ON SOME INFINITE DIMENSIONAL REPRESENTATIONS OF SEMI-SIMPLE LIE ALGEBRAS

HIROSHI KIMURA

1. Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over an algebraically closed field K of characteristic 0. For finite dimensional representations of $\mathfrak{g}$, the following important results are known;

1) $H^1(\mathfrak{g}, V) = 0$ for any finite dimensional $\mathfrak{g}$ space V. This is equivalent to the complete reducibility of all the finite dimensional representations.

2) Determination of all irreducible representations in connection with their highest weights.

3) Weyl's formula for the character of irreducible representations [9].

4) Kostant's formula for the multiplicity of weights of irreducible representations [6].

5) The law of the decomposition of the tensor product of two irreducible representations [1].

Harish-Chandra [3] studied the infinite dimensional $\mathfrak{g}$-spaces with dominant vectors, and established 2) and 3) for such spaces. The lacking of the complete reducibility 1) necessitates the study of $\mathrm{Ext}^1_{\mathfrak{g}}(U, V)$. A. Hattori determined the structure of $H^1(\mathfrak{g}, V) = \mathrm{Ext}^1_{\mathfrak{g}}(K, V)$ for an irreducible $\mathfrak{g}$-space V with a dominant vector [4]. To study the general case we need more. If U is finite dimensional, we have

$$\mathrm{Ext}^1_{\mathfrak{g}}(U, V) = H^1(\mathfrak{g}, \mathrm{Hom}_K(U, V)) = H^1(\mathfrak{g}, U^* \otimes V)$$

where U^* is the contragredient representation of U, so that we are led to the study of $U^* \otimes V$, a special case of 5). Theorem 1 of § 2 concerns with the structure of the tensor product. It follows from this theorem together with a generalization of Hattori's result (Theorem 2) that $\mathrm{Ext}^1_{\mathfrak{g}}(U, V) = 0$ for certain U and V (Theorem 3 of § 3). In § 4 we obtain a formula for the multiplicity of weights in case $\mathfrak{g} = \mathfrak{sl}(3, K)$.

Received February 4, 1964.

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

2. Tensor product

Let $\mathfrak{g}$ be a semi-simple Lie algebra over an algebraically closed field K of characteristic 0, $\mathfrak{h}$ a Cartan subalgebra and l the rank of $\mathfrak{g}$. Let Δ be the set of roots associated with $\mathfrak{h}$, $\{\alpha_1, \ldots, \alpha_l\}$ the set of the simple positive roots relative to some ordering and Δ^+ and Δ^- the sets of positive and negative roots, respectively. We write

$$\mathfrak{n}_+ = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha,\ \ \mathfrak{n}_- = \sum_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha} \text{ and } \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha\in\Delta$.

Let V_Λ be the representation space of $\mathfrak{g}$ with a dominant weight Λ and M_Λ the set of the weights of V_Λ. V_Λ has following decomposition into the weight spaces

$$V_\Lambda = \sum_{\mu\in M_\Lambda} V_\Lambda(\mu),\ \ \mu = \Lambda - \sum m_i(\mu)\alpha_i,$$

where $m_i(\mu)$ are non-negative integers.

Let V_λ be another representation space with the decomposition

$$V_\lambda = \sum_{\nu\in M_\lambda} V_\lambda(\nu),\ \ \nu = \lambda - \sum n_i(\nu)\alpha_i,$$

where $n_i(\nu)$ are non-negative integers. We consider the tensor product of the representation spaces V_Λ and V_λ.

THEOREM 1. *$V = V_\Lambda\otimes V_\lambda$ has a sequence of $\mathfrak{g}$-suqspaces*

$$(0) = V_0\subset V_1\subset \cdots \subset V_n\subset \cdots$$

such that i) $\bigcup_i V_i = V$ ii) for each $i = 1, 2, \ldots$, V_{i+1}/V_i is a representation space with a dominant vector and iii) the highest weight of V_{i+1}/V_i is of the form $\Lambda+\nu_i$, $\nu_i\in M_\lambda$. If in particular V_λ is finite dimensional, then the sequence consists of a finite number of terms.

