Note on Cubature Formulae and Designs Obtained from Group Orbits

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Abstract. In 1960, Sobolev proved that for a finite reflection group G, a G-invariant cubature formula is of degree t if and only if it is exact for all G-invariant polynomials of degree at most t. In this paper, we make some observations on invariant cubature formulas and Euclidean designs in connection with the Sobolev theorem. First, we give an alternative proof of theorems by Xu (1998) on necessary and sufficient conditions for the existence of cubature formulas with some strong symmetry. The new proof is shorter and simpler compared to the original one by Xu, and, moreover, gives a general interpretation of the analytically-written conditions of Xu's theorems. Second, we extend a theorem by Neumaier and Seidel (1988) on Euclidean designs to invariant Euclidean designs, and thereby classify tight Euclidean designs obtained from unions of the orbits of the corner vectors. This result generalizes a theorem of Bajnok (2007), which classifies tight Euclidean designs invariant under the Weyl group of type B, to other finite reflection groups.

1 Introduction

A central problem in numerical integration is to approximate the integral

$$\int_\Omega f(x)d\mu.$$

Here x is an n-dimensional coordinate vector and μ is a probability measure on a domain Ω in $\mathbb{R}^n$. We search for an approximation formula by taking a positive linear combination of the function values of f at specified points $x_1, \ldots, x_N$, that is,

$$(1.1) \qquad \sum_{i=1}^{N} w_i f(x_i).$$

We call (1.1) a *cubature formula*, with *weights* w_i and *points* x_i. To each formula we assign the set of functions for which it is exact. Most often this set is the space of all polynomials of degree no more than t; in this case a cubature formula is said to be *of degree t*. We refer the readers to the comprehensive monograph [11, 25] for the basic theory of cubature formula.

A fundamental objective of this paper is to construct cubature formulas of large degrees with few points. The requirement that a given cubature formula is exact for

polynomials up to a certain degree can be reduced to the problem of solving a system of algebraic equations. In general, the larger the number of points or the degree of a cubature formula, the greater the size of this system. Sobolev [24] gave a celebrated criterion to diminish the size of the system to be solved. Namely, he proved that an invariant cubature formula is of degree t if and only if it is exact for all polynomials of degree at most t invariant under the group. This is known as the Sobolev theorem. The Sobolev theorem is widely accepted by the cubature community in analysis and related areas; for instance, see [19, 23].

Independent of the line of research in analysis and related areas, Goethals and Seidel [13, Theorem 3.12] developed the invariant theory of Chebyshev-type cubature formulas on the sphere or spherical designs. As a generalization of spherical designs, Neumaier and Seidel [20] considered cubature formulas on several concentric spheres called Euclidean designs. Bajnok [2] classified tight Euclidean designs whose points are the union of the orbits of the corner vectors of the group B_n, and in particular, he obtained several new tight designs. Here a Euclidean design is tight if it is minimal with respect to a lower bound for the number of points. To obtain the results, Bajnok [2, Proposition 14] essentially used the idea of the Sobolev theorem for B_n-invariant Euclidean designs, though he did not mention Sobolev. It seems that some researchers in combinatorics and related areas do not fully recognize the Sobolev theorem.

In this paper we make some observations on invariant cubature formulas in connection with the Sobolev theorem. In Section 2 we explain Sobolev's invariant theory in detail. We also explain some basic facts related to Euclidean designs, *e.g.,* a theorem of Neumaier and Seidel [20] that is well known in algebra and combinatorics. In Section 3 we give an alternative proof of famous theorems by Xu [27, Theorems 1.1 and 1.2] on necessary and sufficient conditions for the existence of cubature formulas with radial symmetry. The original proof by Xu requires some tedious calculations and technical tools in numerical analysis such as Gaussian–Lobatto quadrature and Gaussian–Radau quadrature. Furthermore, it is long, and researchers in other areas may not be familiar with this proof; Our new proof is short and simple compared to the original proof. Moreover it gives a general interpretation of the analytically-written conditions of Xu's theorems, and so will be readable and acceptable for researchers not only in analysis, but also in other areas, such as algebra and combinatorics. In Section 4 we extend the theorem of Neumaier and Seidel to invariant Euclidean designs, and thereby classify tight Euclidean designs obtained from unions of the orbits of the corner vectors. This classification generalizes the result of Bajnok for other finite reflection groups.

2 Preliminaries

Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the linear space of all real homogeneous polynomials of total degree l in n variables. Let

$$\mathcal{P}_l(\mathbb{R}^n) = \sum_{i=0}^{l} \mathrm{Hom}_i(\mathbb{R}^n), \quad \mathcal{P}_l^*(\mathbb{R}^n) = \sum_{i=0}^{\lfloor l/2 \rfloor} \mathrm{Hom}_{l-2i}(\mathbb{R}^n).$$

We denote by $\text{Harm}_l(\mathbb{R}^n)$ the subspace of $\mathcal{P}_l(\mathbb{R}^n)$ of harmonic homogeneous polynomials of degree l. Let $\mathcal{P}_l(A), \mathcal{P}_l^*(A)$ be the space of functions that are the restrictions of the corresponding polynomials to $A \subset \mathbb{R}^n$.

Let G be a finite subgroup of the orthogonal group in $\mathbb{R}^n$ and $f \in \mathcal{P}_l(\mathbb{R}^n)$. We consider the action of $\sigma \in G$ on f as follows:

$$(\sigma f)(x) = f(x^{\sigma^{-1}}), \qquad x \in \mathbb{R}^n.$$

A polynomial f is said to be *G-invariant* if it satisfies that

$$\sigma f = f, \qquad \forall \sigma \in G.$$

We denote by $\mathcal{P}_l(\mathbb{R}^n)^G, \text{Harm}_l(\mathbb{R}^n)^G$ the set of G-invariant polynomials in $\mathcal{P}_l(\mathbb{R}^n)$, $\text{Harm}_l(\mathbb{R}^n)$ respectively.

A cubature formula (1.1) is said to be *invariant under G*, or *G-invariant* if the domain Ω and measure μ of the integral are invariant under G and the set of points is the union of G-orbits and to each point of the same orbit an equal weight is assigned. The following is known as the Sobolev theorem.

Theorem 2.1 ([24]) *With the above set up, a G-invariant cubature formula is of degree t if and only if it is exact for every polynomial $f \in \mathcal{P}_t(\mathbb{R}^n)^G$.*

The Sobolev theorem is widely accepted by the cubature community in analysis and related areas. In particular Russian mathematicians in analysis have developed the Sobolev theorem and employed it to construct many cubature formulas; for instance, see [19,23]. Xu [27] presented beautiful theorems on the existence and structure of cubature formulas for radially symmetric integrals. In Section 3 we review his theorems in detail and give a short proof using the Sobolev theorem.

