GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE

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1. Introduction. Let G be a compact group. According to the celebrated theorem of Peter-Weyl there exists a complete set of finite-dimensional irreducible unitary representations of G, the completeness meaning that for any group element other than the identity there is a representation sending it into a matrix other than the unit matrix. If G is commutative, the representations are necessarily one-dimensional. It is an immediate consequence of the Peter-Weyl theorem that the converse also holds: if every representation is one-dimensional, G is commutative. The main theorem in the present paper is a generalization of this result to the case where the representations have bounded degree. We may illustrate by stating the next simplest case. The representations are one- or two-dimensional if and only if G satisfies the following condition: for any 4 elements of G the 12 ($= 4!/2$) products obtained from even permutations can be paired off in equal pairs with the 12 products obtained from odd permutations. The general result is stated in Theorem 3.

Such groups exist: for example, the group extension of an abelian group by a finite group (Theorem 1). On the other hand, if such a group is connected it is abelian (Theorem 2).

In §§ **2, 3** we present some preliminary remarks on matrices and groups, and in § **4** we review some facts on group representations needed for the extension from the compact to the locally compact case. In § **5** the main theorems appear, and in § **6** a connection with a theorem due to Halmos is described.

2. Matrix identities.[1] For elements $x_1, \ldots, x_r$ in a ring we shall write

$$[x_1, \ldots, x_r] = \sum \pm x_{\pi(1)} \ldots x_{\pi(r)}$$

where the sum runs over all permutations π and the plus or minus sign is prefixed according as π is even or odd.

LEMMA 1. *In any algebra A of order $k - 1$ we have $[x_1, \ldots, x_k] = 0$ for all $x_i \epsilon A$.*

Proof. Since the relation in question is multilinear, it need only be proved when $x_1, \ldots, x_k$ are basis elements. In that case at least one repetition occurs, and consequently a transposition can be performed which leaves $[x_1, \ldots, x_k]$ unchanged. Hence

$$[x_1, \ldots, x_k] = -[x_1, \ldots, x_k] = 0.$$

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[1] I am greatly indebted to E. R. Kolchin for the contents of **2**.

(Formally this argument is invalid for characteristic 2, but the result is still correct and may be proved by the usual device of a reduction mod 2.)

We may apply Lemma 1 to the special case where A is the algebra of n by n matrices over a field. We shall write $r(n)$ for the smallest integer such that $[x_1, \ldots, x_r] = 0$ for all n by n matrices.[2] By Lemma 1 we have $r(n) \leqq n^2 + 1$.

The following argument gives a lower bound for r. Write $t = r(n-1) - 1$. Suppose $x_1, \ldots, x_t$ are $n-1$ by $n-1$ matrices with $[x_1, \ldots, x_t] \neq 0$, and we may suppose to be explicit that $[x_1, \ldots, x_t]$ contains a non-zero term in e_{hk}, where $\{e_{ij}\}$ denote the usual matrix units. Embed the matrix x_i in an n by n matrix y_i by adjoining a row and column of zeros. Then it is evident that

$$[y_1, \ldots, y_t, e_{kn}, e_{nn}] \neq 0.$$

This proves the following result.

LEMMA 2. $r(n) \geqq r(n-1) + 2$.

It is clear that $r(1) = 2$ and by Lemma 2 we deduce the lower bound $r(n) \geqq 2n$. For $n = 2$, $r(2)$ is in fact precisely 4. This apparently exhausts the known facts concerning $r(n)$.

3. A certain class of groups.

Let us say that a group G satisfies the condition $P_n (n \geqq 2)$ if the following is true: for any n elements in G the set of $n!/2$ products obtained from even permutations coincides with the $n!/2$ products obtained from odd permutations. It should be noted that it is not asserted that there is a fixed way of carrying out the pairing once for all; the particular correspondence presumably depends upon the particular n elements in question.

It is fairly evident that P_k implies P_{k+1}. P_2 simply asserts commutativity, and so does P_3 as can be seen by taking one of the three elements to be the identity. Starting at $k = 4$ there exist non-abelian groups satisfying P_k; for example, the symmetric group on three elements satisfies P_4. The following theorem provides us with a substantial class of such groups.