Proof. We can decompose V into the following form

$$V = \sum_\omega V(\omega),\ \ \omega = \Lambda+\lambda - \sum l_i(\omega)\alpha_i,$$

where $V(\omega) = \sum_{\mu+\nu=\omega} V_\Lambda(\mu)\otimes V_\lambda(\nu)$ and $l_i(\omega)$, $i = 1, \ldots, l$, are non-negative integers. Put $\nu_0 = \lambda$. Next, let V_1 be the $\mathfrak{g}$-subspace of V generated by $v_\Lambda\otimes v'_\lambda$, $v_\Lambda\in V_\Lambda(\Lambda)$, $v'_\lambda\in V_\lambda(\lambda)$, and consider the factor space $V/V_1 = W^1$. Let $W^1 = \sum_\omega$

$W^1(\omega)$, $\omega=\mu+\nu$, $\nu\neq\nu_0$ or $\mu\neq\Lambda$, be the decomposition of W^1 into the weight spaces. Let $\Lambda+\nu_1=\Lambda+\lambda-\sum n_i(\nu_1)\alpha_i$ be a maximal weight in $A_1=\{\Lambda+\nu;\ \nu\in M-\{\nu_0\}\}$. It is clear that $W^1(\Lambda+\nu_1)$ is finite dimensional. Let $\{w_1^1, \ldots, w_{p_1}^1\}$ be a base of $W^1(\Lambda+\nu_1)$. Let U_1^1 be the $\mathfrak{g}$-subspace of W^1 generated by w_1^1. Then there is a $\mathfrak{g}$-subspace V_2 of V such that $U_1^1=V_2/V_1$. Let U_2^1 be the subspace of V/V_2 generated by $w_2^1 \pmod{V_2}$. Then there exists a subspace V_3 of V such that $U_2^1=V_3/V_2$. Similarly we get subspaces $V_4, \ldots, V_{1+p_1}$ of V. In considering $A_2=\{\Lambda+\nu;\ \nu\in M-\{\lambda, \nu_1\}\}$ by the same manner, we get subspaces $V_{1+p_1+1} \subset \cdots \subset V_{1+p_1+p_2}$. By repeating this process we get a sequence of subspaces of V.

$$(0)=V_0\subset V_1\subset \cdots \subset V_n\subset \cdots$$

It is clear from the construction of the sequence that V_{i+1}/V_i has a dominant vector and its highest weight is of the form $\Lambda+\nu$, $\nu\in M_\lambda$ $(i=1, 2, \ldots)$. To see $V^x=\bigcup_i V_i$ coincides with V, it is sufficient to show $V(\omega)\subset V^x$ for all ω. $V(\omega)$ is spanned by following vectors,

$$e_{-\alpha}^a e_{-\beta}^b \cdots e_{-\gamma}^c v_\Lambda \otimes e_{-\alpha'}^{a'} e_{-\beta'}^{b'} \cdots e_{-\gamma'}^{c'} v'_\lambda$$

where $\alpha, \beta, \ldots, \gamma, \alpha', \beta', \ldots, \gamma'$ are positive roots, $e_{-\alpha}, e_{-\beta}, \ldots, e_{-\gamma}, e_{-\alpha'}, e_{-\beta'}, \ldots, e_{-\gamma'}$ are non-zero vectors corresponding to $\alpha, \beta, \ldots, \gamma, \alpha', \beta', \ldots, \gamma'$ respectively, $a, b, \ldots, c, a', b', \ldots, c'$ are non-negative integers and $\Lambda+\lambda-(a\alpha+b\beta+\cdots+c\gamma+a'\alpha'+b'\beta'+\cdots c'\gamma')=\omega$.

Case 1. $\omega=\Lambda+\nu$, $\nu\in M_\lambda$. It is clear that V^x contains such vectors, i.e. $V(\omega)\subset V^x$.