Next let us explain a combinatorial object called Euclidean design that was introduced by Neumaier and Seidel [20]. Let X be a finite set in $\mathbb{R}^n$. Let $r_1, r_2, \ldots, r_p$ be the norms of the vectors in X. For $i = 1, 2, \ldots, p$ we denote by S_i^{n-1} the sphere of radius r_i centered at the origin, namely, $S_i^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = r_i\}$, and let $X_i = X \cap S_i^{n-1}$. The set X is said to be *supported by p concentric spheres* $S = \bigcup_{i=1}^p S_i^{n-1}$. To each S_i we assign the surface measure ρ_i. Let $|S_i^{n-1}| = \int_{S_i^{n-1}} d\rho_i(x)$, with the convention that

$$\frac{1}{|S_i^{n-1}|} \int_{S_i^{n-1}} f(x) d\rho_i(x) = f(0)$$

if $S_i^{n-1} = \{0\}$.

Definition 2.2 With the same notations as in the above paragraph, we say that X is a *Euclidean t-design supported by S* if there exists a positive weight function $w(x)$ on X such that

$$\sum_{i=1}^p \frac{\sum_{x \in X_i} w(x)}{|S_i^{n-1}|} \int_{S_i^{n-1}} f(x) d\rho_i(x) = \sum_{x \in X} w(x) f(x)$$

for every polynomial $f \in \mathcal{P}_t(S)$.

We can regard a Euclidean design as a cubature formula on some concentric spheres. Conversely a cubature formula for a class of integral with some symmetry is a Euclidean design (cf. [14, Lemma 3.1]). The following theorem by Neumaier and Seidel is well known in algebra and combinatorics.

Theorem 2.3 ([20])   *With the same notations as in Definition* 2.2, *the following are equivalent:*

(i)   *X is a Euclidean t-design with a weight function w;*

(ii)   *$\sum_{x \in X} w(x) f(x) = 0$ for every $f \in \|x\|^{2j} \operatorname{Harm}_l(\mathbb{R}^n)$ with $1 \leq l \leq t, 0 \leq j \leq \lfloor \frac{t-l}{2} \rfloor$.*

In Section 4 we give a stronger theorem than Theorem 2.3 for invariant Euclidean designs, intended for researchers in algebra and combinatorics.

Define $p' = p - \varepsilon_S$, where $\varepsilon_S = 1$ if $0 \in S$, and $\varepsilon_S = 0$ otherwise. The dimensions of $\mathcal{P}_l(S)$ and $\mathcal{P}_l^*(S)$ are well known.

Theorem 2.4 ([10, 12])   *Let $S \subset \mathbb{R}^n$.*

(i) $$\dim \mathcal{P}_l(S) = \begin{cases} \varepsilon_S + \sum_{i=0}^{2p'-1} \binom{n+l-i-1}{n-1} & \text{if } l \geq 2p', \\ \dim \mathcal{P}_l(\mathbb{R}^n) = \binom{n+l}{l} & \text{if } l \leq 2p' - 1. \end{cases}$$

(ii) $$\dim \mathcal{P}_l^*(S) = \begin{cases} \varepsilon_S + \sum_{i=0}^{p'-1} \binom{n+l-2i-1}{n-1} & \text{if } l \text{ is even}, l \geq 2p', \\ \sum_{i=0}^{p'-1} \binom{n+l-2i-1}{n-1} & \text{if } l \text{ is odd}, l \geq 2p', \\ \dim \mathcal{P}_l^*(\mathbb{R}^n) = \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \binom{n+l-2i-1}{n-1} & \text{if } l \leq 2p' - 1. \end{cases}$$

The following lower bounds are known as the Fisher-type inequality for the size of a Euclidean design [5, 10, 16, 18]; the latter one is also called the Möller bound.

Theorem 2.5   (i)   *Let X be a Euclidean $2e$-design supported by S. Then*

$$|X| \geq \dim \mathcal{P}_e(S).$$

(ii)   *Let X be a Euclidean $(2e-1)$-design supported by S. Then,*

$$|X| \geq \begin{cases} 2 \dim \mathcal{P}_{e-1}^*(S) - 1 & \text{if } e \text{ is odd and } 0 \in X, \\ 2 \dim \mathcal{P}_{e-1}^*(S) & \text{otherwise.} \end{cases}$$

A Euclidean t-design is said to be *tight* if the equality holds in one of the bounds in Theorem 2.5.

Hereafter we assume that G is a finite irreducible reflection group in $\mathbb{R}^n$. It is known that finite irreducible reflection groups are classified completely [8]. Let integers $1 = m_1 \leq m_2 \leq \cdots \leq m_n$ be the exponents of G (see [8, Ch.V, §6]).

Theorem 2.6 ([13])   *Let G be a finite irreducible reflection group. Let q_i be the dimension of $\operatorname{Harm}_i(\mathbb{R}^n)^G$. Then*

$$\sum_{i=0}^{\infty} q_i \lambda^i = \prod_{i=2}^{n} \frac{1}{1 - \lambda^{1+m_i}}.$$

Note that for any $x \in \mathbb{R}^n$, the orbit x^G is a spherical m_2-design in S^{n-1} [13].

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the fundamental roots of a reflection group G. The corner vectors $v_1, v_2, \ldots, v_n$ are defined by $v_i \perp \alpha_j$ if and only if $i \neq j$. We may assume that $\|v_k\| = 1$. We consider the set

$$\mathfrak{X}(G, J) = \bigcup_{k \in J} r_k v_k^G,$$

where $J \subset \{1, 2, \ldots, n\}$ and $r_k > 0$. Let R denote the set of r_k.

Bajnok [2] found new tight Euclidean designs in $\mathfrak{X}(B_n, J)$. In Section 4, using the theory of G-invariant harmonic polynomials, we extend Bajnok's method to other reflection groups G, and classify the tight Euclidean designs obtained from $\mathfrak{X}(G, J)$.

3 Xu's Theorem

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq \sqrt{x^2 + y^2} < \infty\}$. Let W be a nonnegative weight function on $[0, \infty)$ with finite moments. We consider the radial weight function defined by $W(\sqrt{x^2 + y^2})$ on Ω. The following integral is said to be *radially symmetric* or *circularly symmetric*:

$$\mathfrak{I}[f] = \int_\Omega f(x, y) W(\sqrt{x^2 + y^2}) \mathrm{d}x \mathrm{d}y$$

$$= \int_0^\infty \left(\int_0^{2\pi} f(r \cos \theta, r \sin \theta) \mathrm{d}\theta \right) r W(r) \mathrm{d}r.$$

To generalize a famous theorem by Verlinden and Cools [26] on the existence of cubature formula for radially symmetric integral, Xu proved the following theorem.

