22) to the form

$$\begin{aligned}\tilde{\mathcal{I}}(\vec{f}) = \tfrac{1}{4}\, |\mathfrak{A}|^4 \{ (A_j^2)^2 + 2(A_j^2)(A_{j'}^2)^2 \cos [2\pi \vec{f}\cdot(\vec{\theta}_j - \vec{\theta}_{j'})] + (A_{j'}^2)^2 \} \\ \times \iint d\vec{r}\, d\vec{r}\,'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}\,')\, W(\vec{r}\,'+\lambda\vec{f}) \\ \times \exp\left(- \{\mathcal{D}(\vec{r}-\vec{r}\,') + \mathcal{D}(\lambda\vec{f}) - \tfrac{1}{2}[\mathcal{D}(\vec{r}-\vec{r}\,'-\lambda\vec{f}) + \mathcal{D}(\vec{r}-\vec{r}\,'+\lambda\vec{f})]\}\right) \quad .\end{aligned} \tag{24}$$

The interpretation of this expression is very straightforward. The quantity in the first curly brackets is just the power spectrum of the two point source pattern if it were able to be imaged perfectly, while the double integral (together with the $\frac{1}{4}\,|\mathfrak{A}|^4$ factor), is proportional to the transfer function which we evaluated in the previous section — as defined by Eq. (18). We can see that so long as we assume isoplanatism, [i. e., can assume the validity of Eq. (23)] , there is nothing significantly different in our analysis of the Labeyrie speckle interferometry technique, whether we consider the one point source or two point source case. In the next section, however, we shall treat the two point source case dropping the assumption of isoplanatism, and will see how the results change in that case.

4. LABEYRIE SPECKLE INTERFEROMETRY: TWO POINT SOURCES: ANISOPLANATISM

Our starting point for this case can be Eq. (22), presented in the last section just before the assumption of isoplanatism was imposed. In this section we shall not make that assumption, but rather shall allow anisoplanatism effects.

To accommodate anisoplanatism in our statistical treatment, we shall make use of the hyper wave-structure function, $\mathfrak{D}(\vec{\vartheta}, \vec{\rho})$, which is defined by the equation

$$\mathfrak{D}(\vec{\vartheta}, \vec{\rho}) = \langle [\phi(\vec{\theta} + \vec{\vartheta}; \vec{r} + \vec{\rho}) - \phi(\vec{\theta} + \vec{\vartheta}; \vec{r})] \times [\phi(\vec{\theta}; \vec{r} + \vec{\rho}) - \phi(\vec{\theta}; \vec{r})] \rangle \quad . \tag{25}$$

This quantity has previously been studied[11] and shown to be expressible as

$$\mathfrak{D}(\vec{\vartheta}, \vec{\rho}) = 8.16 \left(\frac{k}{2\pi}\right)^2 \int_{PATH} dv \quad C_N^2 \int d\vec{\kappa} \; [1 - \exp(i\vec{\kappa} \cdot \vec{\rho})] \times \kappa^{-11/3} [\exp(i\vec{\kappa} \cdot \vec{\vartheta} v) + \exp(-i\vec{\kappa} \cdot \vec{\vartheta} v)] , \tag{26}$$

where the v-integration is over the propagation path, with v = 0 at the measurement, (i.e., the aperture) plane, and $\vec{\kappa}$ is a three-dimensional spatial frequency associated with the turbulence pattern.

By means of some algebraic manipulations and making use of the stationarity of the wavefront distortion statistics, it can be shown that

$$\begin{aligned}
&- \tfrac{1}{2} \langle [\phi(\vec{\theta}_j; \vec{r}) - \phi(\vec{\theta}_{j'}; \vec{r}')]^2 \rangle - \tfrac{1}{2} \langle [\phi(\vec{\theta}_j; \vec{r}') - \phi(\vec{\theta}_{j'}; \vec{r})]^2 \rangle \\
&+ \tfrac{1}{4} \langle [\phi(\vec{\theta}_j; \vec{r}) - \phi(\vec{\theta}_{j'}; \vec{r}' + \lambda\vec{f})]^2 \rangle + \tfrac{1}{4} \langle [\phi(\vec{\theta}_j; \vec{r}') - \phi(\vec{\theta}_{j'}; \vec{r} + \lambda\vec{f})]^2 \rangle \\
&+ \tfrac{1}{4} \langle [\phi(\vec{\theta}_j; \vec{r} + \lambda\vec{f}) - \phi(\vec{\theta}_{j'}; \vec{r}')]^2 \rangle + \tfrac{1}{4} \langle [\phi(\vec{\theta}_j; \vec{r}' + \lambda\vec{f}) - \phi(\vec{\theta}_{j'}; \vec{r})]^2 \rangle \\
&= - \langle [\phi(\vec{\theta}_j; \vec{r}) - \phi(\vec{\theta}_j; \vec{r}')] [\phi(\vec{\theta}_{j'}; \vec{r}) - \phi(\vec{\theta}_{j'}; \vec{r}')] \rangle \\
&+ \tfrac{1}{2} \langle [\phi(\vec{\theta}_j; \vec{r}) - \phi(\vec{\theta}_j; \vec{r}' + \lambda\vec{f})] [\phi(\vec{\theta}_{j'}; \vec{r}) - \phi(\vec{\theta}_{j'}; \vec{r}' + \lambda\vec{f})] \rangle \\
&+ \tfrac{1}{2} \langle [\phi(\vec{\theta}_j; \vec{r}') - \phi(\vec{\theta}_j; \vec{r} + \lambda\vec{f})] [\phi(\vec{\theta}_{j'}; \vec{r}') - \phi(\vec{\theta}_{j'}; \vec{r} + \lambda\vec{f})] \rangle \quad .
\end{aligned} \tag{27}$$