Case 2. $\lambda-\sum l_i(\omega)\alpha_i\notin M_\lambda$ and there exists $\nu_{j_2}\in M_\lambda$ such that $\omega>\Lambda+\nu_{j_2}$. Let $\nu_{j_1}\in M_\lambda$ be the minimal weight such that $\Lambda+\nu_{j_1}>\omega$.

i) The case $a'\alpha'+\cdots+c'\gamma'=\sum n_i(\nu_{j_1})\alpha_i$. By assumption on j_1 we have $e_{-\delta}e_{-\alpha'}^{a'}e_{-\beta'}^{b'}\cdots e_{-\gamma'}^{c'}v'_\lambda=0$ for $\delta=\alpha, \beta, \ldots, \gamma$. Therefore

$$e_{-\alpha}^a \cdots e_{-\gamma}^c(v_\Lambda\otimes e_{-\alpha'}^{a'}, \ldots, e_{-\gamma'}^{c'} v'_\lambda)=e_{-\alpha}^a \cdots e_{-\gamma}^c v_\Lambda\otimes e_{-\alpha'}^{a'}, \ldots, e_{-\gamma'}^{c'} v'_\lambda.$$

From this we have $V(\omega)\subset V^x$.

ii) The case $j_1=0$.

$$e_{-\alpha}^a \cdots e_{-\gamma}^c(v_\Lambda\otimes v'_\lambda)=e_{-\alpha}^a \cdots e_{-\gamma}^c v_\Lambda\otimes v'_\lambda.$$

Therefore $V(\omega)\subset V^x$.

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

iii) The case $\Lambda+\lambda-\{(a-1)\alpha+b\beta+\cdots+c\gamma+a'\alpha'+\cdots+c'\gamma'\}\geqq\Lambda+\nu_{j_1}$. We may assume that if $\omega_1\geqq\Lambda+\nu_{j_1}$, $V(\omega_1)\subset V^x$ and that $e_{-\alpha}^{a-1}e_{-\beta}^{b}\cdots e_{-\gamma}^{c}v_\Lambda\otimes e_{-\alpha}e_{-\alpha'}^{a'}\cdots e_{-\gamma'}^{c'}v'_\lambda\in V^x$. From the equality,

$$\begin{aligned}&e_{-\alpha}(e_{-\alpha}^{a-1}e_{-\beta}^{b}\cdots e_{-\gamma}^{c}v_\Lambda\otimes e_{-\alpha'}^{a'}\cdots e_{-\gamma'}^{c'}v'_\lambda)\\ &\quad-e_{-\alpha}^{a-1}e_{-\beta}^{b}\cdots e_{-\gamma}^{c}v_\Lambda\otimes e_{-\alpha}e_{-\alpha'}^{a'}\cdots e_{-\gamma'}^{c'}v'_\lambda\\ &=e_{-\alpha}^{a}e_{-\beta}^{b}\cdots e_{-\gamma}^{c}v_\Lambda\otimes e_{-\alpha'}^{a'}\cdots e_{-\gamma'}^{c'}v'_\lambda,\end{aligned}$$

we have $V(\omega)\subset V^x$.

iv) The general case. By i), iii) we can assume that $e_{-\alpha}^{x}\cdots e_{-\gamma}^{z}v_\Lambda\otimes e_{-\alpha'}^{x'}\cdots e_{-\gamma'}^{z'}v'_\lambda\in V^x$ if $x\alpha+\cdots z\gamma$ is smaller than $a\alpha+\cdots+c\gamma$ or $x'\alpha'+\cdots z'\gamma'$ is greater than $a'\alpha'+\cdots+c'\gamma'$. From the same equality as in iii) we get $V(\omega)\subset V^x$.

Case 3. There exists no weight $\nu_j\in M_\lambda$ such that $\omega>\Lambda+\nu_i$. Let ν_{j_1} be a minimal weight such that $\Lambda+\nu_{j_1}>\omega$.

i) The case $a'\alpha'+\cdots+c'\gamma'=\sum n_i(\nu_{j_1})\alpha_i$. This case is treated similarly as case 2, i)

ii) The case $\Lambda+\lambda-\{(a-1)\alpha+b\beta+\cdots+c\gamma+a'\alpha'+\cdots+c'\gamma'\}\geqq\Lambda+\nu_{j_1}$. By case 1 and 2, $V(\omega_1)\subset V^x$ for all $\omega_1\geqq\Lambda+\nu_{j_1}$. Therefore the result comes from the same equality as in case 2, iii).

iii) The general case. By use of i), ii) the proof is similar to case 2, iv).