Theorem 3.1 ([27, Theorems 2.1 and 2.2])   (i) *Let*

$$(3.1) \qquad \mathfrak{I}_{2m}(f) = \frac{\pi}{m+1} \sum_{i=1}^m \lambda_i \sum_{j=0}^{2m+1} f\left(r_i \cos \frac{(2j + \sigma_i)\pi}{2m+2}, r_i \sin \frac{(2j + \sigma_i)\pi}{2m+2} \right),$$

$$\mathfrak{I}_{2m+1}(f) = \lambda_0 f(0, 0)$$

$$+ \frac{\pi}{m+2} \sum_{i=1}^m \lambda_i \sum_{j=0}^{2m+3} f\left(r_i \cos \frac{(2j + \sigma_i)\pi}{2m+4}, r_i \sin \frac{(2j + \sigma_i)\pi}{2m+4} \right),$$

where σ_i takes the value 0 if $m + i$ is even and the value 1 if $m + i$ is odd. Then $\mathfrak{I}_n$ forms a cubature formula of degree $2n - 1$ for $\mathfrak{I}$ if and only if the following two types of conditions are satisfied:

$$(3.2) \qquad \int_0^\infty r^{2j+1} W(r) \mathrm{d}r = \sum_{i=1}^{\lfloor n/2 \rfloor} \lambda_i r_i^{2j}, \qquad j = 1, \ldots, n-1,$$

$$(3.3) \qquad \sum_{i=1}^{\lfloor n/2 \rfloor} \lambda_i r_i^{2j}(-1)^i = 0, \qquad j = \lfloor (n+3)/2 \rfloor, \ldots, n-1.$$

(ii) *With the same symbol σ_i as in* (i), *let*

$$\mathfrak{I}_n(f) = \frac{2\pi}{m+1} \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} \lambda_i \sum_{j=0}^{2m} f\left(r_i \cos \frac{(2j+\sigma_i)\pi}{2m+1}, r_i \sin \frac{(2j+\sigma_i)\pi}{2m+1} \right),$$

where $n = 2m - 1$ or $2m$. Then $\mathfrak{I}_n$ forms a cubature formula of degree $2n$ for $\mathfrak{I}$ if and only if the following two types of conditions are satisfied:

$$(3.4) \qquad \int_0^\infty r^{2j+1} W(r) \mathrm{d}r = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} \lambda_i r_i^{2j}, \qquad j = 1, \ldots, n,$$

$$(3.5) \qquad \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} \lambda_i r_i^{2j}(-1)^i = 0, \qquad j = \lfloor (n+1)/2 \rfloor, \cdots, n-1.$$

The aim of this section is to give a proof of Theorem 3.1 different from the original one by Xu.

Proof of Theorem 3.1 Let D_ℓ be the dihedral group of order 2ℓ and let $f(x_1, x_2)$ be a D_ℓ-invariant polynomial. Using the polar coordinate system, we let $f(x_1, x_2) = f(r\cos\theta, r\sin\theta)$. It is shown that f can be represented as a polynomial in two variables $r^2, r^\ell \cos \ell\theta$. We now consider the case where $n = 2m$ in (i); the reader will easily see that the same argument as below works in the remaining cases. The conditions (3.2) and (3.3) respectively mean to substitute the monomials $r^2, \ldots, r^{2n-2}$ and $r^{n+2}\cos(n+2)\theta, r^{n+4}\cos(n+2)\theta, \ldots, r^{2n-2}\cos(n+2)\theta$ into (3.1). Thus the result follows by Theorem 2.1. ∎

Remark 3.2 (i) Xu's proof of Theorem 3.1 required approximately 2 pages for (i) with $n = 2m$. With this in mind, we tried to find a shorter proof using the Sobolev theorem. Another advantage of our proof is its simplicity. Namely, the Sobolev theorem is the only advanced tool used in our proof, whereas Xu's proof requires some technical and advanced tools in numerical analysis such as Gaussian–Lobatto quadrature and Gaussian–Radau quadrature. The proof by Xu also requires some tedious calculations. In summary, our proof is shorter and simpler than the original. Another advantage of our proof is that while conditions (3.2) and (3.3) (or (3.4) and (3.5)) are considerably analytic, and so researchers in other fields like combinatorics and algebra will not be familiar with them, whereas, our new proof gives a general interpretation of the above analytic conditions, and will allow researchers in these areas to enjoy Theorem 3.1 well. The authors hope that researchers in many different fields know the importance of Theorem 3.1 and will be more interested in classical theories of cubature formulas developed in numerical analysis.

(ii) Bajnok [1, Theorem 9] found a tight Euclidean t-design of $\mathbb{R}^2$ that has the same structure of points as Xu's formula, as a generalization of a tight 4-design by Bannai and Bannai [3]. To do this, he implicitly used the same idea as in Theorem 3.1; for instance equation (10) in his paper corresponds to equation (3.3) (or equation (3.5)) here.

4 Orbits of a Reflection Group as Euclidean Designs

In this section we classify the tight Euclidean designs obtained from $\mathfrak{X}(G, J)$ for a finite irreducible reflection group G. A finite set $X \subset \mathbb{R}^d$ is said to be *antipodal* if $X = -X$. A tight Euclidean $2e$-design has a weight function that is constant on each X_i [3], and so does an antipodal tight Euclidean $(2e - 1)$-design [7]. Throughout this section we assume a weight function is constant on each G-orbit.

First, we look at a stronger theorem than Theorem 2.3 for G-invariant Euclidean designs. A Euclidean t-design X is said to be *G-invariant* if X is a union of G-orbits and an equal weight is assigned to each point of the same orbit.

Lemma 4.1 *Let G be a subgroup of $O(\mathbb{R}^n)$. Let f be a G-invariant polynomial and x^G be a G-orbit. Then, $f(y) = f(z)$ for any $y, z \in x^G$.*

Proof This is straightforward. ∎

Let $|G|$ be the order of a group G.

Theorem 4.2 *Let G be a subgroup of $O(\mathbb{R}^n)$. Let $X = \cup_{k=1}^M r_k x_k^G$, where $x_k \in S^{n-1}$ and $r_k > 0$. The following are equivalent:*

(i) *X is a G-invariant Euclidean t-design.*
(ii) *$\sum_{x \in X} w(x) \|x\|^{2j} \varphi(x) = 0$ for any $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)^G$ with $1 \leq l \leq t, 0 \leq j \leq \lfloor \frac{t-l}{2} \rfloor$.*

Proof For $f \in \mathrm{Harm}_l(\mathbb{R}^d)$, the polynomial

$$\varphi(\xi) = \frac{1}{|G|} \sum_{g \in G} f(\xi^g)$$

is an element of $\mathrm{Harm}_l(\mathbb{R}^d)^G$. Let $w(x) = w_k$ for every $x \in x_k^G$. By Lemma 4.1, for any $f \in \mathrm{Harm}_l(\mathbb{R}^d)$, we have

$$
\sum_{x \in X} w(x) \|x\|^{2j} f(x) = \sum_{k=1}^M w_k r_k^{2j} \sum_{x \in x_k^G} f(r_k x)
$$

$$
= \sum_{k=1}^M \frac{w_k r_k^{2j+l} |x_k^G|}{|G|} \sum_{g \in G} f(x_k^g) = \sum_{k=1}^M w_k r_k^{2j+l} |x_k^G| \varphi(x_k)
$$

$$
= \sum_{k=1}^M w_k r_k^{2j} \sum_{x \in x_k^G} \varphi(r_k x) = \sum_{x \in X} w(x) \|x\|^{2j} \varphi(x).
$$

The result thus follows by Theorem 2.3. ∎

Remark 4.3 The radii r_k are not necessarily mutually distinct in Theorem 4.2. Goethals and Seidel [13] stated Theorem 2.1 for spherical designs. Theorem 4.2 with all $r_k = 1$ becomes the theorem of G-invariant spherical designs. The approach using the orbits under subgroups of $O(\mathbb{R}^d)$ has long been considered [4]. Theorem 4.2 reduces the computational cost to check the strength of a G-invariant Euclidean design less than using Theorem 2.3.