THEOREM 1. *A group extension of an abelian group by a finite group of order n satisfies P_{n^2+1}.*

Proof. We suppose that G is abelian, H of order n, and $K/G \cong H$. Choose fixed representatives $k_1, \ldots, k_n \epsilon K$ for the cosets of K mod G. Every element of K can be uniquely written gk_i, $g \epsilon G$. Let b be a product of $n^2 + 1$ elements of K. In such a product some k, say k_1, must be repeated at least $n + 1$ times. Let gk_1 be the element appearing at the first occurrence of k_1, and $g'k_1$ one of the later occurrences. Write x for the product of the k's intervening between these two instances of k_1. The interchange of the pair gk_1 and $g'k_1$ will leave b unchanged provided that $k_1 x$ lies in G. Since we have $n + 1$ or more occurrences of k_1 and only n cosets of K mod G, it will have to happen at least once that an interchange of two of the terms comprising b leaves b unchanged.

[2]It is conceivable that $r(n)$ depends on the coefficient field, or rather on the characteristic of the latter. To be explicit, one may take the characteristic 0 case throughout the paper.

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We now specifically pick out the *first* element in the product b whose interchange with a later element is legal. In all the $(n^2+1)!$ permutations we do the same thing, and thus set up a one-to-one correspondence between the even permutations and the odd permutations. This proves Theorem 1.

Theorem 1 does not give the best possible result. Indeed we shall show below that a group extension of an abelian group by a group of order n actually satisfies $P_{s(n)}$, where

$$(1) \qquad s(n) = r(n) \text{ for } r(n) \text{ even}$$
$$= r(n)+1 \text{ for } r(n) \text{ odd},$$

and $r(n)$ is the integer defined in § **2**. Thus for $n = 2$ we get P_4 instead of the P_5 of Theorem 1. However I am unable to prove this refinement without the detour to group representations and Banach algebras.

We shall conclude this section by showing that there are no connected non-abelian groups of the kind under discussion. Actually we prove a (formally at least) stronger result, in order to carry through an induction.

THEOREM 2. *Let G be a connected topological group having for some fixed $n \geqq 2$ the following property: any product $a_1a_2 \ldots a_n$ is equal to a proper permutation. Then G is abelian.*

Proof. We shall show that G has the same property for $n - 1$ and hence finally reach $n = 2$. Let then $a_1, \ldots, a_{n-1}$ be elements in G. For any b in G the product $a_1 \ldots a_{n-1}b$ must be equal to a proper permutation. We may suppose that there is a neighbourhood U of the identity such that for b in U the proper permutation in question keeps the order of $a_1, \ldots, a_{n-1}$ fixed; for otherwise we can take the limit as b approaches the identity and conclude that $a_1 \ldots a_{n-1}$ equals a proper permutation. Thus for each b in U we have one of the $n - 1$ possible equations

$$a_1 \ldots a_{n-1}b = a_1 \ldots a_iba_{i+1} \ldots a_{n-1} (i = 0, \ldots, n-2).$$

The ith equation asserts that b commutes with $a_{i+1} \ldots a_{n-1}$ and hence is valid in a closed set. Thus U is covered by a finite number of these closed sets, and one of them must have a non-void interior. This says that the centralizer of $a_{i+1} \ldots a_{n-1}$ is open. Since G is connected, this centralizer must be all of G and hence $a_{i+1} \ldots a_{n-1}$ is in the centre. For $i \geqq 1$ this yields the desired result obviously, while for $i = 0$ the assertion that $a_1 \ldots a_{n-1}$ is in the centre implies

$$a_1a_2 \ldots a_{n-1} = a_2 \ldots a_{n-1}a_1.$$

COROLLARY. *If a connected topological group satisfies P_n it is abelian.*

4. Group representations. In order to formulate our main theorem for a locally compact group G, it would not suffice to assume that the finite-dimensional irreducible unitary representations of G have bounded degree; for there exist groups (e.g. the Lorentz group) for which the only finite-dimensional unitary representation is the trivial one. Thus we must impose a further condition which will entail the existence of a respectable number of finite-dimensional representations. For our purposes a convenient hypothesis of this kind can be formulated in terms of the representations introduced by Segal [4]. We devote this section to a brief statement of the necessary facts.

