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SOME PROPERTIES OF HYPERSPACES WITH APPLICATIONS TO CONTINUA THEORY

J. GRISPOLAKIS, SAM B. NADLER, JR., AND E. D. TYMCHATYN

1. Introduction. In 1972, Lelek introduced the notion of Class (W) in his seminar at the University of Houston [see below for definitions of concepts mentioned here]. Since then there has been much interest in classifying and characterizing continua in Class (W). For example, Cook has a result [**5**, Theorem 4] which implies that any hereditarily indecomposible continuum is in Class (W), Read [**21**, Theorem 4] showed that all chainable continua are in Class (W), and Feuerbacher proved the following result:

(1.1) THEOREM [**7**, Theorem 7]. *A non-chainable circle-like continuum is in* Class (W) *if and only if it is not weakly chainable.*

In [**14**, 4.2 and section 6], a covering property (denoted here and in [**18**] by CP) was defined and studied primarily for the purpose of proving that indecomposability is a Whitney property for the class of chainable continua [**14**, 4.3]. Bruce Hughes has recently shown that CP and Class (W) are related:

(1.2) THEOREM [Bruce Hughes—unpublished]. *If a continuum X has* CP, *then X is in* Class (W).

A proof of (1.2) is in [**18**, (14.73.21)].

We adopt the following notation and definitions. By a *mapping* we mean a continuous function. A space is said to be *nondegenerate* provided that it consists of more than one point. A *continuum* is a nondegenerate compact connected metric space. A *subcontinuum* is a non-empty compact connected subset of a continuum. If X is a continuum, then the *hyperspace* $C(X)$ is the space of all subcontinua of X metrized by the Hausdorff metric. The symbol μ will mean any mapping, called a *Whitney map*, from $C(X)$ into $[0, \infty)$ such that

$\mu(\{x\}) = 0$ for any $x \in X$, and
if $A, B \in C(X)$ such that $A \subset B \neq A$, then $\mu(A) < \mu(B)$.

That such a mapping μ exists was proved in [**22**]. Note that for any $K \in C(X)$, the restriction of μ to $C(K)$ is a Whitney map for $C(K)$. The following two

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facts are stated below for use in later proofs:

(1.3) If $K \in C(X)$, then $\bigcup [C(K) \cap \mu^{-1}(t_0)] = K$ whenever $0 \leqq t_0 \leqq \mu(K)$ (use [**12**, (2.3) and (2.6)]);

(1.4) μ is monotone [**6**, p. 1032].

Hence, for any $K \in C(X)$, $C(K) \cap \mu^{-1}(t_0)$ is a continuum whenever $0 \leqq t_0 < \mu(K)$.

For other facts about Whitney maps, see for example [**18**, Chapter XIV]. We say a continuum X *has the covering property* [**14**], written $X \in$ CP, provided that no proper subcontinuum of $\mu^{-1}(t)$ covers X for any Whitney map μ for $C(X)$ and any $t \in [0, \mu(X)]$. We say that X *has the covering property hereditarily* (see also [**20**]), written $X \in$ CPH, provided that each subcontinuum has CP. For facts about CP and CPH, see [**18**] and [**20**]. We define the function $C^*: C(X) \to C(C(X))$ by $C^*(A) = C(A)$ for each $A \in C(X)$. It was proved in [**18**] that

(1.5) C^* is upper semi-continuous [**18**, (15.2)], or $\operatorname{Lim}_{i\to\infty} \sup C(A_i) \subset C(A)$ whenever $\{A_i\}_{i\in\omega}$ converges to A in $C(X)$.

A continuum X is said to be *C^*-smooth at A*, $A \in C(X)$, provided that C^* is continuous at A. The continuum X is said to be *C^*-smooth* [**18**, (15.5)] provided that it is C^*-smooth at each $A \in C(X)$. For a discussion on C^*-smoothness see [**18**, Ch. 15].

A mapping f from a continuum Y onto a continuum X is said to be *weakly confluent* provided that for each $A \in C(X)$ there exists $B \in C(Y)$ such that $f[B] = A$. A continuum X is said to be in Class (W), (written $X \in$ Class (W)), provided that every mapping from any continuum onto X is weakly confluent.

For definitions of, and facts about, chainable, circle-like, and tree-like continua, see [**1**] and [**2**].

A continuum is said to be *weakly chainable* provided that it is the continuous image of a chainable continuum. This is not the original definition in [**16**] but, by results in [**16**], it is equivalent to the definition in [**16**].

A continuum X is said to be *unicoherent* if and only if whenever A and B are subcontinua of X such that $X = A \cup B$, $A \cap B$ is connected. A continuum is *hereditarily unicoherent* if and only if each of its subcontinua is unicoherent.

For sets A and B, $A \backslash B$ denotes the complement of B in A. For a subset G of continuum X, $\mathrm{Cl}(G)$ denotes the closure of G in X and $\mathrm{Bd}(G)$ denotes the boundary of G in X, i.e., $\mathrm{Bd}(G) = \mathrm{Cl}(G) \cap \mathrm{Cl}(X \backslash G)$.

The primary purpose of this paper is to establish relationships among the concepts mentioned above and, for the case of non-chainable circle-like continua X, to show they are all equivalent to a geometric property [(iv) of section 4] of X. We also answer some questions raised in [**18**].

2. General relationships between CP **and C^*-smoothness.** The main results in this section are (2.2) and (2.5). In (2.2) we show that X is C^*-smooth at A if $A \in$ CP. In (2.5) we show that CP and C^*-smoothness are equivalent for a class of continua which includes, for example, the circle-like continua and, more generally, the continua each proper subcontinuum of which is chainable (see [**9**] and section 4 here).