Finally, if V_λ is finite dimensional, then

$$\text{the length of the sequence}\leqq\sum_{\nu\in M_\lambda}\dim V(\Lambda+\nu)<\infty.$$

This completes the proof of Theorem 1.

3. $\mathrm{Ext}^1(V_\lambda\ V_\Lambda)$

The factor spaces in Theorem 1 are not necessarily irreducible. Therefore we need to extend Hattori's result to such a space.

THEOREM 2. *Let V be a representation space of $\mathfrak{g}$ with a dominant vector and λ its highest weight. Then*

$$H^1(\mathfrak{g},\ V)=\begin{cases}K, & \lambda=-\alpha_i\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let C be the Casimir operator of V. First we assume that $C\neq 0$.

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

Let v_λ be a dominant vector. We have $Cv_\lambda = kv_\lambda$, $k \in K$. Therefore $Cv = kv$ for all $v \in V$, so $CV = V$. It follows that $H^n(\mathfrak{g}, V) = 0$, $n = 1, 2, \ldots$, similarly as in the irreducible case. On the other hand if V is irreducible and $H^1(\mathfrak{g}, V) \neq 0$ it was shown by A. Hattori that $\lambda = -\alpha_i$ for some i and that $H^1(\mathfrak{g}, V) = K$. But his proof in [4] does not need the irreducibility. This completes the proof of Theorem 2.

THEOREM 3. *Let V_λ, V_Λ be the representation spaces with dominant weights λ, Λ respectively, and assume that V_λ is finite dimensional. If there is no simple root α_i such that $\Lambda + \alpha_i \in M_\lambda$, we have*

$$\mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\Lambda) = 0.$$

Proof. Let V_λ^* be the contragredient representation space of V_λ. The set of the weights of V_λ^* is $\{-\mu;\ \mu \in M_\lambda\}$. By Theorem 1 there exists the following sequence of $\mathfrak{g}$-subspaces of $V_\lambda^* \otimes V_\Lambda$,

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_n = V_\lambda^* \otimes V_\Lambda,$$

such that V_{i+1}/V_i has a dominant vector for each i and its highest weight is of the form $\Lambda - \mu$, $\mu \in M_\lambda$. By our assumption, $\Lambda - \mu \neq -\alpha_i$, $i = 1, \ldots, l$, for any $\mu \in M_\lambda$. Hence we have $H^1(\mathfrak{g}, V_{i+1}/V_i) = 0$ by Theorem 2. By the half exactness of H^1, we have $H^1(\mathfrak{g}, V_\lambda^* \otimes V_\Lambda) = 0$. Namely

$$\mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\Lambda) \cong H^1(\mathfrak{g}, \mathrm{Hom}_K(V_\lambda, V_\Lambda)) \cong H^1(\mathfrak{g}, V_\lambda^* \otimes V_\Lambda) = 0.$$

4. The multiplicity of a weight

In this section we consider the multiplicity of weights of an irreducible infinite dimensional representation with a dominant vector for $\mathfrak{g} = \mathfrak{sl}(3, K)$ .Let $\{\alpha_1, \alpha_2\}$ be a fundamental system of the roots with respect to a Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is spanned by H_{α_1}, H_{α_2} and the roots are $\pm\alpha_1$, $\pm\alpha_2$, $\pm(\alpha_1 + \alpha_2)$. Let e_α be a non-zero vector contained in the root space $\mathfrak{g}_\alpha$ of α. Let W be the Weyl group of $\mathfrak{g}$. W is generated by $S_1 = S_{\alpha_1}$ and $S_2 = S_{\alpha_2}$, where S_{α_i} are given by

$$S_{\alpha_i}(\mu) = \mu - 2(\mu, \alpha_i)\alpha_i/(\alpha_i, \alpha_i),\ \mu \in \mathfrak{h}_0$$

where $\mathfrak{h}_0$ is the real vector space spanned by Δ, and $W = \{1, S_1, S_2, S_2S_1, S_1S_2, S_1S_2S_1 = S_2S_1S_2\}$. Let $\{f_1, f_2\}$ be the dual base to the base $\{2\alpha_1/(\alpha_1, \alpha_1), 2\alpha_2/(\alpha_2, \alpha_2)\}$ of $\mathfrak{h}_0$, i.e,