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Making use of Eq.'s (25) and (27), together with Eq. (10) of course, we can now rewrite Eq. (22), (making allowance for anisoplanatism effects) as

$$\tilde{\mathscr{I}}(\vec{f}) = \tfrac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f})$$

$$\times \Big\{\!\Big\{ [(A_j^2)+(A_{j'}^2)]^2 \exp\Big(-\{\mathscr{D}(\vec{r}-\vec{r}')+\mathscr{D}(\lambda\vec{f})-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}'-\lambda\vec{f})+\mathscr{D}(\vec{r}-\vec{r}'+\lambda\vec{f})]\}\Big)$$

$$+\, 2(A_j^2)(A_{j'}^2)\cos[2\pi\vec{f}\cdot(\vec{\theta}_j-\vec{\theta}_{j'})]$$

$$\times \exp\Big(-\{\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}')+\mathscr{D}(\lambda\vec{f})-\tfrac{1}{2}[\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}'-\lambda\vec{f})$$

$$+\,\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}'+\lambda\vec{f})]\}\Big)\Big\}\!\Big\}\,. \qquad (28)$$

At this point it is useful to "focus" the anisoplanatism dependence in a single term, Q, defined as

$$Q(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}',\lambda\vec{f}) = \{[\mathscr{D}(\vec{r}-\vec{r}')-\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}')]$$

$$-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}'-\lambda\vec{f})-\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}'-\lambda\vec{f})]$$

$$-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}'+\lambda\vec{f})-\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}'+\lambda\vec{f})]\}. \qquad (29)$$

Making use of this Q-term we can rewrite Eq. (28) as

$$\tilde{\mathscr{I}}(\vec{f}) = \tfrac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\; W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f})$$

$$\times \Big\{\!\Big\{ [(A_j^2)^2+(A_{j'}^2)^2]\exp\Big(-\{\mathscr{D}(\vec{r}-\vec{r}')+\mathscr{D}(\lambda\vec{f})-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}'-\lambda\vec{f})+\mathscr{D}(\vec{r}-\vec{r}'+\lambda\vec{f})]\}\Big)$$

$$+\, 2(A_j^2)(A_{j'}^2)\cos[2\pi\vec{f}\cdot(\vec{\theta}_j-\vec{\theta}_{j'})] \qquad (30)$$

$$\times \exp\Big(-\{\mathscr{D}(\vec{r}-\vec{r}')+\mathscr{D}(\lambda\vec{f})-\tfrac{1}{2}[\mathscr{D}(\vec{r}-\vec{r}'-\lambda\vec{f})+\mathscr{D}(\vec{r}-\vec{r}'+\lambda\vec{f})]\}+Q(\vec{\theta}_j-\vec{\theta}_{j'},\vec{r}-\vec{r}',\lambda\vec{f})\Big)\Big\}\!\Big\}\,.$$

If we compare Eq. (30) with Eq. (24) we can see that if Q were replaced by zero, our results for the power spectrum would be identical to what we got when we assumed isoplanatism. Anisoplanatism effects derive from the fact that the Q-term is not sufficiently close to zero value over most of the integration range. Before we pursue the relationship between the value of Q and the value of $|\vec{\theta}_j - \vec{\theta}_j'|$ so as to establish an estimate of the size of the isoplanatic patch, it is worth while to study Eq. (30) a bit so as to identify the nature of the anisoplanatism effect.