Let A denote the L_1-algebra of the locally compact group G, that is, the algebra of all complex-valued functions summable with respect to the left Haar measure of G, with convolution as multiplication:

$$fg(x) = \int f(y)g(y^{-1}x)dy.$$

Let E be the algebra of bounded operators on a Banach space. A B-representation [4, p. 79] of G is a multiplicative homomorphism T of G into E which sends the identity into the unit operator, is continuous in the strong topology of E, and is such that $||T(a)||$ is bounded for $a \epsilon G$. A B-representation is irreducible if it admits no proper closed invariant subspaces. Irreducible B-representations may be constructed as follows. Let M be a regular maximal left ideal in A, and associate with $a \epsilon G$ the operator $T_a: u + K \rightarrow u_a + K$, where $u_a(x) = u(a^{-1}x)$. We shall call these representations *primitive*, a designation suggested by the fact that the extension of the representation to A has as its kernel the ideal P consisting of all x with $xA \leqq M$; P is a primitive ideal in the sense of Jacobson [2]. Conversely every primitive ideal in A is associated in this fashion with at least one primitive representation of G.

The following facts are known: (1) all primitive representations are irreducible, (2) any irreducible finite-dimensional unitary representation is similar to a primitive representation, (3) if G is compact or abelian, all primitive representations of G are finite-dimensional. It is an open question whether every irreducible B-representation is similar to a primitive representation.

5. Main theorem. In terms of the concepts introduced in the previous sections, the principal result can be stated as follows.

THEOREM 3. *The following two statements are equivalent for a unimodular locally compact group G:*[3]

(*a*) *All primitive representations of G are finite-dimensional and of degree at most n,*

(*b*) *G satisfies the condition $P_{s(n)}$, where $s(n)$ is defined by* (1).

It is to be observed that if G is compact, the theorem simplifies perceptibly: compact groups are unimodular, and their primitive representations are automatically finite-dimensional.

Proof. Suppose that (*a*) holds. Then it follows virtually from the definition of the primitive representations that for every primitive ideal P in $A = L_1(G)$, $A - P$ is finite-dimensional and is in fact a total matrix algebra of degree at most n. Hence $A - P$ satisfies the identity $[x_1, \ldots, x_k] = 0$ for $k = r(n)$ and *a fortiori* for $k = s = s(n)$. Now the intersection of the primitive ideals of A is 0: this is a consequence of the semi-simplicity of A: [4, Th. 1.5]

[3]A group is unimodular if its right and left Haar measures coincide—cf. [5, p. 39].

and [2, Th. 25]. Hence $[f_1, \ldots, f_s] = 0$ holds for all $f_i \epsilon A$. The s-fold convolution of functions is given by

$$(2) \qquad f_1 \ldots f_s(x) = \int \ldots \int f_1(y_1) f_2(y_2) \ldots f_{s-1}(y_{s-1}) f_s[(y_1 \ldots y_{s-1})^{-1} x] dy_1 \ldots dy_{s-1}.$$

We shall now study the effect of a permutation π on $f_1 \ldots f_s$. If π does not involve the letter s, its effect on (2) may be described as carrying out π on the y's in $(y_1 \ldots y_{s-1})^{-1}$, and otherwise leaving the right side of (2) unchanged. Next we try the case $\pi = (is)$. We carry out the interchange of f_i and f_s in (2) and then replace y_i by

$$(3) \qquad (y_1 \ldots y_{i-1})^{-1} x y_i^{-1} (y_{i+1} \ldots y_{s-1})^{-1},$$

(a legal change of variable in view of the assumed unimodularity of G). This replaces

$$(4) \qquad (y_1 \ldots y_{s-1})^{-1} x$$

by y_i and so finally gives us the right side of (2) changed by the substitution of (3) for (4). In view of the fact that s is even, it can be verified that the permutation (4) $\rightarrow$ (3) is odd.