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

$$2(\alpha_i, f_j)/\alpha_i, \alpha_i) = \delta_{ij}.$$

Let $\lambda = c_1 f_1 + c_2 f_2$ be a highest weight, where the c_i are the integers and put $\lambda_\sigma = \sigma(\lambda+\rho) - \rho$ for $\sigma \in W$, where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = f_1 + f_2$. Then

$$\begin{aligned}
\lambda_{S_1} &= \lambda - (c_1+1)\alpha_1 \\
\lambda_{S_2} &= \lambda - (c_2+1)\alpha_2 \\
\lambda_{S_2S_1} &= \lambda - (c_1+1)\alpha_1 - (c_1+c_2+2)\alpha_2 \\
\lambda_{S_1S_2} &= \lambda - (c_2+1)\alpha_2 - (c_1+c_2+2)\alpha_1 \\
\lambda_{S_1S_2S_1} &= \lambda - (c_1+c_2+2)(\alpha_1+\alpha_2).
\end{aligned}$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra and J_λ the left ideal generated by $\mathfrak{n}_+$ and $H - \lambda(H)$, $H \in \mathfrak{h}$. We study the structure of $U(\mathfrak{g})/J_\lambda = V_\lambda$.

LEMMA 4.1. *Let V' be a $\mathfrak{g}$-subspace of V_λ with a dominant vector and λ' its highest weight. Then $\lambda \geqq \lambda'$ and there exists $\sigma \in W$ such that $\lambda' = \lambda_\sigma$.*

Proof. The first part is trivial. Let χ_λ and $\chi_{\lambda'}$ be the characters of V_λ and $V_{\lambda'}$ respectively. By the definition of the character ([8] Exposé 18)

$$\chi_\lambda = \chi_{\lambda'}$$

Therefore, by ([8] Exposé 19, Théorèm 1), there is $\sigma \in W$ such that

$$\sigma(\lambda+\rho) = \lambda' + \rho.$$

This completes the proof of Lemma 4.1.

LEMMA 4.2. *For any λ_σ such that $\lambda \geq \lambda_\sigma$, there is one and only one subspace V_{λ_σ}.*

Proof. If $\lambda_\sigma = \lambda$, this is trivial. Therefore we may assume that $\lambda > \lambda_\sigma$.

(1) The case $c_1 + 1 > 0$, $c_2 + 1 > 0$. For any $\sigma \in W$, $\sigma \neq 1$, $\lambda_\sigma < \lambda$. For λ_σ we consider the following vectors,

$$\begin{aligned}
v_{\lambda_{S_1}} &= e_{-\alpha_1}^{c_1+1} \\
v_{\lambda_{S_2}} &= e_{-\alpha_2}^{c_2+1} \\
v_{\lambda_{S_2S_1}} &= e_{-\alpha_2}^{c_1+c_2+2} e_{-\alpha_1}^{c_1+1} \\
v_{\lambda_{S_1S_2}} &= e_{-\alpha_1}^{c_1+c_2+2} e_{-\alpha_2}^{c_2+1} \\
v_{\lambda_{S_1S_2S_1}} &= e_{-\alpha_1}^{c_2+1} e_{-\alpha_2}^{c_1+c_2+2} e_{-\alpha_1}^{c_1+1}.
\end{aligned}$$

We can easily verify the relations

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

$$e_{\alpha_i} v_{\lambda_\sigma} = 0, \ i = 1, 2.$$

Therefore

$$\mathfrak{n}_+ v_{\lambda_\sigma} = 0,$$

so that there exists a subspace V_{λ_σ} with the dominant vector v_{λ_σ}.