We begin with the following general lemma about convergence of levels.

(2.1) LEMMA. *Let X be a continuum and let $\{A_i\}_{i \in \omega}$ be a sequence of subcontinua A_i of X converging to a subcontinuum A of X. If $\{C(A_i)\}_{i\in\omega}$ converges to some $\Lambda \subset C(X)$ then, for any $0 \leqq t_0 < \mu(A)$, $\{C(A_i) \cap \mu^{-1}(t_0)\}_{i\in\omega}$ converges to $\Lambda \cap \mu^{-1}(t_0)$. Furthermore, $\Lambda \cap \mu^{-1}(t_0)$ is a subcontinuum of $C(A) \cap \mu^{-1}(t_0)$.*

Proof. Since $\{C(A_i)\}_{i\in\omega}$ converges to Λ, clearly

$$\operatorname*{Lim\,sup}_{i\to\infty} [C(A_i) \cap \mu^{-1}(t_0)] \subset \Lambda \cap \mu^{-1}(t_0).$$

Thus, to prove the first part of (2.1), it suffices to prove

$$(\#) \quad \Lambda \cap \mu^{-1}(t_0) \subset \operatorname*{Lim\,inf}_{i\to\infty} [C(A_i) \cap \mu^{-1}(t_0)].$$

To prove (#), let $B \in [\Lambda \cap \mu^{-1}(t_0)]$. Since $\{C(A_i)\}_{i\in\omega}$ converges to Λ, there exist $B_i \in C(A_i)$ for each $i \in \omega$ such that $\{B_i\}_{i\in\omega}$ converges to B. Since $\{A_i\}_{i\in\omega}$ converges to A and $t_0 < \mu(A)$, we assume without loss of generality that $C(A_i) \cap \mu^{-1}(t_0) \neq \emptyset$ for each $i \in \omega$. Then (by using [**12**, 2.3 and 2.6]), for each $i \in \omega$, there exists $B_i' \in [C(A_i) \cap \mu^{-1}(t_0)]$ such that $B_i' \supset B_i$ or $B_i' \subset B_i$. From these properties of B_i' and the fact that $\{B_i\}_{i\in\omega}$ converges to B, it is easy to prove that any convergent subsequence of $\{B_i'\}_{i\in\omega}$ converges to B. Hence, $\{B_i'\}_{i\in\omega}$ converges to B. Since $B \in [\Lambda \cap \mu^{-1}(t_0)]$ was arbitrary, we have proved (#). The second part of (2.1) follows easily using (1.4).

We now give the first main result of this section.

(2.2) THEOREM. *Let X be a continuum and let $A \in C(X)$. If $A \in$* CP, *then X is C^*-smooth at A. Hence if $X \in$* CPH, *then X is C^*-smooth.*

Proof. Let $\{A_i\}_{i\in\omega}$ be a sequence of subcontinua A_i of X converging to A. If $\mu(A) = 0$, then it is easy to see that $\{C(A_i)\}_{i\in\omega}$ converges to $C(A)$. Thus, for the purpose of proof, assume $\mu(A) > 0$. By compactness of $C(X)$, it suffices to prove that any convergent subsequence of $\{C(A_{i_j})\}_{j\in\omega}$ converges to $C(A)$. To do this, let $\{C(A_{i_j})\}_{j\in\omega}$ be a convergent subsequence of $\{C(A_i)\}_{i\in\omega}$ and let Λ denote the limit of $\{C(A_{i_j})\}_{j\in\omega}$. Let $t_0 \in [0, \mu(A))$. Then, by (2.1), $\{C(A_{i_j}) \cap \mu^{-1}(t_0)\}_{j\in\omega}$ converges to $\Lambda \cap \mu^{-1}(t_0)$. Thus, since

$$\bigcup [C(A_{i_j}) \cap \mu^{-1}(t_0)] = A_{i_j}$$

for all but finitely many $j \in \omega$ [by (1.3)], we have by continuity of union [**12**, p. 23] that $\bigcup [\Lambda \cap \mu^{-1}(t_0)] = A$. Hence, since $A \in$ CP and $\Lambda \cap \mu^{-1}(t_0)$ is a subcontinuum of $C(A) \cap \mu^{-1}(t_0)$ [by the second part of (2.1)],

$$(*) \quad \Lambda \cap \mu^{-1}(t_0) = C(A) \cap \mu^{-1}(t_0).$$

Therefore, since $t_0 \in [0, \mu(A))$ was arbitrary, it follows from (*) that $\Lambda = C(A)$. This completes the proof of (2.2).

Let us note the following aspect of (2.2). In general, as is certainly to be expected, the continuity of C^* at A depends on the embedding of A in X. For example, the continuum X in Figure 1 is C^*-smooth at the simple triod A_1 but not at the simple triod A_2. However, by (2.2), C^* is continuous at any $A \in CP$ no matter how A is embedded in X. Thus, for example, this is true when A is any chainable continuum, any non-planar circle-like continuum, any hereditarily indecomposable continuum, or any of the continua in (4.2).