We start by remarking that the Q-term will be non-positive. However, we immediately call attention to the fact that although the Q-term appears in the integral in Eq. (30) as an exp (Q) factor, the fact that Q is not equal to zero does not necessarily reduce the magnitude of $\tilde{\mathscr{I}}(\vec{f})$. Depending on the value of $\cos[2\pi\, \vec{f}\cdot(\vec{\theta}_j - \vec{\theta}_j')]$, a nonzero value of Q could possibly increase the value of $\tilde{\mathscr{I}}(\vec{f})$. But this increase is not of any particular use to us. What we need is the proper indication of the power spectrum associated with the object and in practice we get this from the value of $\tilde{\mathscr{I}}(\vec{f})$ measured when viewing a complex object, by dividing by the value of $\tilde{\mathscr{I}}(\vec{f})$ obtained when viewing a single point source. This ratio should yield the unbiased power spectrum of the complex object — and indeed it does when essentially the same degree of isoplanatism applies to both power spectrum.

However, when we have an significantly nonzero value of the Q-term in Eq. (30), the single point source data does not incorporate the same atmospheric turbulence effects as does the two point source data. Thus the ratio does not represent a proper normalization of the power spectrum. High spatial frequency details can appear in the estimated object power spectrum even though there is significant anisoplanatism. The effect of anisoplanatism is not to "wash out" the high spatial frequency contribution to the estimated power spectrum — but rather to simply render it invalid.

The relationship of the measured high spatial frequency part of the power spectrum to the actual object is questionable when anisoplanatism effects are present.

To return now to the question of the relationship between the angular spread $|\vec{\theta}_j - \vec{\theta}_{j'}|$ and the effective value of the Q-term in the integral in Eq. (30), i.e., to the determination of the isoplanatic patch size for the Labeyrie speckle imagery technique, we start by restricting our attention to the case of high spatial frequencies, i.e., $|\vec{f}|$ significantly greater than r_0/λ. We recall from our previous analysis that the value of the $(\vec{r}, \vec{r}')$-integral is dominated by the $(\vec{r}, \vec{r}')$-region where $|\vec{r}, \vec{r}'|$ is less than or about equal to the coherence diameter, r_0. If we combine Eq.'s (29) and (30) and collect terms, we can write the Q-term as

$$Q(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}) = \frac{8.16}{2\pi^2} k^2 \int_{PATH} dv\, C_N{}^2 \int d\vec{\kappa}\, \kappa^{-11/3} \exp(i\vec{\kappa}\cdot\vec{\rho}) \times \sin^2(\tfrac{1}{2}\lambda\vec{f}\cdot\vec{\kappa}) [\exp(i\vec{\kappa}\cdot\vec{\vartheta}v/2) - \exp(-i\vec{\kappa}\cdot\vec{\vartheta}v/2)]^2, \quad (31)$$

where $\vec{\rho}$ corresponds to $\vec{r} - \vec{r}'$ and $\vec{\vartheta}$ corresponds to $\vec{\theta}_j - \vec{\theta}_{j'}$. We know that most of the wavefront distortion of significance comes from turbulence spatial frequencies for which κ is less than, or of the order of $r_0{}^{-1}$. This means, in view of the previously mentioned constraints on the interesting range of $\vec{f}$ and of $\vec{\rho}$ (or rather of $\vec{r} - \vec{r}'$), that in Eq. (31) we can with reasonable accuracy estimate the value of the $\vec{\kappa}$-integration by replacing the $\exp(i\vec{\kappa}\cdot\vec{\rho})$-term by unity and the $\sin^2(\frac{1}{2}\lambda\vec{f}\cdot\vec{\kappa})$ - term by one-half. This allows us to rewrite Eq. (31) as

$$Q(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}) = -\frac{8.16}{\pi^2} k^2 \int_{PATH} dv\, C_N{}^2 \int d\vec{\kappa}\, \kappa^{-11/3} \sin^2(\vec{\kappa}\cdot\vec{\vartheta}v/2). \quad (32)$$

The $\vec{\kappa}$-integration in Eq. (32) can be carried out in closed form yielding

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the result that, for use in Eq. (30), we can write

$$Q(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}) \approx -(\vartheta/\vartheta_0)^{5/3} \quad , \tag{33}$$

where

$$\vartheta_0 = \left\{2.91\, k^2 \int_{PATH} dv\, v^{5/3}\, C_N{}^2\right\}^{-3/5} \quad . \tag{34}$$

Based on our current best estimates[4,5] for the vertical distribution of the optical strength of turbulence, $C_N{}^2$, we have calculated that $\vartheta_0 \approx 8.5\ \mu rad$. It is clear from the nature of Eq. (33) and our previous discussion of the implications of the Q-term in Eq. (30) being significantly nonzero, that ϑ_0 is the outer limit of what we should consider to be the isoplanatic patch size. The smallness of this number, as noted before, raises questions as to either the validity of our data for $C_N{}^2$ or else the validity of our interpretation of some of our speckle interferometry related results. At this point, we feel unable to offer a definitive choice between these two possibilities.