(2) The case $c_2+1<0$, $c_1+c_2+2>0$. $\sigma \in W$ such that $\lambda_\sigma < \lambda$ are S_1, S_2S_1, $S_1S_2S_1$. If $\sigma = S_1$ or S_2S_1 we can construct V_{λ_σ} from v_{λ_σ} in (1). If $\sigma = S_1S_2S_1$, in the expression of v_{λ_σ};

$$\begin{aligned} v_{\lambda_\sigma} &= e_{-\alpha_1}^{c_2+1} e_{-\alpha_2}^{c_1+c_2+2} e_{-\alpha_1}^{c_1+1} \\ &= \sum k_r e_{-\alpha_1}^{c_1+c_2+2-r} e_{-\alpha_2}^{c_1+c_2+2-r} e_{-(\alpha_1+\alpha_2)}^{r} \end{aligned}$$

$c_1+c_2+2-r \geqq 0$ for $k_r \neq 0$, because $e_{-\alpha_2}^{c_1+c_2+2} e_{-\alpha_1}^{c_1+1}$ is a linear combination of elements

$$e_{-\alpha_1}^{c_1+1-r} e_{-\alpha_2}^{c_1+c_2+2-r} e_{-(\alpha_1+\alpha_2)}^{r}, \ r = 0, 1, \ldots, c_1+c_2+2.$$

And we can verify the relations

$$e_{\alpha_i} v_{\lambda_\sigma} = 0. \qquad i = 1, 2$$

Therefore there exists a subspace V_{λ_σ} with the dominant vector v_{λ_σ}

(3) The case $c_1+1<0$, $c_1+c_2+2>0$. The proof is similar to case (2).

(4) The case $c_1+1>0$, $c_1+c_2+2 \leqq 0$. There is only one element $\sigma = S_1$ such that $\lambda_\sigma < \lambda$. Therefore there is the subspace $V_{\lambda_{S_1}}$ with the dominant vector $v_{\lambda_{S_1}}$.

(5) The case $c_2+1>0$, $c_1+2_2+2 \leqq 0$. There is the subspace $V_{\lambda_{S_2}}$ with the dominant vector $v_{\lambda_{S_2}}$.

(6) The case $c_1+1>0$, $c_2+1=0$. $\lambda_\sigma < \lambda$ for any $\sigma \neq 1$, S_2 and $\lambda_{S_1} = \lambda_{S_1S_2}$, $\lambda_{S_2S_1} = \lambda_{S_1S_2S_1}$. Then there are the subspaces $V_{\lambda_{S_1}}$, $V_{\lambda_{S_2S_1}}$ with the dominant vectors $v_{\lambda_{S_1}}$, $v_{\lambda_{S_2S_1}}$ respectively.

(7) The case $c_2+1>0$, $c_1+1=0$. The proof is similar to case (6).

(8) The case $c_1+1 \leqq 0$, $c_2+1 \leqq 0$. There is no element $\sigma \in W$ such that $\lambda_\sigma < \lambda$.

Next we shall show that V_{λ_σ} is unique for $\lambda_\sigma < \lambda$. Write $\lambda_\sigma = \lambda - \sum m_i \alpha_i$ where m_i are non-negative integers and

$$v_\sigma = \sum k_r e_{-\alpha_1}^{m_1-r} e_{-\alpha_2}^{m_2-r} e_{-(\alpha_1+\alpha_2)}^{r} \qquad \text{where } r \leqq \min(m_1, m_2).$$

v_σ is a dominant vector if and only if

https://doi.org/10.1017/S0027763000011545 Published online by Cambridge University Press

$$e_{\alpha_1} v_\sigma = 0 \tag{i}$$

$$e_{\alpha_2} v_\sigma = 0. \tag{ii}$$

We can verify that the coefficients $\{k_r\}$ satisfying (i), (ii) are unique for σ. This completes the proof of Lemma 4.2.

LEMMA 4.3. *Let $P(\mu)$ be the number of ways in which μ may be partitioned into a sum of positive roots. Let $\lambda = c_1 f_1 + c_2 f_2$, where c_i are integers, be the highest weight of an infinite dimensional representation of $\mathfrak{g}$ with a dominant vector and $m_{\lambda_\sigma}(\mu) = \dim V_{\lambda_\sigma}(\mu)$, where V_{λ_σ} is an irreducible $\mathfrak{g}$-space. Then*

$$P(\lambda - \mu) = \sum_{\lambda\sigma \equiv \lambda} m_{\lambda_\sigma}(\mu).$$