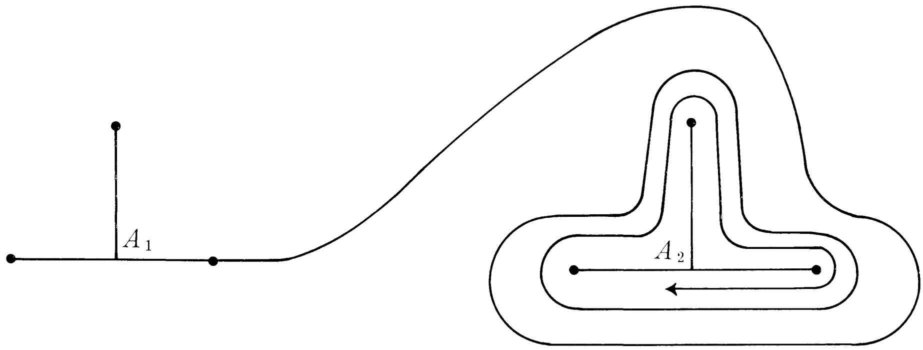

FIGURE 1

Let us note the following two corollaries to (2.2). The first one, originally stated in [**19**, 2.4], was proved in [**18**, (15.13)] but with a different proof than given here. The second one was originally obtained independently in [**8**, 3.1] and [**18**, (1.207.8)] with different proofs than given here.

(2.3) COROLLARY [**18**, (15.13)] and [**19**, 2.4]). *Any chainable continuum is C^*-smooth.*

Proof. Since any subcontinuum of a chainable continuum is chainable and any chainable continuum has CP by [**14**, 4.2], the corollary follows from (2.2).

(2.4) COROLLARY ([**8**, 3.1] and [**18**, (1.207.8)]). *Any hereditarily indecomposable continuum is C^*-smooth.*

Proof. Since hereditarily indecomposable continua have CP[**14**, section 6], the corollary follows from (2.2).

The following theorem is the second main result of this section.

(2.5) THEOREM. *Let X be a continuum such that each proper sub-continuum has* CP. *Then, X is C^*-smooth if and only if $X \in$* CP.

Proof. Assume $X \in$ CP. Then, by the hypotheses of (2.5), $X \in$ CPH and hence, by (2.2), X is C^*-smooth. Conversely, assume X is C^*-smooth. Let t_0

be such that $0 \leqq t_0 < \mu(X)$. By Zorn's lemma or by [**15**, Theorem 2, p. 54], there is a subcontinuum Γ of $\mu^{-1}(t_0)$ such that $\bigcup \Gamma = X$ and such that no proper subcontinuum of Γ covers X. We will show that $\Gamma = \mu^{-1}(t_0)$. Note that Γ is nondegenerate since $\bigcup \Gamma = X$ and $t_0 < \mu(X)$. Hence, there is a sequence $\{\Gamma_i\}_{i \in \omega}$ of proper subcontinua Γ_i of Γ such that $\{\Gamma_i\}_{i \in \omega}$ converges to Γ. For each $i \in \omega$, let $A_i = \bigcup \Gamma_i$. Note that for each $i \in \omega$, A_i is a subcontinuum of X [**12**, 1.2] and, since Γ_i is a proper subcontinuum of Γ, $A_i \neq X$. Hence, by the hypothesis in (2.5), $A_i \in$ CP for each $i \in \omega$. Thus, since $A_i = \bigcup \Gamma_i$ since Γ_i is a subcontinuum of $C(A_i) \cap \mu^{-1}(t_0)$ for each $i \in \omega$, and since the restriction of μ to $C(A_i)$ is a Whitney map for $C(A_i)$ for each $i \in \omega$, we have

(1) $\quad \Gamma_i = C(A_i) \cap \mu^{-1}(t_0)$ for each $i \in \omega$.

Recall that $\{\Gamma_i\}_{i \in \omega}$ converges to Γ, $\bigcup \Gamma_i = A_i$ for each $i \in \omega$, and $\bigcup \Gamma = X$. Hence, since union is continuous [**12**, p. 23], the sequence $\{A_i\}_{i \in \omega}$ converges to X. Thus, since X is C^*-smooth, the sequence $\{C(A_i)\}_{i \in \omega}$ converges to $C(X)$. Therefore, by (2.1),

(2) $\quad \{C(A_i) \cap \mu^{-1}(t_0)\}_{i \in \omega}$ converges to
$\quad C(X) \cap \mu^{-1}(t_0) = \mu^{-1}(t_0)$.

By (1) and (2), $\{\Gamma_i\}_{i \in \omega}$ converges to $\mu^{-1}(t_0)$. Therefore, since $\{\Gamma_i\}_{i \in \omega}$ converges to Γ, $\Gamma = \mu^{-1}(t_0)$. It follows from what we have shown that $X \in$ CP.

(2.6) *Remark.* Since each proper subcontinuum of a circle-like continuum is chainable and chainable continua have CP [**14**, 4.2], we have by (2.5) that CP is equivalent to C^*-smoothness for circle-like continua. More equivalences are in (4.1).

3. Local separating continua. In this section we consider some separation properties of spaces which are C^*-smooth. The following definitions will be used throughout the remainder of this paper.

(3.0) *Definitions.* A subcontinuum K of a continuum X is said to be a *local separating continuum* of X provided that there is an open subset U of X, $K \subset U$, and points p and q in the component of K in U such that $U \backslash K = M \cup N$ where M and N are mutually separated sets with $p \in M$ and $q \in N$. When this happens we say that K *locally separates p and q in X*. We say that K *separates p and q in X* provided we can take $U = X$ in the above definition. If K is degenerate, $K = \{k\}$, then we say k *locally separates* (respectively, k *separates*) p *and* q *in* X. It is easy to see that K is local separating continuum of X if and only if $\{K\}$ is a local separating point [**23**, p. 61] of the quotient space X/K.

