THE SECOND MEAN VALUES OF ENTIRE FUNCTIONS

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1. Let $f(z)$ be an entire function of the complex variable $z = x + iy$ defined by the everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n \exp(\lambda_n z) \qquad (0 < \lambda_n < \lambda_{n+1} \to \infty). \tag{1.1}$$

If

$$m(x,f) = \sup_{-\infty<y<\infty} |f(x+iy)|,$$

then $\log m(x,f)$ is an increasing convex function of x **(2)**, and

$$\rho_{R,f} = \limsup_{x\to\infty} \frac{\log \log m(x,f)}{x}$$

is called the *Ritt order* of $f(z)$. For functions of finite Ritt order **(5)**

$$\log m(x,f') \sim \log m(x,f) \qquad \text{as } x \to \infty. \tag{1.2}$$

Here f' stands for the derivative of f.

Let

$$I_2(x,f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+iy)|^2\, dy.$$

Gupta **(3**, Theorem 2**)** has proved that, for $x > x_0$,

$$I_2(x,f') - I_2(x,f) \left(\frac{\log I_2(x,f)}{2x}\right)^2 \geqslant 0. \tag{1.3}$$

We observe that the difference between $I_2(x,f')$ and

$$I_2(x,f) \left(\frac{\log I_2(x,f)}{2x}\right)^2$$

is much greater for large x. In fact, the zero on the right-hand side of (1.3) can be replaced by

$$\frac{1}{(2x)^2} I_2(x,f) \log I_2(x,f) \log \left(\frac{I_2(x,f)}{2x}\right).$$

We shall deduce various results for $I_2(x,f)$ and $I_2(x,f')$ from corresponding results involving $m(x,f)$ and $m(x,f')$.

To illustrate the method, we are going to prove the proposed refinement of (1.3).

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LEMMA. *If in* (1.1) *all the coefficients* $\{a_n\}$, $n = 1, 2, \ldots$, *are non-negative, then for large values of* x,

$$m(x, f') \geqslant m(x, f)\, \frac{\log m(x, f)}{x}.$$

From the fact that $\log m(x, f)$ is a non-decreasing convex function of x, it follows that

$$g(x) = \frac{\log m(x, f)}{x}$$

is non-decreasing for large values of x, say $x > x_0$. Since the coefficients $\{a_n\}$, $n = 1, 2, \ldots$, are non-negative, $m(x, f) = f(x)$ and $m(x, f') = f'(x)$. If $x > x_0$, then

$$\begin{aligned} m(x, f') = f'(x) &= \lim_{h \to +0} \frac{f(x) - f(x - h)}{h} \\ &= \lim_{h \to +0} \frac{m(x, f) - m(x - h, f)}{h} \\ &= \lim_{h \to +0} \frac{\exp\{xg(x)\} - \exp\{(x - h)g(x - h)\}}{h} \\ &\geqslant \lim_{h \to +0} \frac{\exp\{xg(x)\} - \exp\{(x - h)g(x)\}}{h} \\ &= \exp\{xg(x)\}g(x) = m(x, f)\, \frac{\log m(x, f)}{x}, \end{aligned}$$

and the lemma is proved.

Now let $f(z)$ be an entire function defined by (1.1), where the coefficients are not restricted to be non-negative. Note that the functions represented by the series

$$\sum_{n=1}^{\infty} |a_n|^2 \exp(\lambda_n z), \qquad \sum_{n=1}^{\infty} \lambda_n |a_n|^2 \exp(\lambda_n z)$$

satisfy the hypothesis of our lemma. The coefficients are clearly non-negative. The fact that

$$\sum_{n=1}^{\infty} |a_n|^2 \exp(\lambda_n z)$$

represents an entire function $\phi(z)$ follows, for example, from the fact that, for every $X < \infty$,

$$\max_{\mathrm{Re}\, z < 2X} \big| |a_n|^2 \exp(\lambda_n z) \big| \leqslant |a_n|^2 \exp(2\lambda_n X),$$