With the analysis of anisoplanatism effects for the Labeyrie speckle interferometry technique complete we are now ready to take up consideration of anisoplanatism effects in the Knox-Thompson speckle imagery algorithim. In the next section we shall start this analysis by presenting the relevant analysis, when isoplanatism is assumed. In the section after that we shall extend our results to the case where anisoplanatism effects are allowed. In both sections we shall work with a two point source object.

5. KNOX-THOMPSON SPECKLE IMAGERY ALGORITHIM: ISOPLANATISM ASSUMED

The Knox-Thompson speckle imagery algorithim starts at the same point as the Labeyrie speckle interferometry technique, i.e., with the spatial frequency fourier transform of the recorded intensity pattern. For a pair of point sources Eq. (19), originally stated for speckle interferometry, represents the starting point for speckle imagery as well. However, the Knox-Thompson speckle imagery algorithim calculates from this not only the power spectrum, $\tilde{\mathscr{I}}(\vec{f})$, which we have just been considering, but also the bispectrum

$$\mathscr{S}(\vec{f}\ ,\ \vec{f}') = \langle S^{*}(\vec{f}')\ S(\vec{f})\rangle \quad . \tag{35}$$

From the phase shifts in the bispectrum the Knox-Thompson algorithim calculates the phase shift to be associated with each spatial frequency component of the objects spectrum. We shall be concerned here to see under what conditions the phase shift of the bispectrum, $\mathscr{S}(\vec{f}\ ,\ \vec{f}')$, is equal to the difference in the phase shift to be associated with the two spatial frequency components of the object at $\vec{f}$ and $\vec{f}'$ — as it is the assumption of equality that allows the phase shift of each component to be calculated. In this section we shall assume there are no anisoplanatism effects.

If we substitute Eq. (19) into Eq. (35) and interchange the order of integration and ensemble averaging after first rewriting the product of integrals as a double integral, we get

$$\mathscr{S}(\vec{f}, \vec{f}_{\mathscr{S}}) = \tfrac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\ d\vec{r}'\ W(\vec{r})\ W(\vec{r}+\lambda\vec{f})\ W(\vec{r}')\ W(\vec{r}'+\lambda\vec{f}')$$

$$\times \Big\langle \Big\{ (A_j{}^2)^2 \exp\left[-2\pi i\,(\vec{f}-\vec{f}_{\pi})\cdot\vec{\theta}_j\right] \exp\{ i[\phi(\vec{\theta}_j;\vec{r}) - \phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f})$$

$$+ \phi(\vec{\theta}_j;\vec{r}'+\lambda\vec{f}') - \phi(\vec{\theta}_j;\vec{r}')]\}$$

$$+ A_j{}^2\ A_{j'}{}^2 \exp\left[-2\pi i\,(\vec{f}\cdot\vec{\theta}_j - \vec{f}\cdot\vec{\theta}_{j'})\right] \exp\{\phi(\vec{\theta}_j;\vec{r})$$

$$- \phi(\vec{\theta}_j;\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_{j'};\vec{r}'+\lambda\vec{f}') - \phi(\vec{\theta}_{j'};\vec{r}')]\}$$

$$+ A_j{}^2\ A_{j'}{}^2 \exp\left[-2\pi i\,(\vec{f}\cdot\vec{\theta}_{j'} - \vec{f}'\ \vec{\theta}_j)\right] \exp\{ i[\phi(\vec{\theta}_{j'};\vec{r})$$

$$- \phi(\vec{\theta}_{j'};\vec{r}+\lambda\vec{f}) + \phi(\vec{\theta}_j;\vec{r}'+\lambda\vec{f}') - \phi(\vec{\theta}_j;\vec{r}')]\}$$

$$+ (A_{j'}^2)^2 \exp\left[-2\pi i (\vec{f}-\vec{f}')\cdot\vec{\theta}_{j'}\right] \exp\left\{i\left[\phi(\vec{\theta}_{j'};\vec{r}) - \phi(\vec{\theta}_{j'};\vec{r}+\lambda\vec{f}) - \phi(\vec{\theta}_{j'};\vec{r}'+\lambda\vec{f}') - \phi(\vec{\theta}_{j'};\vec{r}')\right]\right\}\Big\}\Big\rangle \quad . \qquad (36)$$