From now on, let G be a finite irreducible reflection group, and let v_k be a corner vector. Put $N_k = |v_k^G|$. The following is immediate by Lemma 4.1 and Theorem 4.2.

Corollary 4.4   *We have that $\mathfrak{X}(G, J)$ is a Euclidean t-design if and only if there exist $w_k > 0$ and $r_k > 0$ such that the equation $\sum_{k \in J} w_k r_k^{2j+i} N_k f(v_k) = 0$ holds for any $f \in \mathrm{Harm}_i(\mathbb{R}^d)^G$, where i, j are positive integers satisfying $1 \le 2j + i \le t$.*

The dimension of $\mathrm{Harm}_i(\mathbb{R}^d)^G$ is clear by Theorem 2.6. We can determine the basis of $\mathrm{Harm}_i(\mathbb{R}^d)^G$ by harmonic polynomials f satisfying $f(x^g) = f(x)$ for each generator g of G. By the basis of $\mathrm{Harm}_i(\mathbb{R}^d)^G$ and Corollary 4.4, we obtain a necessary and sufficient condition for $\mathfrak{X}(G, J)$ to be a Euclidean t-design. Bajnok [2] found an explicit such condition for the group B_n by using Corollary 4.4. For other groups, it is possible to give the conditions, but the statements are not simple. Therefore we do not inlude them in this paper.

Now, let us classify the tight Euclidean designs obtained from $\mathfrak{X}(G, J)$. For each group, we determine the possible maximum strength of $\mathfrak{X}(G, J)$ for any J and radii R. Since the cardinality of v_k^G is easily calculated, we can check whether the total size of a union of several orbits attains the fisher type inequality. For the set attaining the bound, we give its maximum strength by Corollary 4.4.

Hereafter let $e_i \in \mathbb{R}^n$ be the row vector whose i-th entry is 1 and other entries are 0. Let S_n be the symmetric group. Define

$$\mathrm{sym}(f) := \frac{1}{|(S_n)_f|} \sum_{g \in S_n} f(x^g),$$

where $(S_n)_f := \{g \in S_n \mid f(x^g) = f(x)\}$.

4.1   Group A_n

Dynkin diagram
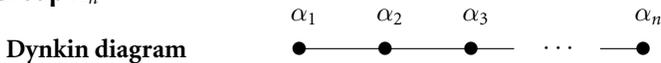
Exponents $1, 2, \ldots, n$

Fundamental roots $\alpha_i := e_i - e_{i+1}$ for $1 \le i \le n-1$. $\alpha_n := [a, a, \ldots, a, b]$, where $a = (-1 + \sqrt{n+1})/n$ and $b = (n - 1 + \sqrt{n+1})/n$.

Corner Vectors $v_k = [c_k, \ldots, c_k, d_k, \ldots, d_k]$ whose first k coordinates are equal to c_k, and last $n - k$ coordinates are equal to d_k, where

$$c_k = \frac{n + 1 - k + \sqrt{n+1}}{\sqrt{k(n+1-k)(n+2+2\sqrt{n+1})}}, \quad d_k = \frac{-k}{\sqrt{k(n+1-k)(n+2+2\sqrt{n+1})}}.$$

Reflection group   The reflection group $A_n \subset O(\mathbb{R}^n)$ is generated by the following:

$$r(\alpha_i) = \left[{}^t e_1, \ldots, {}^t e_{i-1}, {}^t e_{i+1}, {}^t e_i, {}^t e_{i+2}, \ldots, {}^t e_n \right] \text{ for } 1 \le i \le n-1,$$

$$r(\alpha_n) = \begin{bmatrix} I_{n-1} - a^2({}^t j j) & -ab {}^t j \\ -abj & 1 - b^2 \end{bmatrix},$$

where I_n is the identity matrix of size n, and j is the all-ones row vector.

Orbits   Let U_1 be the set of all vectors such that k coordinates are equal to c_k, and other $n - k$ coordinates are equal to d_k. Let U_2 be the set of all vectors such that $k - 1$ coordinates are equal to c'_k, and other $n + 1 - k$ coordinates are equal to d'_k, where

$$c'_k = \frac{n+1-k}{\sqrt{k(n+1-k)(n+2+2\sqrt{n+1})}}, \quad d'_k = \frac{-k-\sqrt{n+1}}{\sqrt{k(n+1-k)(n+2+2\sqrt{n+1})}}.$$

Then the orbit $v_k^{A_n} = U_1 \cup U_2$. Furthermore, we have $N_k = \binom{n+1}{k}$ and $v_k^{A_n} = -v_{n+1-k}^{A_n}$.

Harmonic Molien Series

$$\frac{1}{(1-t^3)(1-t^4)\cdots(1-t^{n+1})} = \begin{cases} 1 + t^3 + t^6 + \cdots, & \text{if } n = 2, \\ 1 + t^3 + t^4 + t^6 + \cdots, & \text{if } n = 3, \\ 1 + t^3 + t^4 + t^5 + \cdots, & \text{if } n \ge 4. \end{cases}$$

G-invariant harmonic polynomials

1. *Degree* 3: Note that $\dim \operatorname{Harm}_3(\mathbb{R}^n)^{A_n} = 1$ for any $n \ge 2$. $\operatorname{Harm}_3(\mathbb{R}^n)^{A_n}$ is spanned by the following:

(i) $n = 2$.

$$f_3 = x_1^3 - 3x_1^2 x_2 - 3x_1 x_2^2 + x_2^3.$$

(ii) $n = 3$.

$$f_3 = \operatorname{sym}(x_1^3) - \frac{3}{2}\operatorname{sym}(x_1 x_2^2) - \frac{3}{4}\operatorname{sym}(x_1 x_2 x_3).$$

(iii) $n \ge 4$.

$$f_3 = \operatorname{sym}(x_1^3) - \frac{3}{n-1}\operatorname{sym}(x_1 x_2^2) + \frac{6(2-\sqrt{n+1})}{(n-1)(n-3)}\operatorname{sym}(x_1 x_2 x_3).$$

2. *Degree* 4: Note that $\dim \operatorname{Harm}_4(\mathbb{R}^2)^{A_2} = 0$ and $\dim \operatorname{Harm}_4(\mathbb{R}^n)^{A_n} = 1$ for any $n \ge 3$. The following are S_n-invariant harmonic polynomials:

$$h_{4,1} = \operatorname{sym}(x_1^4) - \frac{6}{n-2}\operatorname{sym}(x_1^2 x_2^2),$$

$$h_{4,2} = \operatorname{sym}(x_1 x_2 x_3 x_4),$$

$$h_{4,3} = \operatorname{sym}(x_1 x_2^3) - \frac{6}{n-2}\operatorname{sym}(x_1 x_2 x_3^2).$$