and

$$\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n X)$$

is convergent, its sum being $I_2(X, f)$. The series

$$\sum_{n=1}^{\infty} \lambda_n |a_n|^2 \exp(\lambda_n z)$$

represents the function $\phi'(z)$. Applying the lemma to the functions $\phi(z)$ and $\phi'(z)$, we conclude that for large values of x

$$(1.4)\qquad \sum_{n=1}^{\infty} \lambda_n |a_n|^2 \exp(2\lambda_n x) \geqslant \left(\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right) \frac{\log \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)}{2x}$$

and

$$(1.5)\qquad \sum_{n=1}^{\infty} \lambda_n^2 |a_n|^2 \exp(2\lambda_n x) \geqslant \left(\sum_{n=1}^{\infty} \lambda_n |a_n|^2 \exp(2\lambda_n x)\right) \frac{\log \sum_{n=1}^{\infty} \lambda_n |a_n|^2 \exp(2\lambda_n x)}{2x}.$$

Thus if x is large enough, say $x > x_0$, then

$$\begin{aligned} I_2(x, f') &= \sum_{n=1}^{\infty} \lambda_n^2 |a_n|^2 \exp(2\lambda_n x) \\ &\geqslant \left(\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right)\left(\frac{\log \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)}{2x}\right)^2 \\ &+ \frac{1}{(2x)^2}\left(\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right)\left(\log \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right) \\ &\times \log\left(\frac{\log \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)}{2x}\right) = I_2(x, f)\left(\frac{\log I_2(x, f)}{2x}\right)^2 \\ &+ \frac{1}{(2x)^2} I_2(x, f) \log I_2(x, f) \log\left(\frac{\log I_2(x, f)}{2x}\right). \end{aligned}$$

This gives the desired refinement of (1.3).

THEOREM 1. *If the Ritt order of f is finite, then*

$$\log I_2(x, f') \sim \log I_2(x, f), \qquad x \to \infty.$$

We have proved elsewhere that if the function $f(z)$ defined by (1.1) is of finite Ritt order $\rho_{R,f}$, then for every $\epsilon > 0$,

$$(1.6)\qquad m(x, f') \leqslant m(x, f) \exp\{x(\rho_{R,f} + \epsilon)\},$$

if x is sufficiently large. Since

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+iy)|^2\, dy \leqslant \{m(x, f)\}^2,$$

the Ritt order of the function represented by the series

$$\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n z)$$

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is at most $\rho_{R,f}$. Hence, for every $\epsilon > 0$ and sufficiently large x,

$$(1.7)\quad \sum_{n=1} 2\lambda_n |a_n|^2 \exp(2\lambda_n x) \leqslant \left(\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right) \exp\{x(\rho_{R,f} + \epsilon)\}.$$

We know **(5)** that the Ritt order of a function is the same as the Ritt order of its derivative. Therefore the Ritt order of the function represented by the series

$$\sum_{n=1}^{\infty} 2\lambda_n |a_n|^2 \exp(2\lambda_n z)$$

is not greater than $\rho_{R,f}$. From (1.6), we obtain

$$(1.8)\quad \sum_{n=1}^{\infty} 4\lambda_n^2 |a_n|^2 \exp(2\lambda_n x) \leqslant \left(\sum_{n=1}^{\infty} 2\lambda_n |a_n|^2 \exp(2\lambda_n x)\right) \exp\{x(\rho_{R,f} + \epsilon)\},$$
$$\epsilon > 0,\ x > X(\epsilon).$$

Inequalities (1.7) and (1.8) are equivalent to

$$(1.9)\quad I_2(x, f') = \sum_{n=1}^{\infty} 4\lambda_n^2 |a_n|^2 \exp(2\lambda_n x)$$
$$\leqslant \left(\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)\right) \exp\{2x(\rho_{R,f} + \epsilon)\}$$
$$= I_2(x, f) \exp\{2x(\rho_{R,f} + \epsilon)\}, \qquad \epsilon > 0,\ x > X(\epsilon).$$