Assuming isotropy, and making use of the gaussian nature of the statistics of the phase fluctuations, we can recast Eq. (36) in the form

$$\mathscr{S}(\vec{f},\vec{f}') = \frac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')$$
$$\times \exp\Big\{-\tfrac{1}{2}\,\{\mathcal{D}(\lambda\vec{f}) + \mathcal{D}(\lambda\vec{f}') + \mathcal{D}(\vec{r}-\vec{r}') + \mathcal{D}[\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')]$$
$$- \mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f}) - \mathcal{D}(\vec{r}-\vec{r}' - \lambda\vec{f}')\}\Big\}$$
$$\times \{[(A_j^2) \exp(-2\pi i\vec{f}\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i\vec{f}\cdot\vec{\theta}_{j'})]$$
$$\times [(A_j^2) \exp(2\pi i\vec{f}'\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i\vec{f}'\cdot\vec{\theta}_{j'})]\} \quad . \qquad (37)$$

Making use of Eq. (12), and introducing various approximations based on the facts that, 1) we are only interested in spatial frequencies large enough that $|\vec{f}|$ and $|\vec{f}'|$ are much greater than r_0/λ , and that, 2) on the basis of our experience in evaluating a nearly equivalent integral in Eq. (24), we know that the significant part of the contribution to the value of the integral will come from the region where $|\vec{r} - \vec{r}'|$ is less than or about equal to r_0 , we can write in place of Eq. (37)

$$\mathscr{S}(\vec{f},\vec{f}') \approx \frac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')$$
$$\times \exp\{-3.44\, r_0^{-5/3}[\,|\vec{r}-\vec{r}'|^{5/3} + |\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')|^{5/3}]\}$$
$$\times \{[(A_j^2) \exp(-2\pi i\vec{f}\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i\vec{f}\cdot\vec{\theta}_{j'})]$$
$$\times [(A_j^2) \exp(2\pi i\vec{f}'\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i\vec{f}'\cdot\vec{\theta}_{j'})]\} \quad . \qquad (38)$$

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If the two frequencies in the bispectrum, $\vec{f}$ and $\vec{f}'$ are close enough together, this result reduces to

$$\mathscr{I}(\vec{f},\vec{f}') \approx \frac{1}{4} |\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')$$
$$\times \exp\left[-6.88\, (|\vec{r}-\vec{r}'|/r_0)^{5/3}\right]$$
$$\times \{[(A_j^2) \exp(-2\pi i \vec{f}\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i \vec{f}\cdot\vec{\theta}_{j'})]$$
$$\times [(A_j^2) \exp(2\pi i \vec{f}'\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i \vec{f}'\cdot\vec{\theta}_{j'})]\} \quad . \qquad (39)$$

In this case, the integral is the same one we evaluated in Eq. (17) to obtain Eq. (18). It follows from this that the normalized bispectrum, which the Knox-Thompson algorithim calculates, can be written as

$$\frac{\mathscr{I}(\vec{f},\vec{f}')}{[\tilde{\mathscr{I}}(\vec{f})\,\tilde{\mathscr{I}}(\vec{f}')]^{1/2}} = N\, [(A_j^2) \exp(-2\pi i \vec{f}\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i \vec{f}\cdot\vec{\theta}_{j'})]$$
$$\times [(A_j^2) \exp(2\pi i \vec{f}'\cdot\vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i \vec{f}'\cdot\vec{\theta}_{j'})] \; , \qquad (40)$$

where N is a real constant of proportionality of no particular concern to us here. Clearly this result has the desired phase shift corresponding to the difference of the phase of the two components of the object pattern at frequencies $\vec{f}$ and $\vec{f}'$

If we do not make the assumption that the two spatial frequencies $\vec{f}$ and $\vec{f}'$ are very close together, i.e., $\lambda|\vec{f}-\vec{f}'|$ is smaller than r_0, but rather assume that it is larger than r_0, then instead of obtaining Eq. (39) from Eq. (38), we get

$$\mathscr{I}(\vec{f},\vec{f}') = \exp\left[-3.44\, (\lambda|\vec{f}-\vec{f}'|/r_0)^{5/3}\right] \frac{1}{4} |\mathfrak{A}|^4$$
$$\times \iint d\vec{r}\, d\vec{r}'\, \exp\left[-3.44\, (|\vec{r}-\vec{r}'|/r_0)^{5/3}\right]$$
$$\times W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')$$

$$\times \{[(A_j^2) \exp(-2\pi i \vec{f} \cdot \vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i \vec{f} \cdot \vec{\theta}_{j'})]$$

$$\times [(A_j^2) \exp(2\pi i \vec{f}' \cdot \vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i \vec{f}' \cdot \vec{\theta}_{j'})]\} \quad . \qquad (41)$$