$\operatorname{Harm}_4(\mathbb{R}^n)^{A_n}$ is spanned by the following:

(i) $n = 3$.

$$f_4 = h_{4,1} - \frac{20}{13} h_{4,3}.$$

(ii) $n \geq 4$.

$$f_4 = h_{4,1} + c_{4,2} h_{4,2} + c_{4,3} h_{4,3},$$

where

$$c_{4,2} = \frac{24(n+2)(n^2 - 5n - 12 + 4\sqrt{n+1})}{(n-1)(n-2)(n^3 - 2n^2 - 15n - 16)},$$

$$c_{4,3} = -\frac{4(n+2)(n^2 - 2n - 7 - (n-1)\sqrt{n+1})}{(n-1)(n^3 - 2n^2 - 15n - 16)}.$$

3. *Degree* 5: Note that $\dim \mathrm{Harm}_5(\mathbb{R}^n)^{A_n} = 0$ for $n = 2, 3$, and $\dim \mathrm{Harm}_5(\mathbb{R}^n)^{A_n} = 1$ for any $n \geq 4$. The following are S_n-invariant harmonic polynomials:

$$h_{5,1} = \mathrm{sym}(x_1^5) - \frac{10}{n-1} \mathrm{sym}(x_1^2 x_2^3) + \frac{30}{(n-1)(n-2)} \mathrm{sym}(x_1 x_2^2 x_3^2),$$

$$h_{5,2} = \mathrm{sym}(x_1^5) - \frac{10}{n-1} \mathrm{sym}(x_1^2 x_2^3) + \frac{5}{n-1} \mathrm{sym}(x_1 x_2^4),$$

$$h_{5,3} = \mathrm{sym}(x_1 x_2 x_3^3) - \frac{9}{n-3} \mathrm{sym}(x_1 x_2 x_3 x_4^2),$$

$$h_{5,4} = \mathrm{sym}(x_1 x_2 x_3 x_4 x_5).$$

$\mathrm{Harm}_4(\mathbb{R}^n)^{A_n}$ is spanned by the following:
(i) $n = 4$.

$$f_5 = h_{5,1} + \frac{17 - 20\sqrt{5}}{58} h_{5,2} + \frac{10(18 + \sqrt{5})}{87} h_{5,3}.$$

(ii) $n \geq 5$.

$$f_5 = h_{5,1} + c_{5,2} h_{5,2} + c_{5,3} h_{5,3} + c_{5,4} h_{5,4},$$

where

$$c_{5,2} = -\frac{2n^3 + 5n^2 - 21n - 90 - n(n+6)\sqrt{n+1}}{4n^3 + 3n^2 - 60n - 180},$$

$$c_{5,3} = \frac{20(2n^3 + 6n^2 - 32n - 168 + (n^2 - 8n + 12)\sqrt{n+1})}{(n-1)(n-2)(4n^3 + 3n^2 - 60n - 180)},$$

$$c_{5,4} = -\frac{120(n+6)(n^2 - 11n - 78 + (2n^2 - 2n + 12)\sqrt{n+1})}{(n-1)(n-2)(n-3)(4n^3 + 3n^2 - 60n - 180)}.$$

Substitute v_k for G-invariant Harmonic Polynomials

1. *Degree 3:* For $n = 2, n \geq 4$,

$$f_3(v_k) = -\frac{k - \frac{n+1}{2}}{\sqrt{k(n+1-k)}}\phi_3(n),$$

where

$$\phi_3(n) = \frac{2(n^3 + 3n^2 - 12n - 16 + (3n^2 - 4n - 16)\sqrt{n+1})}{(n-1)(n-3)(n+2+2\sqrt{n+1})^{\frac{3}{2}}}.$$

For $n = 3$,

$$f_3(v_1) = \frac{729}{4}, \qquad f_3(v_2) = 0, \qquad f_3(v_3) = -\frac{729}{4}.$$

2. *Degree 4:* For $n \geq 3$,

$$f_4(v_k) = \frac{(k-\alpha)(k-\beta)}{k(n+1-k)}\phi_4(n),$$

where

$$\phi_4(n)$$
$$= \frac{6(n+1)(n^5 + 7n^4 - 24n^3 - 160n^2 - 256n - 128 + 4(n^4 - 20n^2 - 48n - 32)\sqrt{n+1})}{(n-1)(n-2)(n^3 - 2n^2 - 15n - 16)(n+2+2\sqrt{n+1})^2},$$

$$\alpha = \frac{n+1}{2} - \frac{\sqrt{3(n^2-1)}}{6}, \qquad \beta = \frac{n+1}{2} + \frac{\sqrt{3(n^2-1)}}{6}.$$

3. *Degree 5:* For $n \geq 4$,

$$f_5(v_k) = -\frac{(k-\frac{n+1}{2})(k-\alpha')(k-\beta')}{(k(n+1-k))^{\frac{3}{2}}}\phi_5(n),$$

where

$$\phi_5(n) = \frac{24(n+1)(2n^6 + 31n^5 + 50n^4 - 448n^3 - 2144n^2 - 3200n - 1536)}{(n-1)(n-2)(n-3)(4n^3 + 3n^2 - 60n - 180)(n+2+2\sqrt{n+1})^{\frac{5}{2}}}$$
$$+ \frac{24(n+1)(11n^5 + 50n^4 - 96n^3 - 1120n^2 - 2432n - 1536)\sqrt{n+1}}{(n-1)(n-2)(n-3)(4n^3 + 3n^2 - 60n - 180)(n+2+2\sqrt{n+1})^{\frac{5}{2}}},$$

$$\alpha' = \frac{n+1}{2} - \frac{\sqrt{3(n+1)(2n-3)}}{6}, \qquad \beta' = \frac{n+1}{2} + \frac{\sqrt{3(n+1)(2n-3)}}{6}.$$

Theorem 4.5 *There is no choice of J, R, and w for which $(\mathfrak{X}(A_n, J), w)$ is a Euclidean 6-design.*