(3.1) THEOREM. *Let X be a C^*-smooth continuum. Let A and B be subcontinua of X such that $A \subset B$ and A locally separates p and q in B. Then A separates p and q in B.*

Proof. Suppose the theorem fails. Since C^*-smoothness is a hereditary property, we suppose that $B = X$. Let U_0 be a neighbourhood of A in X such that p and q lie in the component R of A in U_0 and $U_0\backslash A = M_0 \cup N_0$ where M_0 and N_0 are separated sets such that $p \in M_0$ and $q \in N_0$. Let A_0 be the component of A in $X\backslash M_0$. If A_0 were disjoint from $\mathrm{Bd}(U_0) \cap \mathrm{Bd}(M_0)$, then there would exist an open and closed neighbourhood V of A_0 in $X\backslash M_0$ such that $V \cap \mathrm{Bd}(U_0) \cap \mathrm{Bd}(M_0) = \emptyset$. Then $V\backslash A$ and $X\backslash V$ would give a separation of $X\backslash A$ between p and q which is a contradiction. Hence, A_0 meets $\mathrm{Bd}(U_0) \cap \mathrm{Bd}(M_0)$.

Let P_0 be the component of $\mathrm{Cl}(R)\backslash A$ which contains p. Then $A \cup \mathrm{Cl}(P_0)$ is a continuum. Let Q_0 be the component of $\mathrm{Cl}(R)\backslash A$ which contains q. Then $A \cup \mathrm{Cl}(Q_0)$ is a continuum. Let U_1 be a neighbourhood of A such that $\mathrm{Cl}(U_1) \subset U_0$ and $\mathrm{Bd}(U_1)$ meets both P_0 and Q_0. Let R_1 be the component of A in $\mathrm{Cl}(U_1) \cap (A \cup \mathrm{Cl}(P_0) \cup \mathrm{Cl}(Q_0)$. Then R_1 meets both $\mathrm{Bd}(U_1) \cap M_0$ and $\mathrm{Bd}(U_1) \cap M_0$. Let P_1 be a component of $R_1\backslash A$ whose closure meets both $\mathrm{Bd}(U_1) \cap N_0$ and A and let Q_1 be a component of $R_1\backslash A$ whose closure meets both $\mathrm{Bd}(U_1) \cap N_0$ and A. Then $P_1 \subset P_0 \cap M_0$ and $Q_1 \subset Q_0 \cap N_0$. Hence, A locally separates every point of P_1 from every point of Q_1 in X but A does not separate any point of P_1 from any point of Q_1 in X.

By induction, there exist for each $i \in \omega$ a neighbourhood U_i of A, connected sets P_i and Q_i and open sets M_i and N_i such that

i) $\mathrm{Cl}(U_{i+1}) \subset U_i$,

ii) $\bigcap_{i=0}^{\infty} U_i = A$,

iii) $M_i = U_i \cap M_0$ and $N_i = U_i \cap N_0$,

iv) $P_{i+1} \subset P_i \cap \mathrm{Cl}(M_i)$ and $Q_{i+1} \subset Q_i \cap \mathrm{Cl}(N_i)$,

v) P_i is a connected set in $\mathrm{Cl}(M_i)\backslash A$ such that $\mathrm{Cl}(P_i)$ meets both $\mathrm{Bd}(U_i)$ and A and Q_i is a connected set in $\mathrm{Cl}(N_i)\backslash A$ such that $\mathrm{Cl}(Q_i)$ meets both $\mathrm{Bd}(U_i)$ and A,

vi) $P_i \cup A \cup Q_i$ is a continuum,

vii) $\mathrm{Bd}(U_{i+1})$ meets both P_i and Q_i, and

viii) A locally separates in X every point of Q_{i+1} from every point of P_{i+1} but A does not separate any point of Q_{i+1} from any point of P_{i+1} in X.

For each $i \in \omega$ let A_i be the component of A in $X\backslash M_i$. Then A_i meets $\mathrm{Bd}(U_i) \cap \mathrm{Bd}(M_i)$ by the same argument as in the case for U_0 and A_0. Also $A_{i+1} \supset A_i$ for each $i \in \omega$. Let $E = \mathrm{Lim}_{i\to\infty} \sup A_i = \mathrm{Lim}_{i\to\infty} A_i$ and let D be the component of A in $E \cap \mathrm{Cl}(U_1)$. Then D meets both $\mathrm{Bd}(U_1) \cap \mathrm{Bd}(M_1)$ and $\mathrm{Bd}(U_1) \cap \mathrm{Bd}(N_1)$. It is now clear that there does not exist for each $i \in \omega$ a continuum $D_i \subset A_i$ such that $D = \mathrm{Lim}_{i\to\infty} D_i$. This contradicts the assumption that X is C^*-smooth.

(3.2) COROLLARY. *If X is a C^*-smooth continuum and p is a local separating point of X then p is a cut-point of X.*

(3.3) LEMMA. *If X is a non-unicoherent continuum, then there exists a sub-*

continuum P of X and $p, q \in X\backslash P$ such that P locally separates p and q in X but P does not separate p and q in X.

Proof. Since X is not unicoherent, there exist continua P and Q in X such that $P \cup Q = X$, $P \cap Q = M \cup N$ where M and N are non-empty disjoint closed sets. Let Q' be a continuum in Q which is irreducible with respect to meeting both M and N. By [**17**, (I.47)], $Q'\backslash P$ is connected and its closure meets both M and N. Let U and V be disjoint neighbourhoods of M and N respectively. Then $P \cup U \cup V$ is a neighbourhood of P in X. As in the proof of (3.1), there exist components L_1 and K_1 of $(Q' \cap U)\backslash P$ and $(Q' \cap V)\backslash P$, respectively, such that $\mathrm{Cl}(L_1)$ and $\mathrm{Cl}(K_1)$ each meet P. Let $p \in L_1$ and let $q \in K_1$. Then p and q are the required points and P is the required continuum.