With some minor changes the integral evaluation proceeds essentially the same way as before. This time we obtain for the normalized bispectrum

$$\frac{\mathscr{J}(\vec{f}, \vec{f}')}{[\tilde{\mathscr{J}}(\vec{f}) \tilde{\mathscr{J}}(\vec{f}')]^{1/2}} = \left\{ 2.30 \exp\left[-3.44 \left(\frac{\lambda |\vec{f} - \vec{f}'|}{r_0}\right)^{5/3}\right]\right\}$$

$$\times N [(A_j^2) \exp(-2\pi i \vec{f} \cdot \vec{\theta}_j) + (A_{j'}^2) \exp(-2\pi i \vec{f} \cdot \vec{\theta}_{j'})]$$

$$\times [(A_j^2) \exp(2\pi i \vec{f}' \cdot \vec{\theta}_j) + (A_{j'}^2) \exp(2\pi i \vec{f}' \cdot \vec{\theta}_{j'})] \quad . \qquad (42)$$

We see that in this case also the phase shift of the normalized bispectrum is a valid measure of the difference of the phases to be associated with the two components of the object at spatial frequencies $\vec{f}$ and $\vec{f}'$. However, the quantity in the curly brackets represents an attenuation factor in the magnitude of the bispectrum value we can work with. If we let $|\vec{f} - \vec{f}'|$ be too large in our calculations the bispectrum magnitude will be reduced — but the validity of the calculated phase shift will not be impaired, (except perhaps by additive noise). Clearly we must work with a value of $|\vec{f} - \vec{f}'|$ which is less than r_0/λ , in order to obtain a full strength signal. This fact was originally developed by Knox and Thompson[1] based on some computer simulation results. Assuming a smooth transition in the strength of the normalized bispectrum from the value in Eq. (40), when $|\vec{f} - \vec{f}'|$ is much smaller than r_0/λ , to the value in Eq. (42), when $|\vec{f} - \vec{f}'|$ is much larger than r_0/λ , our results suggest that there is a factor of two loss in the strength of the normalized bispectrum when

$$\lambda |\vec{f} - \vec{f}'|_K = 0.427\, r_0 \quad . \qquad (43)$$

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Accordingly, we suggest that in application of the Knox-Thompson algorithim for speckle imagery, the bispectrum be calculated for pairs of spatial frequencies whose separation is less than about $0.2\ r_0/\lambda$.

This completes our analysis of the Knox-Thompson speckle imagery algorithim for the case where there are no anisoplanatism effects, and set the stage for our evaluation of anisoplanatism effects in speckle imagery. We take this up in the next section.

6. KNOX-THOMPSON SPECKLE IMAGERY ALGORITHIM: ANISOPLANATISM EFFECTS

Our analysis of anisoplanatism effects in the Knox-Thompson speckle imagery algorithim can start with Eq. (36), the bispectrum for a pair of point sources. In this case when we make use of the fact that the wavefront distortion obeys gaussian statistics, and express our results in terms of the wave-structure function and the hyper wave-structure function, we get

$$\begin{aligned}\mathcal{J}(\vec{f},\vec{f}') = \tfrac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')\\ \times \Big\{\Big[(A_j^2)^2 \exp[-2\pi i(\vec{f}-\vec{f}')\cdot\vec{\theta}_j] + (A_{j'}^2)^2 \exp[-2\pi i(\vec{f}-\vec{f}')\cdot\vec{\theta}_{j'}]\Big]\\ \times \exp\Big(-\tfrac{1}{2}\{\mathcal{D}(\lambda\vec{f})+\mathcal{D}(\lambda\vec{f}')+\mathcal{D}(\vec{r}-\vec{r}')+\mathcal{D}[\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')]\\ -\mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f})-\mathcal{D}(\vec{r}-\vec{r}'-\lambda\vec{f}')\}\Big)\\ + (A_j^2)(A_{j'}^2)\{\exp[-2\pi i(\vec{f}\cdot\vec{\theta}_j-\vec{f}'\cdot\vec{\theta}_{j'})] + \exp[-2\pi i(\vec{f}\cdot\vec{\theta}_{j'}-\vec{f}'\cdot\vec{\theta}_j)]\}\\ \times \exp\Big(-\tfrac{1}{2}\{\mathcal{D}(\lambda\vec{f})+\mathcal{D}(\lambda\vec{f}')+\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'};\vec{r}-\vec{r}')+\mathfrak{D}[\vec{\theta}_j-\vec{\theta}_{j'};\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')]\\ -\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'};\vec{r}-\vec{r}'+\lambda\vec{f})-\mathfrak{D}(\vec{\theta}_j-\vec{\theta}_{j'};\vec{r}-\vec{r}'-\lambda\vec{f}')\}\Big)\Big\}\end{aligned} \tag{44}$$