Proof The polynomial of degree 6:

$$f(x_1, x_2, \ldots, x_n) = \text{sym}(x_1 x_2^5) - \frac{10}{3} \text{sym}(x_1^3 x_2^3)$$

is harmonic for any $n \geq 2$. We can calculate

$$\sum_{x \in v_k^{A_n}} f(x) = g_1(n, k) \Big(k(k - n - 1)\big(4k^2 - 4(n+1)k + n^2 + 5n + 4\big) g_2(n) + g_3(n) \Big),$$

where

$$g_1(n, k) = \frac{n(n+1)}{3k^3(n+1-k)^3(n+2+2\sqrt{n+1})^3} \binom{n-1}{k-1},$$

$$g_2(n) = 5(n^2 + 11n + 12 + (6n + 12)\sqrt{n+1}),$$

$$g_3(n) = (n+1)^2(n+2)(2n^2 + 28n + 30 + (15n + 30)\sqrt{n+1}).$$

Note that $g_1(n, k) > 0$ for $1 \leq k \leq n$. Define

$$F(k) = k(k - n - 1)\big(4k^2 - 4(n+1)k + n^2 + 5n + 4\big) g_2(n) + g_3(n).$$

For a fixed n, we prove that $F(k) < 0$ for $1 \leq k \leq n$. We have

$$\frac{d}{dk} F(k) = 16 \left(k - \frac{n+1}{2} \right) (k - \alpha'')(k - \beta'') g_2(n),$$

where

$$\alpha'' = \frac{n+1}{2} - \frac{\sqrt{2(n+1)(n-2)}}{4}, \qquad \beta'' = \frac{n+1}{2} + \frac{\sqrt{2(n+1)(n-2)}}{4}.$$

If $F(1) = F(n) < 0$, $F((n+1)/2) < 0$, and $F(\alpha'') = F(\beta'') < 0$, then $F(k) < 0$ for all $1 \leq k \leq n$. Indeed for $n \geq 2$,

$$F(1) = F(n)$$
$$= -(n-1)(3n^4 + 27n^3 + 10n^2 + 26n + 60 + (15n^3 + 15n^2 + 60)\sqrt{n+1}) < 0,$$

$$F(\alpha'') = F(\beta'')$$
$$= -\frac{1}{16}n(n+1)^2(5n^3 + 63n^2 + 68n - 16 + (30n^2 + 60n)\sqrt{n+1}) < 0,$$

$$F\left(\frac{n+1}{2} \right) = -\frac{1}{4}(n-1)(n+1)^2(7n^2 + 59n + 60 + (30n + 60)\sqrt{n+1}) < 0.$$

Therefore, $\sum_{k \in J} \sum_{x \in v_k^{A_n}} w_k r_k f(x) < 0$ for any J, R, and w.　　　∎

$|R|$	t	n	J	r_i	w_i
1	2	any	$\{1\}$	$r_1 = 1$	$w_1 = 1$
	2	any	$\{n\}$	$r_n = 1$	$w_n = 1$
	3	3	$\{2\}$	$r_2 = 1$	$w_2 = 1$
	5	2	$\{1,2\}$	$r_1 = r_2 = 1$	$w_1 = w_2 = 1$
	5	7	$\{2,6\}$	$r_2 = r_6 = 1$	$w_2 = w_6 = 1$
2	4	2	$\{1,2\}$	$r_1 = 1, r_2 \neq 1$	$w_1 = 1, w_2 = \frac{1}{r_2^3}$
	4	4	$\{1,3\}$	$r_1 = 1, r_3 = \frac{1}{\sqrt{6}}$	$w_1 = 1, w_3 = 27$
	4	4	$\{2,4\}$	$r_4 = 1, r_2 = \frac{1}{\sqrt{6}}$	$w_4 = 1, w_2 = 27$
	4	5	$\{1,4\}$	$r_1 = 1, r_4 = \sqrt{\frac{8}{5}}$	$w_1 = 1, w_4 = \frac{1}{2}$
	4	5	$\{2,5\}$	$r_5 = 1, r_2 = \sqrt{\frac{8}{5}}$	$w_5 = 1, w_2 = \frac{1}{2}$
	4	6	$\{1,5\}$	$r_1 = 1, r_5 = \sqrt{15}$	$w_1 = 1, w_5 = \frac{1}{81}$
	4	6	$\{2,6\}$	$r_6 = 1, r_2 = \sqrt{15}$	$w_6 = 1, w_2 = \frac{1}{81}$
	5	3	$\{1,2,3\}$	$r_1 = r_3 = 1, r_2 \neq 1$	$w_1 = w_3 = 1, w_2 = \frac{9}{8r_2^4}$
	5	5	$\{1,3,5\}$	$r_1 = r_5 = 1, r_3 \neq 1$	$w_1 = w_5 = 1, w_3 = \frac{27}{25r_2^4}$

Table 4.1: Tight Euclidean t-designs from $\mathfrak{X}(A_n, J)$

Theorem 4.6 $\mathfrak{X}(A_n, J)$ *is not a tight Euclidean t-design except for the cases in Table* 4.1.

Proof We prove only the classification of tight Euclidean 4-designs on two concentric spheres obtained from $\mathfrak{X}(A_n, J)$. The other cases can be proved in a similar way.

Define $f_3(v_k)$ and $\phi_3(n)$ as above. Since $\phi_3(n) > 0$ for $n > 1$, $f_3(v_k) = 0$ if and only if $n \equiv 1 \pmod{2}$ and $k = (n+1)/2$. Clearly $f_3(v_k) > 0$ for $k < (n+1)/2$, and $f_3(v_k) < 0$ for $k > (n+1)/2$. Therefore J must contain k_1 and k_2 such that $k_1 < (n+1)/2 < k_2$ by Corollary 4.4.

The size of a tight Euclidean 4-design on two concentric spheres is $(n+1)(n+2)/2$. By noting that $(n+1)(n+2)/2 < N_3 = N_{n-2}$ for $n > 5$, we can determine that $J = \{1, n-1\}$, (or equivalently $J = \{2, n\}$) for any $n > 2$, or $J = \{1, 2\}$ for $n = 2$. For $n = 2$, we can obtain tight Euclidean 4-designs on two concentric spheres as in Table 4.1.

Define $f_4(v_k), \phi_4(n), \alpha$, and β as above. Note that $\phi_4(n) \neq 0, \alpha > 1$, and $\beta < n$ for any integer $n > 2$. Therefore $1 < \alpha < n-1 < \beta$, (or equivalently $\alpha < 2 < \beta < n$) holds by Corollary 4.4. The integers satisfying the condition are only $n = 4, 5, 6$. For $n = 4, 5, 6$, we can obtain tight Euclidean 4-designs on two concentric spheres as in Table 4.1. ∎

4.2 Group B_n

Dynkin diagram
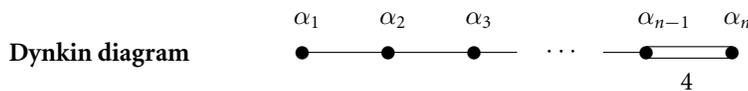
Exponents $1, 3, \ldots, 2n - 1$

Fundamental roots $\alpha_i := e_i - e_{i+1}$ for $1 \le i \le n - 1$ and $\alpha_n := \sqrt{2} e_n$.

Corner Vectors $v_k = [1/\sqrt{k}, \ldots, 1/\sqrt{k}, 0, \ldots, 0]$, where v_k has k coordinates equal to $1/\sqrt{k}$.