The general permutation π which does involve s can be written uniquely as $\pi = (is)\pi_1$, where π_1 is independent of s. The effect of π on the right side of (2) can thus be described as changing the argument of f_s by the permutation (4) $\rightarrow$ (3), followed by the permutation π_1 on the y's. This is a one-to-one correspondence: given the induced permutation on the argument of f_s we can unambiguously reconstruct π; for the position of x (in the ith place) gives us the portion (is), and the position of the y's then determines π_1. Moreover, the correspondence preserves the parity of π, as we have seen.

We may summarize as follows. We have

$$(5) \qquad \int \ldots \int f_1(y_1) f_2(y_2) \ldots f_{s-1}(y_{s-1}) Z dy_1 \ldots dy_{s-1} = 0,$$

where we have written Z for

$$Z = \sum_j \pm f_s(z_j),$$

z_j being the general permutation of (4), and the plus or minus sign being taken according to the parity of the permutation z_j. Since (5) holds for all f_i in A and in particular for all continuous f_i in A we deduce that $Z = 0$ for all continuous f_s in A. Since a continuous function in A can take arbitrary values at any finite subset of G, we conclude that G must satisfy the condition P_s.

We now proceed to the proof of the converse. Suppose that (b) holds. Then the computation above is reversible to the point where we have $[f_1, \ldots, f_s] = 0$ for $f_i \epsilon A$. Let P be a primitive ideal in A; the identity $[x_1, \ldots, x_s] = 0$ is of course inherited by $A - P$. Theorem 1 of [3] asserts that a primitive algebra satisfying a polynomial identity is finite-dimensional over its center. Hence $A - P$ is finite-dimensional over its centre C. By the Gelfand-Mazur theorem on normed fields, C is just the complex numbers. Hence $A - P$ is an algebra of finite order over the complex numbers, and is indeed a full matrix algebra. As for the degree of these matrices, it cannot exceed n; for by Lemma 2, $s \leqq r(n)+1 < r(n+1)$ and consequently matrices of degree $n+1$ fail to satisfy $[x_1, \ldots, x_s] = 0$. This shows that the primitive representations of G are finite-dimensional and of degree at most n, and concludes the proof of Theorem 3.

The criterion provided by Theorem 3 is in many cases easy to apply. For example, let G and H be unimodular locally compact groups whose primitive representations have bounded degree; then from Theorem 3 it follows that the same is true for $G \times H$, any unimodular homomorphic image of G, and any closed unimodular subgroup of G. Also the following result is a corollary of Theorems 2 and 3.

COROLLARY. *Let G be a connected unimodular locally compact group whose primitive representations are finite-dimensional and of bounded degree. Then G is abelian.*

This corollary may be derived in another way which we shall now describe. Let G be a connected locally compact group which is maximally almost-periodic, that is, G has a complete set of finite-dimensional unitary representations. (This hypothesis is weaker than the assumption that the primitive representations of G are finite-dimensional.) By a theorem of Freudenthal [5, p. 129], G is the direct product of a compact group and a finite number of copies of the additive group of real numbers. The question as to when the irreducible unitary representations are of bounded degree is thereby reduced to the compact case; and by considering the images under the representations, we further reduce to the case of a compact Lie group. In fact, our problem becomes precisely the following: prove that a connected compact simple Lie group possesses irreducible representations of arbitrarily high degree. That this is in fact the case follows from known classical results.

The corresponding theorem for Lie algebras asserts that a simple Lie algebra has irreducible representations of arbitrarily high degree. In this form, the theorem has recently been given a purely algebraic proof by Harish-Chandra [6]. It is perhaps worth remarking that, by standard devices, the theorem on Lie algebras can conversely be derived from the group theorem.