We can rewrite this as

$$\begin{aligned}\mathcal{J}(\vec{f},\vec{f}') = \tfrac{1}{4}\,|\mathfrak{A}|^4 \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\, W(\vec{r}')\, W(\vec{r}'+\lambda\vec{f}')\\ \times \exp\Big(-\tfrac{1}{2}\{\mathcal{D}(\lambda\vec{f})+\mathcal{D}(\lambda\vec{f}')+\mathcal{D}(\vec{r}-\vec{r}')+\mathcal{D}[\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')]\\ -\mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f})-\mathcal{D}(\vec{r}-\vec{r}'-\lambda\vec{f}')\}\Big)\end{aligned}$$

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$$\times \Big((A_j{}^2)^2 \exp[-2\pi i (\vec{f}-\vec{f}') \cdot \vec{\theta}_j - \vec{\theta}_{j'}] + (A_{j'}{}^2)^2 \exp[-2\pi i (\vec{f}-\vec{f}') \cdot \vec{\theta}_{j'}]$$

$$+ (A_j{}^2)(A_{j'}{}^2)\{\exp[-2\pi i (\vec{f}\cdot\vec{\theta}_j - \vec{f}'\ \vec{\theta}_{j'})] + \exp[-2\pi i (\vec{f}\cdot\vec{\theta}_{j'} - \vec{f}\cdot\vec{\theta}_j)]\}$$

$$\times \exp[\tfrac{1}{2} \Omega (\vec{\theta}_j - \vec{\theta}_{j'}, \vec{r}-\vec{r}', \lambda\vec{f}, \lambda\vec{f}')]\}\Big) \quad , \qquad (45)$$

where

$$\Omega(\vec{\theta}_j - \vec{\theta}_j, \vec{r}-\vec{r}', \lambda\vec{f}, \lambda\vec{f}') = \Big([\mathcal{D}(\vec{r}-\vec{r}') - \mathfrak{D}(\vec{\theta}_j - \vec{\theta}_{j'}; \vec{r}-\vec{r}']$$

$$+ \{\mathcal{D}[\vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')] - \mathfrak{D}[\vec{\theta}_j - \vec{\theta}_{j'}; \vec{r}-\vec{r}'+\lambda(\vec{f}-\vec{f}')]\}$$

$$- [\mathcal{D}(\vec{r}-\vec{r}'+\lambda\vec{f}) - \mathfrak{D}(\vec{\theta}_j - \vec{\theta}_{j'}; \vec{r}-\vec{r}'+\lambda\vec{f})]$$

$$- [\mathcal{D}(\vec{r}-\vec{r}'-\lambda\vec{f}') - \mathfrak{D}(\vec{\theta}_j - \vec{\theta}_{j'}; \vec{r}-\vec{r}'-\lambda\vec{f}')]\Big) \quad . \qquad (46)$$

A comparison of Eq. (45) with Eq. (37), makes it quite clear that if the anisoplanatic turbulence effects are not to influence the apparent phase of the bispectrum value, then the Q-term in Eq. (45) must have a negligible value over significant range of the $(\vec{r}, \vec{r}')$ - integration. We recall that for the higher spatial frequencies, i.e., for $\lambda|\vec{f}|$ and $\lambda|\vec{f}'|$ both much greater than r_0, (and $\lambda|\vec{f} - \vec{f}'|$ much less than r_0), the significant portion of the $(\vec{r}, \vec{r}')$ range of integration is the region where $|\vec{r} - \vec{r}'|$ is less than or about equal to r_0 .

Making use of the definition of the value of the hyper wave-structure function provided by Eq. (26) , we can rewrite the Q-term as defined in Eq. (46) as

$$\Omega(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}, \lambda\vec{f}') = \frac{16.32}{\pi^2} k^2 \int_{\text{PATH}} dv\, C_N^2 \int d\vec{\kappa}\, \kappa^{-11/3}$$

$$\times \cos\{\vec{\kappa}\cdot[\vec{\rho} + \tfrac{1}{2}\lambda(\vec{f}-\vec{f}')]\}\{1 - 2\sin^2[\tfrac{1}{4}\vec{\kappa}\cdot\lambda(\vec{f}-\vec{f}')]\}$$

$$\times \sin^2(\vec{\kappa}\cdot\vec{\vartheta}\, v/2) \quad . \qquad (47)$$

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In view of the above mentioned constraints and equating $\vec{\rho}$ with $\vec{r} - \vec{r}'$ we can make use of the same sort of approximations as were used to obtain Eq. (32) from Eq. (31). This allows us to write

$$\Omega(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}, \lambda\vec{f}') = -\frac{16.32}{\pi^2} k^2 \int_{\text{PATH}} dv\, C_N{}^2 \int d\vec{\kappa}\, \kappa^{-11/3} \sin^2(\vec{\kappa}\, v/2), \quad (48)$$