Reflection group   The reflection group $B_n \subset O(\mathbb{R}^n)$ is generated by the following:

$$r(\alpha_i) = \left[{}^t e_1, \ldots {}^t e_{i-1}, {}^t e_{i+1}, {}^t e_i, {}^t e_{i+2}, \ldots, {}^t e_n\right] \text{ for } 1 \le i \le n - 1,$$

$$r(\alpha_n) = \left[{}^t e_1, \ldots {}^t e_{n-1}, -{}^t e_n\right].$$

Orbits   The orbit $v_k^{B_n}$ is the set of vectors with exactly k nonzero coordinates equal to $\pm 1/\sqrt{k}$. Note that $v_k^{B_n}$ is antipodal and $N_k = 2^k \binom{n}{k}$.

Harmonic Molien series

$$\frac{1}{(1 - t^4)(1 - t^6) \cdots (1 - t^{2n})} = \begin{cases} 1 + t^4 + t^8 + \cdots, & \text{if } n = 2, \\ 1 + t^4 + t^6 + \cdots, & \text{if } n \ge 3. \end{cases}$$

G-invariant harmonic polynomials

1. *Degree* 4:   Note that $\dim(\mathrm{Harm}_4(\mathbb{R}^n)^{B_n}) = 1$ for any $n \ge 2$. The following is a B_n-invariant harmonic polynomial of degree 4:

$$f_4 = \mathrm{sym}(x_1^4) - \frac{6}{n - 1} \mathrm{sym}(x_1^2 x_2^2).$$

2. *Degree* 6:   Note that $\dim(\mathrm{Harm}_6(\mathbb{R}^2)^{B_2}) = 0$ and $\dim(\mathrm{Harm}_6(\mathbb{R}^n)^{B_n}) = 1$ for any $n \ge 3$. The following is a B_n-invariant harmonic polynomial of degree 6:

$$f_6 = \mathrm{sym}(x_1^6) - \frac{15}{n - 1} \mathrm{sym}(x_1^2 x_2^4) + \frac{180}{(n - 1)(n - 2)} \mathrm{sym}(x_1^2 x_2^2 x_3^2).$$

Substitute v_k for G-invariant Harmonic Polynomials

1. *Degree* 4:

$$f_4(v_k) = \frac{1}{k}\left(1 - 3\frac{k - 1}{n - 1}\right).$$

2. *Degree* 6:

$$f_6(v_k) = \frac{1}{k^2}\left(1 - 15\frac{k - 1}{n - 1} + 30\frac{(k - 1)(k - 2)}{(n - 1)(n - 2)}\right).$$

Theorem 4.7 ([2])   *There is no choice of R, J, and w for which $(\mathfrak{X}(B_n, J), w)$ is a Euclidean* 8-*design.*

$|R|$	t	n	J	r_i	w_i
1	3	any	$\{1\}$	$r_1 = 1$	$w_1 = 1$
	3	2	$\{2\}$	$r_2 = 1$	$w_2 = 1$
	7	2	$\{1,2\}$	$r_1 = r_2 = 1$	$w_1 = w_2 = 1$
2	5	2	$\{1,2\}$	$r_1 = 1, r_2 \neq 1$	$w_1 = 1, w_2 = \frac{1}{r_2^4}$
	5	3	$\{1,3\}$	$r_1 = 1, r_3 \neq 1$	$w_1 = 1, w_3 = \frac{9}{8r_2^4}$
	7	4	$\{1,2,4\}$	$r_1 = r_4 = 1, r_2 \neq 1$	$w_1 = w_4 = 1, w_2 = \frac{1}{r_2^6}$
3	7	3	$\{1,2,3\}$	$r_1 = 1, r_2 = \sqrt{\frac{2r_3^2}{5r_3^2-3}}, (r_i \neq r_j)$	$w_1 = 1, w_2 = \frac{4}{5r_2^6}, w_3 = \frac{27}{40r_3^6}$

Table 4.2: Tight Euclidean t-designs from $\mathfrak{X}(B_n, J)$

Theorem 4.8 ([2])   *We have that $\mathfrak{X}(B_n, J)$ is not a tight Euclidean t-design except for the cases in Table* 4.2.

Remark 4.9   We can also prove Theorems 4.7 and 4.8 by the B_n-invariant harmonic polynomials.

4.3   Group D_n

Dynkin diagram
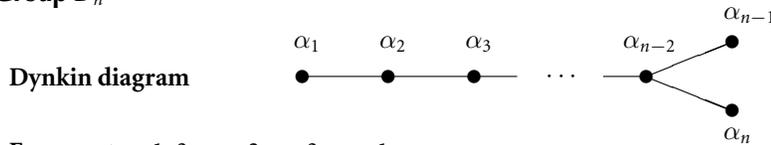
Exponents $1, 3, \ldots, 2n-3, n-1$

Fundamental roots $\alpha_i := e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n := e_{n-1} + e_n$.

Corner Vectors $v_k = [1/\sqrt{k}, \ldots, 1/\sqrt{k}, 0, \ldots, 0]$, where v_k has k coordinates equal to $1/\sqrt{k}$ for $1 \leq k \leq n-2$, $v_{n-1} = [1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n}]$ and $v_n = [1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}]$.

Reflection group   The reflection group $D_n \subset O(\mathbb{R}^n)$ is generated by the following:

$$r(\alpha_i) = \left[{}^te_1, \cdots {}^te_{i-1}, {}^te_{i+1}, {}^te_i, {}^te_{i+2}, \cdots, {}^te_n\right] \text{ for } 1 \leq i \leq n-1,$$

$$r(\alpha_n) = \left[{}^te_1, \cdots {}^te_{n-2}, -{}^te_n, -{}^te_{n-1}\right].$$

Orbits   For $1 \leq k \leq n-2$, $v_k^{D_n} = v_k^{B_n}$. The orbit $v_n^{D_n}$ (resp. $v_{n-1}^{D_n}$) consists of the vectors $\{\pm 1/\sqrt{n}\}^n$ with an even (resp. odd) number of negative coordinates. Note that $v_n^{D_n} = -v_{n-1}^{D_n}$ for odd n, and both $v_n^{D_n}$ and $v_{n-1}^{D_n}$ are antipodal for even n. Furthermore, $|N_{n-1}| = |N_n| = 2^{n-1}$.