The following Corollary answers in the affirmative a question that was raised in [**18**, (15, 18)]. Some special cases had been proved in [**18**, (15.18)] and [**18**, (15.19)].

(3.4) COROLLARY. *If X is a C^*-smooth continuum, then X is hereditarily unicoherent.*

Proof. The Corollary follows immediately from (3.1), (3.3) and the observation that C^*-smoothness is hereditary [**18**, (15.6)].

(3.5) COROLLARY. *If X is a C^*-smooth continuum and if Y is a monotone image of X, then Y is C^*-smooth.*

Proof. Assume X is a C^*-smooth continuum and let $f: X \to Y$ be a monotone mapping from X onto a continuum Y. Let $\{B_i\}_{i\in\omega}$ be a sequence of subcontinua B_i of Y converging to a subcontinuum B of Y. Since $\{B_i\}_{i\in\omega}$ and B are arbitrary, it suffices by compactness of $C(Y)$ and (1.5) to show that

(#) $\quad C(B) \subset \operatorname*{Lim\,sup}_{i\to\infty} C(B_i).$

To prove (#), let $K \in C(B)$. Since f is monotone, $A_i = f^{-1}(B_i) \in C(X)$ for each $i \in \omega$. By compactness of $C(X)$, the sequence $\{A_i\}_{i\in\omega}$ has a convergent subsequence $\{A_{i_j}\}_{j\in\omega}$. Let A denote the limit of $\{A_{i_j}\}_{i\in\omega}$. Since $f[A_{i_j}] = B_{i_j}$ for each $j \in \omega$ and since $\{A_{i_j}\}_{j\in\omega}$ and $\{B_{i_j}\}_{j\in\omega}$ converge to A and B respectively, it follows that $f[A] = B$. Thus, since $K \subset B$, it follows that $f[f^{-1}(K) \cap A] = K$. Since f is monotone, $f^{-1}(K) \in C(X)$. Therefore, since X is C^*-smooth and $A \in C(X)$, we have by (3.4) that $[f^{-1}(K) \cap A] \in C(A)$. Hence, since $\{C(A_{i_j})\}_{j\in\omega}$ converges to $C(A)$ by C^*-smoothness of X, there exists $G_{i_j} \in C(A_{i_j})$ for each $j \in \omega$ such that $\{G_{i_j}\}_{j\in\omega}$ converges to $f^{-1}(K) \cap A$. Then, $f[G_{i_j}] \in C(B_{i_j})$ for each $j \in \omega$ and $\{f[G_{i_j}]\}_{j\in\omega}$ converges to $f[f^{-1}(K) \cap A] = K$. It follows from this that $K \in \mathrm{Lim}_{i\to\infty} \sup C(B_i)$. Since $K \in C(B)$ was arbitrary, we have proved (#). This completes the proof of (3.5).

(3.6) THEOREM. *If X is a continuum in* Class (*W*) *and A is a subcontinuum that locally separates p and q in X, then A separates p and q in X.*

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Proof. Suppose that the theorem fails. Let U be a neighbourhood of A such that p and q lie in the component of A in U, A separates p and q in U and $\mathrm{Cl}(U) \neq X$. Then, $U \backslash A = M \cup N$ where M and N are separated sets, $p \in M$ and $q \in N$.

We define a set $Y = A_1 \cup B \cup A_2$ and $f\colon Y \to X$ a function such that f carries B one to one onto $X \backslash A$, f carries A_1 one to one onto A and f carries A_2 one to one onto A. We topologize Y as follows. A basic neighbourhood for a point $y \in A_1$ (respectively, $y \in A_2$) is given by

$$f^{-1}(V)\backslash (A_2 \cup f^{-1}(\mathrm{Cl}(N)\backslash A))$$

(respectively,

$$f^{-1}(V)\backslash (A_1 \cup f^{-1}(\mathrm{Cl}(M)\backslash A))),$$

where V is a neighbourhood of $f(y)$ in X. If $y \in B$, then a basic neighbourhood of y in Y is of the form $f^{-1}(V)$ where V is a neighbourhood of $f(y)$ in X. It is easy to check that Y is a compact metric space and f is continuous. Notice that A_1 and A_2 are homeomorphic to A. Since A does not separate p and q in X, there is a component L of $X \backslash A$ such that both $\mathrm{Cl}(L \cap M)$ and $\mathrm{Cl}(L \cap N)$ meet A. It is now easy to see that Y is connected.

Let R be the component of A in $\mathrm{Cl}(U)$. As in the proof of (3.1), R meets both $\mathrm{Bd}(U) \cap \mathrm{Bd}(M)$ and $\mathrm{Bd}(U) \cap \mathrm{Bd}(N)$. It follows that R is not the image of any continuum in Y. Thus, f is not weakly confluent. This is a contradiction and the theorem is proved.

The following corollary is known [**18**, (14.73.25)].

(3.7) COROLLARY. *Every continuum in* Class (W) *is unicoherent.*

Proof. The theorem follows from (3.3) and (3.5).

In [**14**, section 6] it was proved that a space with the covering property is unicoherent. The following corollary gives a somewhat stronger result.

(3.8) COROLLARY. *If X is a continuum with the covering property and A is a subcontinuum of X such that A locally separates p and q in X then A separates p and q in X.*

Proof. The corollary follows from (1.2) and (3.6).

The following theorem is an analogue for Class (W) of (3.5). Notice that (3.5) and (3.9) fail for open mappings by (4.9), since there exist open mappings of any non-planar solenoid onto S^1.