We return to the study of the group K of Theorem 1, and shall derive the purely group-theoretic fact that K satisfies $P_{s(n)}$. We give K the discrete topology, which assures its local compactness and unimodularity. Then by Theorems 1 and 3 we have that the primitive representations are finite-dimensional and of bounded degree. Theorems 1 and 3 also yield a bound for the degree in question, but a better bound can be obtained by a simple direct argument. In fact we assert that any finite-dimensional irreducible unitary representation T of K is of degree at most n. For the induced representation of G decomposes into one-dimensional representations, since G is abelian. Let α be a non-zero vector in one of these G-invariant one-dimensional subspaces. Then for $g \epsilon G$, $\alpha T(g)$ is a multiple of α. Using the notation of the proof of Theorem 1, we deduce that the invariant subspace generated by $\boldsymbol{a}$ is spanned by $\boldsymbol{a}T(k_1), \ldots, \boldsymbol{a}T(k_n)$ and hence is at most n-dimensional, as desired. Quotation of Theorem 3 proves that K satisfies $P_{s(n)}$.

We shall conclude this section with a variant of Theorem 3:

THEOREM 4. *The following two statements are equivalent for a unimodular locally compact group G:*

(a) The primitive representations of G are finite-dimensional and of bounded degree.

(b) The L_1-algebra of G satisfies a polynomial identity.

Proof. The proof coincides with the corresponding portions of the proof of Theorem 3, except for the following remark. In proving that (*b*) implies (*a*) we take a primitive ideal P in $A = L_1(G)$ and quote Theorem 1 of [3] to sustain the claim that $A - P$ is finite-dimensional. But more than that: Theorem 1 of [3] shows that the dimension of $A-P$ has a fixed upper bound depending only on the polynomial identity in question (cf. remark (*b*) on p. 580 of [3]). The rest of the proof proceeds unchanged.

6. A theorem of Halmos. The study of groups with bounded representations arose in connection with an attempt to generalize a theorem of Halmos, which we shall now describe. Let G be a compact group and S a continuous automorphism of G. The uniqueness of Haar measure shows that S induces a measure preserving transformation on G, which in turn induces a unitary operator $U: f \rightarrow f^S$ on $L_2(G)$. We say that S is ergodic if the only solutions of $f^S = f$ are constant. In [1] Halmos studied the case where G is commutative, and showed (Th. 3) that if S is ergodic, the spectral type of U is entirely determined by the cardinal number of the character group of G. We refer the reader to [1] for the precise result.

Now let G be a compact group which is not necessarily commutative. The automorphism S induces in a natural way a permutation π_S of the irreducible representations of G. This permutation leaves the trivial representation ψ fixed (ψ sends every element into the matrix (1)). The analogue of Halmos's [1, Th. 1] is now valid: S is ergodic if and only if π_S has no finite orbits other than ψ. The proof is virtually the same as that given by Halmos: one uses the coordinates of irreducible representations in place of characters.

Supposing that S is ergodic, we can now proceed to discuss the spectral type of U. Of course $U(\psi) = \psi$. By appropriate choice of the remaining coordinates of irreducible representations, which together with ψ form an orthonormal base of $L_2(G)$, we can arrange them in a double array $\phi_{i,j}$ such that $U(\phi_{i,j}) = U(\phi_{i,\,j+1})$. Here the index j runs over all integers, and the index i over the orbits of π_S. If we let c denote the number of orbits in question, we have proved Halmos's [1, Th. 3] except for the assertion that c is infinite.[4] If the

[4]If there are an uncountable number of irreducible representations, it is clear that c is infinite (and equal to that number). Thus further discussion is really needed only for the case of a countable number of irreducible representations.

representations have unbounded degree, then it is clear that c is infinite, for the permutation π_S necessarily preserves degree. At the other extreme, if all the representations are one-dimensional (G commutative), Halmos provided a group-theoretic argument on the character group to show that c is infinite [1, Th. 2]. There remains the case of representations of bounded degree, where it would be necessary to generalize suitably Halmos's argument. I have been unable to supply such an argument, but possibly the results in this paper will point the way toward the completion of this problem.

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Since this manuscript was completed, a paper by F. W. Levi has appeared: "On Skew Fields of a Given Degree," *J. Indian Math. Soc.*, vol. II (1947), 85-88. Reference is made there to a paper to be published in the *Mathematische Annalen*. In the notation of §2, this latter paper proves (among other things) that $r(n)$ is even and $r(3) = 6$. The distinction between $r(n)$ and $s(n)$ may therefore be suppressed.

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