Proof. In the decomposition to the weight spaces

$$U(\mathfrak{g})/J_\lambda = \sum_\mu V_\lambda(\mu),$$

it is clear that

$$\dim V_\lambda(\mu) = P(\lambda - \mu).$$

First we shall prove that any subspace V of V_λ is the union of subspaces with dominant vectors. If $V = V_\lambda$, then this is clear. So we assume that $V \neq V_\lambda$. Then

$$V = \sum_\mu V \cap V(\mu) \text{ and } V \cap V_\lambda(\lambda) = 0.$$

Put ${}^1V = \bigcup_{V_{\lambda_\sigma} \subset V} V_{\lambda_\sigma}$. We consider $V/{}^1V = V'$. Let ν be the maximal weight of V' and $v(\nu) (\bmod\ {}^1V)$ a non-zero vector belonging to ν. Let V'_ν be the subspace of V' generated by $v(\nu) (\bmod\ {}^1V)$. The characters of V_λ and V'_ν are equal. Therefore there is $\sigma \in W$ such that $\nu = \lambda_\sigma$. Write

$$e_{\alpha_1} v(\lambda_\sigma) = \sum h_r^1 e_{-\alpha_1}^{m_1 - 1 - r} e_{-\alpha_2}^{m_2 - r} e_{-(\alpha_1 + \alpha_2)}^r \qquad r \leqq \min(m_1 - 1, m_2),$$

$$e_{\alpha_2} v(\lambda_\sigma) = \sum h_r^2 e_{-\alpha_1}^{m_1 - r} e_{-\alpha_3}^{m_2 - 1 - r} e_{-(\alpha_1 + \alpha_2)}^r \qquad r \leqq \min(m_1, m_2 - 1).$$

Then $h_r^1 = -h_r^2$ for all r. On the other hand if $\lambda_\sigma > \lambda_\tau$, then by the construction of v_{λ_σ} we have $V_{\lambda_\sigma} \supset V_{\lambda_\tau}$. Therefore

$${}^1V = \bigcup_{\lambda\sigma} V_{\lambda_\sigma},$$

where any λ_σ is a maximal weight in the set $\{\lambda_\tau;\ V_{\lambda_\tau} \subset V\}$.

(1) The case $c_2 + 1 < 0$, $c_1 + c_2 + 2 > 0$. In this case $\lambda > \lambda_{S_1} > \lambda_{S_2 S_1}$ and $\lambda >$

$\lambda_{s_1s_2s_1} > \lambda_{s_2s_1}$. When ${}^1V = V_{\lambda s_1}$, $V_{\lambda s_2s_1}$ or $V_{\lambda s_1s_2s_1}$ there is no element $v(\lambda_\sigma)$. If ${}^1V = V_{\lambda s_1} \cup V_{\lambda s_2s_1}$, then the vectors of ${}^1V$ belonging to the weights $\lambda_{s_2s_1}+\alpha_1$, $\lambda_{s_2s_1}+\alpha_2$ are $e_{-\alpha_1}^{-1}e_{-\alpha_2}^{c_1+c_2+2}e_{-\alpha_1}^{c_1+1}$, $e_{-\alpha_2}^{c_1+c_2+1}e_{-\alpha_1}^{c_1+1}$ respectively. From this we obtain that $h_r^1 \neq -h_r^2$ for some r if h_r^1, h_r^2 are not zero for all r. Therefore $v(\lambda_\sigma) = v_{\lambda_\sigma}$ and $V = {}^1V$.

(2) The case $c_1+1<0$, $c_1+c_2+2>0$. The proof is similar to case (1).

(3) The case $c_1+1>0$, $c_1+c_2+2<0$. λ_σ such that $\lambda_\sigma<\lambda$ is λ_{s_1}. Therefore the result is trivial.

(4) The case $c_2+1>0$, $c_1+c_2+2<0$. The proof is similar to the case (3).

(5) The case $c_1+1>0$, $c_2+1=0$. $\lambda>\lambda_{s_1}>\lambda_{s_2s_1}$. If ${}^1V = V_{\lambda_1}$, then

$$e_{\alpha_1}v(\lambda_{s_2s_1}) = 0.$$

Therefore $v(\lambda_{s_2s_1}) \in V_{\lambda s_2s_1}$

(6) The case $c_2+1>0$, $c_1+1=0$. The proof is similar to case (5).

(7) The case $c_1+1\leqq 0$, $c_2+1\leqq 0$. There is no λ such that $\lambda_\sigma<\lambda$. Therefore V_λ is a simple $\mathfrak{g}$-module.