(3.9) THEOREM. *If $X \in$ Class (W) and if f is a monotone mapping from X onto a continuum Y, then $Y \in$ Class (W).*

Proof. To show $Y \in$ Class (W), let g be any mapping of a continuum Z onto Y. Let

$$E = \{(x, z) \in X \times Z\colon f(x) = g(z)\}.$$

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Since f and g map onto Y, the projections π_X: $E \to X$ and π: $E \to Z$ are onto, i.e., $\pi_X[E] = X$ and $\pi_Z[E] = Z$. Note that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\pi_Z"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that, since $\pi_Z^{-1}(z_0) = f^{-1}[g(z_0)] \times \{z_0\}$ for each $z_0 \in Z$ and since f is monotone, π_Z is monotone. Hence, E is a continuum. Thus, since $X \in$ Class (W), π_X is weakly confluent. Now, to see that g is weakly confluent, let K be a subcontinuum of Y. Then, since f is monotone, $f^{-1}(K)$ is a subcontinuum of X. Thus, since π_X is weakly confluent, there exists a subcontinuum A of E such that $\pi_X[A] = f^{-1}(K)$. Therefore, since $f(\pi_X[A]) = K$ and the diagram above commutes, we have that $g(\pi_Z[A]) = K$. This proves that there is a subcontinuum of Z, namely $\pi_Z[A]$, which g maps onto K. Hence, g is weakly confluent. Therefore, since g was an arbitrary mapping of a continuum onto Y, $Y \in$ Class (W).

The following example shows that the hypothesis in (3.1) cannot be weakened by assuming only that X is C^*-smooth at X.

(3.10) *Example.* We construct a continuum X which is C^*-smooth at X but which is not unicoherent. Let A and B be two disjoint pseudo arcs and let h be a homeomorphism of A onto B. Let p and q be two points which lie in the same composant of A. Define an equivalence relation $\sim$ on $A \cup B$ by setting $x \sim y$ if and only if $x = y$ or $\{x, y\} = \{p, h(p)\}$ or $\{x, y\} = \{q, h(q)\}$. Let X be the quotient space $A \cup B/\sim$. Then X is a continuum which is not unicoherent. Let R be the irreducible continuum in A between p and q. Then, if $\{X_i\}_{i \in \omega}$ is a sequence of continua in X which converges to X it is easy to see that $R \cup h(R)$ is eventually in each X_i. Now, using the fact that A and B are C^*-smooth one checks that X is C^*-smooth at X. Notice that X is not a triod although X contains triods.

4. Applications to circle-like continua and smooth dendroids. Consider the following conditions that may be or may not be satisfied by a continuum X.

(i) $X \in$ CPH;
(ii) $X \in$ CP;
(iii) X is in Class (W);
(iv) X contains no local separating continuum;
(v) X is C^*-smooth;
(vi) X is not weakly chainable.

It is known that chainable continua satisfy conditions (i), (ii), (iii) and (v) (see [**14**], [**21**] and (2.3)), and that hereditarily indecomposable continua satisfy conditions (i)–(v) (see [**14**], [**5**], [**15**] and (2.4)). However, there exist examples of continua which do not have to satisfy any of the above conditions. Continua which satisfy condition (vi) have been characterized by A. Lelek in [**16**]. For a continuum X a subset A or X and an open cover $\mathscr{U}$ of X, we denote by $\mathrm{St}(A, \mathscr{U})$ the set $\bigcup\{U \in \mathscr{U} \mid A \cap U \neq \emptyset\}$.

(4.1) THEOREM. *Let X be a non-chainable circle-like continuum. Then conditions (i) through (vi) are equivalent.*

Proof. (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (iii) follows from (1.2). (iii) $\Rightarrow$ (iv) is true by (3.6) and [**4**, Theorem 4]. (iv) $\Rightarrow$ (v): By (2.6), it suffices to prove that if $\{A_i\}_{i\in\omega}$ is a sequence of proper subcontinua of X which converges to X, then

$$\operatorname*{Lim\,sup}_{i\to\infty} C(A_i) = C(X).$$

Let $A \in C(X)$ be a non-degenerate proper subcontinuum of X. Let $\mathscr{U}$ be a finite open cover for X whose nerve is a simple closed curve, $\mathrm{St}(A, \mathscr{U}) = U_1 \cup \ldots \cup U_n$ for some $U_1, \ldots, U_n \in \mathscr{U}$, where $n > 3$ and $\mathrm{Cl}(U_i) \cap \mathrm{Cl}(U_j) \neq \emptyset$ if and only if $|i - j| \leqq 1$, and such that

$$\mathrm{St}[\mathrm{St}(A, \mathscr{U}), \mathscr{U}] \neq X.$$

Since A is chainable, to prove that $A \in \mathrm{Lim}_{i\to\infty} \sup C(A)_i$ it suffices, since the mesh of $\mathscr{U}$ can be taken to be arbitrarily small, to prove that for some $i \in \omega$ there exists a component of $A_i \cap \mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ which meets both U_1 and U_n.