This fact, together with (1.3) and **(3**, Theorem 3**)**, implies that for functions of finite Ritt order

$$\log I_2(x, f') \sim \log I_2(x, f) \qquad \text{as } x \to \infty.$$

Azpeitia **(1)** has proved that if

$$(1.10)\quad \lim_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log n} = \infty,$$

then

$$\rho_{R,f} = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log(1/|a_n|)}.$$

Hence if $f(z)$, defined by (1.1), is of order $\rho_{R,f}$ $(0 \leqslant \rho_{R,f} \leqslant \infty)$, the order of the function defined by the series

$$\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n z)$$

is also $\rho_{R,f}$ if (1.10) holds. Consequently, if (1.10) is satisfied, then

$$(1.11)\quad \limsup_{x\to\infty} \frac{\log\log I_2(x, f)}{x} = \limsup_{x\to\infty} \frac{\log\log \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x)}{x} = \rho_{R,f}.$$

If

$$(1.12)\quad \liminf_{n\to\infty} \frac{\lambda_n}{\log n} = \frac{1}{D} > 0,$$

then for every $\epsilon > 0$

$$\{m(x,f)\}^2 \leqslant \left\{\sum_{n=1}^{\infty} |a_n|^2 \exp\{2\lambda_n(x + \tfrac{1}{2}D + \epsilon)\}\right\} \sum_{n=1}^{\infty} \exp\{-2\lambda_n(\tfrac{1}{2}D + \epsilon)\}$$
$$< KI_2(x + \tfrac{1}{2}D + \epsilon, f)$$

where K is a constant. From this and the inequality

$$I_2(x,f) \leqslant \{m(x,f)\}^2$$

it follows that

$$\liminf_{x\to\infty} \frac{\log\log I_2(x,f)}{x} = \lambda_{R,f}, \tag{1.13}$$

where

$$\lambda_{R,f} = \liminf_{x\to\infty} \frac{\log\log m(x,f)}{x}$$

is the lower order of f.

THEOREM 2. *If* (1.10) *is satisfied, then*

$$\limsup_{x\to\infty} \frac{\log\{I_2(x,f')/I_2(x,f)\}}{x} = 2\rho_{R,f}$$

and

$$\liminf_{x\to\infty} \frac{\log\{I_2(x,f')/I_2(x,f)\}}{x} = 2\lambda_{R,f}, \tag{1.14}$$

provided (1.12) *holds.*

From (1.3), (1.9), and (1.11), we can deduce the first equation, whereas, from (1.3), (1.9), and (1.13) follows the inequality

$$\liminf_{x\to\infty} \frac{\log\{I_2(x,f')/I_2(x,f)\}}{x} \geqslant 2\lambda_{R,f}. \tag{1.15}$$

The sign of inequality in (1.15) can, in fact, be replaced by the equality sign. For this we refer to the inequality (**5**, (14))

$$m(x,f') \leqslant \frac{1}{\delta} m(x + \delta, f), \qquad \delta > 0,$$

valid for the entire function f defined by (1.1). Applying this result successively to the entire functions

$$\sum_{n=1}^{\infty} 2\lambda_n |a_n|^2 \exp(2\lambda_n(z - 2\delta)), \qquad \sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n(z - \delta)),$$

we obtain

$$(1.16)\qquad I_2(x-2\delta,f') = \sum_{n=1}^{\infty} 4\lambda_n^{\,2}\,|a_n|^2 \exp(2\lambda_n(x-2\delta))$$

$$\leqslant \frac{1}{\delta}\sum_{n=1}^{\infty} 2\lambda_n\,|a_n|^2 \exp(2\lambda_n(x-\delta))$$

$$\leqslant \frac{1}{\delta^2}\sum_{n=1}^{\infty} |a_n|^2 \exp(2\lambda_n x) = \frac{1}{\delta^2} I_2(x,f), \qquad \delta > 0.$$