Harmonic Molien series

$$\frac{1}{(1-t^4)(1-t^6)\cdots(1-t^{2n-2})(1-t^n)} = \begin{cases} 1 + 2t^4 + t^6 + 3t^8 + \cdots, & \text{if } n = 4, \\ 1 + t^4 + t^5 + t^6 + 2t^8 + \cdots, & \text{if } n = 5, \\ 1 + t^4 + 2t^6 + 2t^8 + \cdots, & \text{if } n = 6, \\ 1 + t^4 + t^6 + t^7 + 2t^8 \cdots, & \text{if } n = 7, \\ 1 + t^4 + t^6 + 2t^8 \cdots, & \text{if } n \geq 8. \end{cases}$$

G-invariant Harmonic Polynomials

1. *Degree* 4: Note that $\dim(\mathrm{Harm}_4(\mathbb{R}^4)^{D_4}) = 2$ and $\dim(\mathrm{Harm}_4(\mathbb{R}^n)^{D_n}) = 1$ for any $n \geq 5$. The following are D_n-invariant harmonic polynomials of degree 4:

$$f_4 = \mathrm{sym}(x_1^4) - \frac{6}{n-1}\,\mathrm{sym}(x_1^2 x_2^2).$$

The following is a D_4-invariant harmonic polynomial of degree 4, which is linearly independent of f_4:

$$f_{4,2} = x_1 x_2 x_3 x_4.$$

2. *Degree* 5: Note that $\dim(\mathrm{Harm}_5(\mathbb{R}^5)^{D_5}) = 1$ and $\dim(\mathrm{Harm}_5(\mathbb{R}^n)^{D_n}) = 0$ for any $n \neq 5$. The following is a D_5-invariant harmonic polynomial of degree 5:

$$f_5 = x_1 x_2 x_3 x_4 x_5.$$

3. *Degree* 6: Note that $\dim(\mathrm{Harm}_6(\mathbb{R}^6)^{D_6}) = 2$ and $\dim(\mathrm{Harm}_6(\mathbb{R}^n)^{D_n}) = 1$ for any $n \neq 6$. The following is a D_n-invariant harmonic polynomial of degree 6:

$$f_6 = \mathrm{sym}(x_1^6) - \frac{15}{n-1}\,\mathrm{sym}(x_1^2 x_2^4) + \frac{180}{(n-1)(n-2)}\,\mathrm{sym}(x_1^2 x_2^2 x_3^2).$$

The following is a D_6-invariant harmonic polynomial of degree 6, which is linearly independent of f_6:

$$f_{6,2} = x_1 x_2 x_3 x_4 x_5 x_6.$$

Substitute v_k for G-invariant harmonic polynomials

1. *Degree* 4: For $1 \leq k \leq n - 2$,

$$f_4(v_k) = \frac{1}{k}\left(1 - 3\frac{k-1}{n-1}\right).$$

For $k = n - 1, n$,

$$f_4(v_k) = -\frac{2}{n}.$$

For $n = 4$,

$$f_{4,2}(v_1) = 0, \qquad f_{4,2}(v_2) = 0, \quad f_{4,2}(v_3) = -\frac{1}{16}, \quad f_{4,2}(v_4) = \frac{1}{16}.$$

2. *Degree 5*: For $n = 5$,

$$f_5(v_1) = 0, \quad f_5(v_2) = 0, \quad f_5(v_3) = 0, \quad f_5(v_4) = -\frac{1}{25\sqrt{5}}, \quad f_5(v_1) = \frac{1}{25\sqrt{5}}.$$

3. *Degree 6*: For $1 \le k \le n - 2$,

$$f_6(v_k) = \frac{1}{k^2}\left(1 - 15\frac{k-1}{n-1} + 30\frac{(k-1)(k-2)}{(n-1)(n-2)}\right).$$

For $k = n - 1, n$, $f_6(v_k) = \frac{16}{n^2}$. For $n = 6$,

$$f_{6,2}(v_1) = 0, \quad f_{6,2}(v_2) = 0, \quad f_{6,2}(v_3) = 0, \quad f_{6,2}(v_4) = 0,$$

$$f_{6,2}(v_5) = -\frac{1}{216}, \quad f_{6,2}(v_6) = \frac{1}{216}.$$

Theorem 4.10 *There is no choice of J, R, and w for which $(\mathcal{X}(D_n, J), w)$ is a Euclidean 8-design.*

Proof The following is a D_n-invariant harmonic polynomial of degree 8:

$$f_8 = \mathrm{sym}(x_1^8) - \frac{28}{n-1}\,\mathrm{sym}(x_1^2 x_2^6) + \frac{70}{n-1}\mathrm{sym}(x_1^4 x_2^4).$$

For $1 \le k \le n - 2$,
$$f_8(v_k) = \frac{1}{k^3}\left(1 + 7\frac{k-1}{n-1}\right),$$

and for $k = n - 1, n$, $f_8(v_k) = \frac{8}{n^3}$. Therefore, $f_8(v_k) > 0$ for all k. ∎

Theorem 4.11 *Assume that J contains n or $n - 1$. Then $\mathcal{X}(D_n, J)$ is not a tight Euclidean design except for the cases in Table 4.3.*

Proof By Theorem 4.10 and D_n-invariant harmonic polynomials, the proof is similar to that of Theorem 4.6. ∎

Remark 4.12 The tight Euclidean designs in Tables 4.1, 4.2, and 4.3 are already known in [2, 3, 5–7, 9].

Remark 4.13 For each $G = F_4, H_3, H_4, E_6, E_7, E_8$, by checking the cardinality of a union of several v_k^G, we can prove that $\mathcal{X}(G, J)$ is not a tight Euclidean design except for known tight spherical designs [21, 22].

$\lvert R\rvert$	t	n	J	r_i	w_i
1	7	8	$\{2,7\}$	$r_2 = r_7 = 1$	$w_2 = w_7 = 1$
	7	8	$\{2,8\}$	$r_2 = r_8 = 1$	$w_2 = w_8 = 1$
2	5	6	$\{1,5\}$	$r_1 = 1, r_5 \neq 1$	$w_1 = 1, w_5 = \frac{9}{8r_2^4}$
	5	6	$\{1,6\}$	$r_1 = 1, r_6 \neq 1$	$w_1 = 1, w_6 = \frac{9}{8r_2^4}$
	7	4	$\{1,2,3,4\}$	$r_1 = r_3 = r_4 = 1, r_2 \neq 1$	$w_1 = w_3 = w_4 = 1, w_2 = \frac{1}{r_2^6}$

Table 4.3: Tight Euclidean t-designs from $\mathfrak{X}(D_n, J)$, where n or $n-1 \in J$

5 Concluding Remarks

In this paper we made some observations on invariant cubature formulas and Euclidean designs in connection with the Sobolev theorem. First, we gave an alternative proof of the celebrated theorems by Xu on necessary and sufficient conditions for the existence of cubature formulas with radial symmetry. The new proof is much shorter and simpler compared to the original. Thus researchers in analysis will be reminded of the importance of the Sobolev theorem. Moreover, our proof gives a general interpretation of the analytically-written conditions of Xu's theorems, and so will allow researchers in algebra and combinatorics to become more familiar with Xu's theorems. Second we extended the Neumaier–Seidel theorem to invariant Euclidean designs, and thereby classified tight Euclidean designs obtained from unions of the orbits of the corner vectors. The classification generalizes Bajnok's theorem to other finite reflection groups beside groups of type *B*. Bajnok's theorem and results obtained in Section 4 may imply that invariant cubature formulas of high degree could hardly exist. Xu pointed out, however, that the general Lie groups have been used for studying cubature formulas in a different setting (cubature rules on the fundamental domain of the group, which are for exponential or trigonometric functions), and they yield Gaussian type cubature for algebraic polynomials of very high orders; for instance see [15, 17] for details. We believe this direction of research in analysis will also motivate the study of cubature formulas in other areas of mathematics.

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