Let A' be the component of A in $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$. Suppose, on the contrary, that no component of $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ other than A' meets both U_1 and U_n. Let K be the union of the components of $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]\backslash A'$ which meet U_1 and let L be the union of the components of $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]\backslash A'$ which meet U_n. Then

$$\mathrm{St}(A, \mathscr{U}) \subset A' \cup K \cup L.$$

Since A' is not a local separating continuum of X, there exists $x \in [\mathrm{Cl}(K) \cap \mathrm{Cl}(L)]\backslash A'$. Otherwise, if $[\mathrm{Cl}(K) \cap \mathrm{Cl}(L)]\backslash A' = \emptyset$, $\mathrm{St}[\mathrm{St}(A, \mathscr{U}), \mathscr{U}]$ would be separated by A' between some points $a \in U_0 \cap (A''\backslash A')$ and $b \in U_{n+1} \cap (A''\backslash A')$, where A'' is the component of A in $\mathrm{St}[\mathrm{St}(A, \mathscr{U}), \mathscr{U}]$ and $U_0, U_{n+1} \in \mathscr{U}$ are such that

$$U_0 \cap U_1 \neq \emptyset = U_0 \cap \bigcup_{i=2}^{n} U_i \quad \text{and} \quad U_{n+1} \cap U_n = \emptyset = U_{n+1} \cap \bigcup_{i=1}^{n-1} U_i.$$

By the definition of K and L, $x \in U_2 \cup \ldots \cup U_{n+1}$. Without loss of generality, we may assume that $x \in K$. Let B be the component of K which contains x. Let $\{C_i\}_{i\in\omega}$ be a sequence of components of L such that $x \in \mathrm{Lim}_{i\to\infty} C_i$. Then $C = \mathrm{Lim}_{i\to\infty} \sup C_i$ is a continuum in $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ which contains x and meets $\mathrm{Cl}(U_n)$. Since $x \notin A'$, we infer that $C \cap A' = \emptyset$. Thus, $B \cup C$ is a continuum in $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ which is disjoint from A' and which meets both U_1 and U_n.

This contradiction shows that there exists a component of $\mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$, say D, which meets both U_1 and U_n.

Let $\mathscr{W}$ be a finite open cover for X whose nerve is a simple closed curve, $\mathrm{St}(\mathscr{W}) < \mathscr{U}$, and

$$\mathrm{St}(\mathrm{St}(A, \mathscr{W}), \mathscr{W}) \cap \mathrm{St}(D, \mathscr{W}) = \emptyset.$$

Let $\mathrm{St}(A, \mathscr{W}) = W_1 \cup \ldots \cup W_m$ and $\mathrm{St}(D, \mathscr{W}) = V_1 \cup \ldots \cup V_r$ for some $W_1, \ldots, W_m, V_1, \ldots, V_r \in \mathscr{W}$ such that $\mathrm{Cl}(W_i) \cap \mathrm{Cl}(W_j) \neq \emptyset$ if and only if $|i - j| \leqq 1$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_j) \neq \emptyset$ if and only if $|i - j| \leqq 1$.

Choose i large enough so that A_i meets both W_1 and W_m. If a component Q of $A_i \cap \mathrm{Cl}[\mathrm{St}(A, \mathscr{W})]$ meets both W_1 and W_n, then we are done since Q is contained in a component of $A_i \cap \mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ and Q meets both U_1 and U_n. If no component of $A_i \cap \mathrm{Cl}[\mathrm{St}(A, \mathscr{W})]$ meets both W_1 and W_m, then $\mathrm{St}(D, \mathscr{W})$ disconnects A_i. It follows that some component Q of $A_i \cap \mathrm{St}(D, \mathscr{W})$ meets both V_1 and V_r. Hence, $Q \subset \mathrm{St}(A, \mathscr{U})$ and $A \subset \mathrm{St}(Q, \mathscr{U})$, which implies that there is a component of $A_i \cap \mathrm{Cl}[\mathrm{St}(A, \mathscr{U})]$ which contains Q and meets U_1 and U_n. Since $\mathscr{U}$ can be taken to be an ϵ-cover for arbitrarily small $\epsilon > 0$, it follows that $A \in \mathrm{Lim}_{i \to \infty} \sup C(A_i)$. (v) $\Rightarrow$ (i) follows from (2.5) and from the fact that every proper subcontinuum of a circle-like continuum is chainable. Finally, the equivalence of (iii) and (vi) has been proved by G. A. Feuerbacher [see (1.1) above].

Somewhat similar methods are used in [**9**] to prove the first part of the following result. The second part is a consequence of the first part, (2.5), and (1.2).

(4.2) THEOREM ([**9**, Theorem 1]). *Let X be a tree-like continuum such that every proper subcontinuum is chainable. Then $X \in CP$. Hence, X is C^*-smooth and in* Class (W).

(4.3). *Remark.* In 1972, A. Lelek asked for a characterization of Class (W). In [**18**, (14.73.25)], Hughes asked whether condition (iii) implies condition (ii). Theorem (4.1) gives a solution to both problems in the case of circle-like continua, as well as to a question of Nadler [**18**, (15.15)]. A partial answer to the latter question was obtained in [**18**, (15.16)].

(4.4) COROLLARY. *Let X be a circle-like continuum with a local separating continuum. Then X is planar.*

Proof. By (4.1), X is the continuous image of a chainable continuum. By Ingram [**11**, Theorem 5] X is planar (For a shorter proof of [**11**, Theorem 5] see also [**10**]).

(4.5) *Example.* This is a planar unicoherent, indecomposable, circle-like continuum X', with a local separating point. Thus, by (4.1), X' is not C^*-smooth. Let X be the Knaster indecomposable chainable continuum with two end-points (see [**15**, §48, V, 3]). By identifying the two end-points we obtain the desired example.

The converse of (4.4) fails. The pseudo-circle X as defined in [R. H. Bing, "Concerning hereditarily indecomposable continua", Pacific J. Math. 1 (1951), pp. 43–51] is a planar, circle-like continuum. It is hereditarily indecomposable and, hence, has no local separating continuum.