Now, since $\log I_2(x,f)$ is an increasing convex function of x, we have

$$(1.17)\qquad \log I_2(x,f) = \log I_2(x_0,f) + \int_{x_0}^{x} w(t)\,dt,$$

where $x_0 < x$ and $w(t)$ is a non-decreasing function of t. From this it follows that if $\lambda_{R,f} < \infty$ and ϵ is a fixed positive number, then, for a sequence of values of x tending to infinity,

$$\int_{x}^{x+2} w(t)\,dt < \log I_2(x+2,f) < \exp\{(x+2)(\lambda_{R,f}+\epsilon)\}, \qquad x = x_1, x_2, \ldots.$$

Since $w(t)$ is non-decreasing, we obtain

$$2w(x_n) < \exp\{(x_n+2)(\lambda_{R,f}+\epsilon)\}, \qquad n = 1, 2, \ldots.$$

Since ϵ is arbitrary, we can write

$$w(x_n) < \exp\{x_n(\lambda_{R,f}+\epsilon)\}$$

for sufficiently large n. Equality (1.17) then gives

$$\log I_2(x_n,f) = \log I_2(x_n-2\delta,f) + \int_{x_n-2\delta}^{x_n} w(t)\,dt,$$

where the integral is smaller than

$$2\delta \exp\{x_n(\lambda_{R,f}+\epsilon)\},$$

and if we take

$$\delta = \tfrac{1}{2}\exp\{-x_n(\lambda_{R,f}+\epsilon)\},$$

we obtain

$$\log I_2(x_n,f) < \log I_2(x_n-2\delta,f) + 1,$$

for sufficiently large n. Substituting this value of δ and the corresponding estimate for $I_2(x_n,f)$ in (1.16), we see that, $\epsilon > 0$ being given, there exists a sequence of values of x such that

$$(1.18)\qquad I_2(x-2\delta,f') \leqslant I_2(x-2\delta,f)\exp(2x(\lambda_{R,f}+\epsilon)).$$

Since $\delta < 1$, we can even write

$$I_2(x-2\delta,f') \leqslant I_2(x-2\delta,f)\exp\{2(x-2\delta)(\lambda_{R,f}+\epsilon)\}$$

instead of (1.18). It follows that

$$\liminf_{x\to\infty} \frac{\log\{I_2(x,f')/I_2(x,f)\}}{x} \leqslant 2\lambda_{R,f}. \tag{1.19}$$

Thus, if $\lambda_{R,f} < \infty$, the inequality in (1.15) can be replaced by equality. If $\lambda_{R,f} = \infty$, then, from (1.15),

$$\liminf_{x\to\infty} \frac{\log\{I_2(x,f')/I_2(x,f)\}}{x} = \infty.$$

2. In this section, we are going to make a certain remark concerning the second mean value

$$\mu_2(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta\right)^{\frac{1}{2}}$$

of $|f(z)|$ on the circle $|z| = r$.

It has been proved by Lakshminarasimhan **(4**, Lemma 2**)** that if $f(z)$ is an entire function, then for $r > r_0 \geqslant 1$

$$\mu_2(r,f') > \frac{\mu_2(r,f)}{r}\,\frac{\log \mu_2(r,f) - \log \mu_2(r_0,f)}{\log r}.$$

By the argument used in the preceding section, we can deduce a somewhat better result from an inequality of Vijayaraghavan **(6)** which states that if

$$M(r,f) = \max_{|z|=r} |f(z)|,$$

then for $r > r_0$

$$M(r,f') \geqslant \frac{M(r,f)}{r}\,\frac{\log M(r,f)}{\log r}.$$

We note that if $f(z)$ has the power series representation

$$\sum_{n=0}^{\infty} a_n z^n,$$

then

$$(\mu_2(r,f))^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

and if $f(z)$ is an entire function, then the series

$$\sum_{n=0}^{\infty} |a_n|^2 z^n, \qquad \sum_{n=0}^{\infty} n|a_n|^2 z^n$$

also represent entire functions. We obtain the following result:

If $f(z)$ is an entire function, then

$$\mu_2(r,f') \geqslant \frac{\mu_2(r,f)}{r}\,\frac{\log \mu_2(r,f)}{\log r}$$

for $r > r_0$, where r_0 is a number depending on f.

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