with the understanding that this quantity must have a value lesser in magnitude than minus two, [recall the factor of one-half in the exponent in Eq. (4) and not included in the definition of Q], if anisoplanatism effects are not to influence the measured phase of the bispectrum, and thus the accuracy of the phase shifts calculated for the spatial frequency components of the object. Exactly the same sort of calculations can now be performed to evaluate Eq. (48), as were performed to obtain Eq. (33) from Eq. (32), so that we obtain the result that, for use in evaluation of Eq. (45), we can write

$$\tfrac{1}{2}\,\Omega\,(\vec{\vartheta}, \vec{\rho}, \lambda\vec{f}, \lambda\vec{f}') \approx -(\vartheta/\vartheta_0)^{5/3} \quad , \quad (49)$$

where ϑ_0 is defined, as for speckle interferometry, by Eq. (34).

We may consider ϑ_0 to be the isoplanatic patch size, though the nature of our results suggest that it is more of an outer bound than an allowable extent. It is to be noted that, if the isoplanatic condition is violated and we apply the Knox-Thompson speckle imagery algorithim, the calculated bispectrum will have a phase shift, seemingly indicative of the difference of phase shifts for the two spatial frequency components of the object at spatial frequencies $\vec{f}$ and $\vec{f}'$ — only it will not be a true indication of the value of this difference of phase shifts! According to how badly the isoplanatism conditions are violated the resulting image, obtained with the Knox-Thompson algorithim, may be only slightly distorted or totally meaningless. While we may get an indication of this from the image, we would get no warning of a violation of isoplanatism conditions from the calculated bispectrum.

REFERENCES

1. K. T. Knox and B. J. Thompson, "Recovery of Images from Atmospherically Degraded Short-Exposure Photographs", Ap. J. 193, L45 (1974).

2. A. Labeyrie, "High Resolution Techniques in Optical Astronomy", in Progress in Optics Vol. XV, E. Wolf, ed., (North Holland, Amsterdam 1976).

3. R. V. Stachrik, P. Nisenson, et al, "Solar Speckle Image Reconstruction, Nature (1977).

4. M. G. Miller and P. L. Zieske, Turbulence Environment Characterization, (Interim Technical Report for 15 October 1975 to 15 September 1976, by Avco Evertee Corp., on Contract F30602-76-C-0054).

5. D. P. Greenwood, private communications.

6. D. L. Fried, Theoretical Study of Non-Standard Imaging Concepts, RADC Technical Report, RADC-TR-74-185, May 1974.

7. D. L. Fried, Theoretical Study of Non-Standard Imaging Concepts, RADC Technical Report, RADC-TR-74-276, October 1974.

8. D. L. Fried, "Varieties of Isoplanatism", Proc. of SPIE 75, 20 (1976).

9. D. L. Fried, "Statistics of a Geometric Representation of Wavefront Distortion", J. Opt. Soc. Am. 55, 1427 (1965).

10. D. Korff, "Analysis of a Method for Obtaining Near-Diffraction-Limited Information in the Presence of Atmospheric Turbulence", J. Opt. Soc. Am. 63, 971 (1973).

11. D. L. Fried, "Spectral and Angular Covariance of Scintillation for Propagation in a Randomly Inhomogeneous Medium", Appl. Opt. 10, 721 (1971).

DISCUSSION

D. Korff: Is the rolloff of $<\tau(f;0)\tau^*(f;\theta)>$ for $f > r_o/\lambda$ frequency independent?

D. L. Fried: Yes. There are two parts to $<\tau\tau^*>$, one which is frequency independent and θ-independent, and the other which is frequency dependent and θ-dependent. Though the frequency and θ-dependence of this latter part of $<\tau\tau^*>$ is factorable, for the sum of the two parts, I don't think we can say that the frequency and θ dependencies are separable. This is true a fortiori when we consider objects more complex than just a pair of point sources.

D. Korff: Wouldn't speckle still be possible for an extended object if the isoplanatic patch angle were greater than the seeing angle?

D. L. Fried: Possibly. Nisenson and Stachnik's results for solar granulation suggest that masking to reduce the effective field of view is possible. I would have thought that introducing a mask in the focal plane would result in spurious data, but Nisenson and Stachnik's results suggest otherwise.

W. Waller: What can you say concerning the speckle observation of extended objects whose size exceeds that of the isoplanatic patch?

D. L. Fried: It may be possible if focal plane masking is used.

P. Connes: May I again make a plea for the turbulence specialists to turn some of their attention towards the so far unfashionable problem of small apparent stellar motion detection. The potential scientific returns are much greater than the ones expected from apparent diameter measurements. The problem is different; speckles and isoplanatic patch size are irrelevant. What one would like is to predict anomalous refraction differential fluctuations within a field of the order of 1^o.