Now we consider a composition series, and we obtain

$$P(\lambda-\mu) = \sum_{\lambda\geqq\lambda_\sigma} m_{\lambda_\sigma}(\mu).$$

This completes the proof of Lemma 4.3.

THEOREM 3. *Let* $\mathfrak{g}$ *be* $\mathfrak{sl}(3, K)$. *We assume that* $\sigma(\lambda+\rho)\neq\lambda+\rho$ *unless* $\sigma=1$. *Then we have*

$$m_\lambda(\mu) = \sum_{\lambda\geqq\lambda_\sigma} \mathrm{Sg}(\sigma)P(\lambda_\sigma-\mu).$$

Proof. (1) The case $c_1+1>0$, $c_2+1>0$. This is Kostant's formula.

(2) The other case. We may easily examine that

$$\sum_{\lambda\geqq\lambda_\sigma\geqq\lambda_\tau} \mathrm{Sg}(\sigma) = \begin{cases} 1, & \tau=1 \\ 0, & \tau\neq 1 \end{cases}$$

By these relations and Lemma 4.3, we have

$$\begin{aligned} m_\lambda(\mu) &= \sum_{\lambda\geqq\lambda_\tau} m_{\lambda_\tau}(\mu) \sum_{\lambda\geqq\lambda_\sigma\geqq\lambda_\tau} \mathrm{Sg}(\sigma) \\ &= \sum_{\lambda\geqq\lambda_\sigma} \mathrm{Sg}(\sigma) \sum_{\lambda_\sigma\geqq\lambda_\tau} m_{\lambda_\tau}(\mu) \\ &= \sum_{\lambda\geqq\lambda_\sigma} \mathrm{Sg}(\sigma)P(\lambda_\sigma-\mu). \end{aligned}$$

This completes the proof of Theorem 3.

Finally we consider the case that there is $\sigma(\neq 1)\in W$ such that $\sigma(\lambda+\rho)=\lambda+\rho$. By Lemma 4.3,

$$m_\lambda(\mu)=P(\lambda-\mu)-\sum_{\lambda<\lambda_\sigma} m_{\lambda_\sigma}(\mu).$$

(1) The case $c_1+1>0$, $c_1+c_2+2=0$.

$$m_\lambda(\mu)=P(\lambda-\mu)-P(\lambda_{S_1}-\mu).$$

(2) The case $c_2+1>0$, $c_1+c_2+2=0$.

$$m_\lambda(\mu)=P(\lambda-\mu)-P(\lambda_{S_2}-\mu).$$

(3) The case $c_1+1>0$, $c_2+1=0$.

$$m_\lambda(\mu)=P(\lambda-\mu)-P(\lambda_{S_1}-\mu).$$

(4) The case $c_1+1=0$, $c_2+1>0$.

$$m_\lambda(\mu)=P(\lambda-\mu)-P(\lambda_{S_2}-\mu).$$

References

[1] R. Brauer, Sur la multiplication des caractéristiques des groupes continus et semi-simples, C. R. Acad. Sci. Paris, **204** (1937), 1784-1786.

[2] P. Cartier, On H. Weyl's character formula, Bull. Amer. Math. Soc., **67** (1961), 228-230.

[3] Harish Chandra, On some applications of the universal enveloping algebra of a semi-simple Lie algebras, Trans. Amer. Math. Soc., **70** (1951), 28-96.

[4] A. Hattori, On 1-cohomology groups of infinite dimensional representations of semi-simple Lie algebras, J. Math. Soc. Japan, **16** (1964), 226-229.

[5] N. Iwahori, Theory of Lie groups (In Japanese), Iwanami-Shoten (1957).

[6] B. Kostant, A formula for the multiplicity of a weight, Trans. Amer. Math. Soc., **93** (1959), 53-73.

[7] Y. Matsushima, Theory of Lie algebras (In Japanese), Kyoritsu-Press (1956).

[8] Séminaire "Sophus Lie", Ecole Normale Supérieure, Paris (1955).

[9] H. Weyl, Über die Darstellungen der halbeinfachen Gruppen durch lineare Transformationen, Math. Z., **24** (1926), 328-395.

Tokyo University of Education