The following two corollaries were obtained by A. Petrus in [**20**, Propositions 19 and 20] by a method different from that given here.

(4.6) COROLLARY ([**20**, Proposition 19]). *Non-planar circle-like continua have the covering property hereditarily.*

Proof. By Ingram [**11**, Theorem 5], non-planar circle-like continua are not the continuous images of chainable continua. By (4.1), they have the covering property hereditarily.

(4.7) COROLLARY. ([**20**, Proposition 20]). *If X is a solenoid different from S^1, then X has the covering property hereditarily.*

(4.8) COROLLARY. *Non-planar circle-like continua are C^*-smooth.*

The proof follows from (4.6), the fact that proper subcontinua of circle-like continua are chainable, and (2.5).

A continuum X is said to be a *dendroid*, provided that X is arcwise connected and hereditarily unicoherent. A dendroid is said to be *smooth* at the point $p \in X$, provided for any sequence of points $a_0, a_1, a_2, \ldots$ in X with $\lim_{i\to\infty} a_i = a_0$ we have that $\operatorname{Lim}_{i\to\infty} [p, a_i] = [p, a]$, where $[x, y]$ denotes the unique arc in X with end-points x and y. It is well-known that smooth dendroids are partially ordered spaces, where the partial order is closed (see [**13**]). J. H. Carruth [**4**] has proved that partially ordered compacta admit a radially convex metrix ρ (i.e. If $x \leqq y \leqq z$, then $\rho(x, z) = \rho(x, y) + \rho(y, z)$).

(4.9) THEOREM. *A dendroid D which is smooth at p is C^*-smooth if and only if for each pair of sequences $\{a_i\}_{i\in\omega}$ and $\{b_i\}_{i\in\omega}$ such that $\lim_{i\to\infty} a_i = a_i$ $\lim_{i\to\infty} b_i = b$, $a \notin [p, b]$ and $b \notin [p, a]$ we have that* $\operatorname{Lim}_{i\to\infty} [a_i, b_i] = [a, b]$.

Proof. It is obvious that the condition is sufficient. To show that it is necessary, suppose that D is smooth at the point $p \in D$. Let ρ be a radially convex metric for D. Notice that $\operatorname{Lim}_{i\to\infty} [p, a_i] = [p, a]$ and $\operatorname{Lim}_{i\to\infty} [p, b_i] = [p, b]$, since D is a smooth dendroid. Suppose that $\operatorname{Lim}_{i\to\infty} [a_i, b_i] = T \neq [a, b]$. Since D is uniquely arcwise connected $[a_i, b_i] \subset [p, a_i] \cup [p, b_i]$. Let $\epsilon > 0$ such that $S(a, \epsilon) \cap [p, b] = \emptyset$ (where $S(a, \epsilon)$ is the ϵ-ball about a). If for each $i \in \omega$, $x_i \in [p, a_i]$ such that $\rho(a, x_i) \geqq \epsilon$ and $\lim_{i\to\infty} x_i = x$, then $x \in [p, a]$ and $p(x, a) \geqq \epsilon$. Since ρ is radially convex, we have

$$\rho(p, x) \leqq \rho(p, a) - \epsilon.$$

Hence, if $y \in [p, x_i]$, then $\rho(p, y) \leqq \rho(p, a) - \epsilon/2$ for sufficiently large i. Thus, for sufficiently large i

$$[p, x_i] \cap S(a, \epsilon/2) = \emptyset.$$

It follows that if for each i, $x_i \in [p, a_i]$ and $y_i \in [p, b_i]$ such that $\mathrm{Lim}_{i\to\infty}[x_i, y_i] \supset [a, b]$, then $\mathrm{Lim}_{i\to\infty} x_i = a$ and $\mathrm{Lim}_{i\to\infty} y_i = b$. Hence, $\mathrm{Lim}_{i\to\infty}[x_i, y_i] = T \neq [a, b]$. This contradicts the assumption that D is C^*-smooth.

(4.10) *Remark.* Notice that there is no relation between the notions of a "smooth dendroid" and of a "C^*-smooth dendroid". For a discussion on that see [**18**, (15, 20)].

5. Questions. To a large extent this paper came about because of our interest in the following question (stated in [**18**, (14.73.25)]):

(5.1) *Question* [Bruce Hughes]. Is the converse of (1.2) true?

We have given some partial answers to (5.1). The following questions are related to our work.

(5.2) *Question.* If $X \in$ Class (W), then is X C^*-smooth at X? By (2.2), a negative answer would give a negative answer to (5.1) and an affirmative answer would exhibit another property that Class (W) and CP have in common.

(5.3) *Questions.* In (2.2), can the assumption that "$A \in$ CP" be replaced by the assumption that $A \in$ Class (W)? Also in (2.2), can the assumption that "$X \in$ CPH" be replaced by the assumption that $X \in$ Class (W) hereditarily? Of course [by (2.2)], an affirmative answer to (5.1) would give affirmative answers to these questions.

The questions in (5.4) below are motivated by the comments in the paragraph following the proof of (2.2).

(5.4) *Questions.* What "internal" conditions characterize those continua A such that whenever A is embedded in a continuum X, X is C^*-smooth at A? In (2.2) we showed that $A \in$ CP is a sufficient condition. Is it also a necessary condition?

Other questions about Class (W), CP, C^*-smooth, etc. can be found in Chapters XIV and XV of [**18**].

Questions (5.1)–(5.4) have been recently answered in the affirmative by J. Grispolakis and E. D. Tymchatyn.

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University of Saskatchewan,
Saskatoon, Saskatchewan