ON GENERATORS OF NEW METHODS OF THE PERTURBATION THEORY

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Abstract. In this article are discussed classical and modern interpretations of the perturbation theory methods.

1. On the classical perturbation theory

Let us consider an n-dimensional differential equation with small parameter μ

$$\frac{dz}{dt} = Z(z, t, \mu), \qquad z(0) = z_0 \tag{1}$$

where the vector-function $Z(z, t, \mu)$ is determined and has properties guaranteeing the existence and uniqueness of the solutions of the Cauchy problem (1) in a $(n+1)$-dimensional domain $G_{(n+1)} = \{z \in G \times R \ni t\}$ of the Euclidean space. Our purpose is to construct this solution. Together with equation (1), we consider an equivalent one

$$\frac{dz}{dt} = \bar{Z}(z, t, \mu) + Z(z, t, \mu) - \bar{Z}(z, t, \mu), \qquad z(0) = z_0, \tag{2}$$

in which $\bar{Z}(z, t, \mu)$ is an arbitrary function. We write the linear equality

$$z(t, \mu) = \bar{z}(t, \mu) + u(t, \mu) \tag{3}$$

where $\bar{z}, u$ are some new unknown functions. The solution of a Cauchy problem can be found by solving the following two Cauchy problems:

$$\frac{d\bar{z}}{dt} = \bar{Z}(\bar{z}, t, \mu), \qquad \bar{z}(0) = \bar{z}_0 \in G_n \tag{4}$$

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$$\frac{du}{dt} = Z(\bar{z}+u,t,\mu) - \bar{Z}(\bar{z},t,\mu), \qquad u(0) = z_0 - \bar{z}_0 \tag{5}$$

where $\bar{z}_0$ is some new initial point. The equation (4) defines the choice of the initial approximation $\bar{z}(t,\mu)$ and equation (5) defines the total perturbation $u(t,\mu)$. From problem (5), one can see that perturbation $u(t,\mu)$ depends on the choice of function $\bar{Z}(\bar{z},t,\mu)$, initial point $\bar{z}_0$ and, moreover, its finding is possible only after the solution of a equation (4). Thus, for a Cauchy problem (1), it is possible to construct a set of variants of the perturbation theory with parameters $\bar{Z}$ and $\bar{z}_0$. It is necessary that the solutions of equation (5) be "small" under the norm. We call $\bar{Z}(\bar{z},t,\mu)$ and $\bar{z}_0$ the generators of the perturbation theory for problem (1) and equation (4) the generating equation for equation (1).

2. New variants of the perturbation theory

Now, we assume that the perturbation u depends on $\bar{z}, t$ and μ, that is, instead of (3) we have an equality

$$z(t,\mu) = \bar{z}(t,\mu) + u(\bar{z},t,\mu). \tag{6}$$

Therefore, instead of equations (4) and (5) we shall have equations (4) and (7):

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial \bar{z}}, \bar{Z}(z,t,\mu)\right) = Z(\bar{z}+u,t,\mu) - \bar{Z}(\bar{z},t,\mu), u(0) = z_0 - \bar{z}_0. \tag{7}$$

The perturbation theory based on equations (4) and (7) differs from the classical perturbation theory in an essential point: the determination of perturbation $u(\bar{z},t,\mu)$ from the equation (7) does not require the preliminary solving of a generating equation (4).

So, let a problem of classical dynamics be described by a multifrequency system of $(m+n)$-order

$$\begin{cases} \frac{dx}{dt} &= \mu X(x,y) \\ \frac{dy}{dt} &= \omega(x) + \mu Y(x,y). \end{cases} \tag{8}$$

where

$$X(x,y) = \sum_{\|k\|\in I} X_k(x) e^{i(k,y)}, \qquad Y(x,y) = \sum_{\|k\|\in I} Y_k(x) e^{i(k,y)}, \tag{9}$$

$$i = \sqrt{-1}, \quad (k,y) = \sum_{s=1}^{n} k_s y_s, \quad \|k\| = \sum_{s=1}^{n} |k_s|, \quad I = \{0,1,2,\cdots\},$$

$$k_s = 0, \pm 1, \cdots$$

We choose, corresponding to (8), a generating system of the form

$$\begin{cases} \frac{d\bar{x}}{dt} = \mu \bar{X}(\bar{x}, \bar{y}) + \sum_{k \geq 2} \mu^k A_k(\bar{x}, \bar{y}), \\ \frac{d\bar{y}}{dt} = \omega(x) + \mu \bar{Y}(\bar{x}, \bar{y}) + \sum_{k \geq 2} \mu^k B_k(\bar{x}, \bar{y}), \end{cases} \tag{10}$$

where $\bar{X}, \bar{Y}, A_k, B_k$ are arbitrary functions of their arguments. Let's look for the replacement of variable (6) as formal series

$$x = \bar{x} + \sum_{k \geq 1} \mu^k u_k(\bar{x}, \bar{y}), \qquad y = \bar{y} + \sum_{k \geq 1} \mu^k v_k(\bar{x}, \bar{y}), \tag{11}$$

with unknown functions $u_k(\bar{x}, \bar{y}), v_k(\bar{x}, \bar{y})$. We have infinite system of linear partial differential equations of first order

$$\begin{cases} \left(\frac{\partial u_1}{\partial \bar{y}}, \omega(\bar{x})\right) = X(\bar{x}, \bar{y}) - \bar{X}(\bar{x}, \bar{y}), \\ \left(\frac{\partial v_1}{\partial \bar{y}}, \omega(\bar{x})\right) = \left(\frac{\partial \omega}{\partial \bar{x}}, u_1\right) + Y(\bar{x}, \bar{y}) - \bar{Y}(\bar{x}, \bar{y}), \\ \left(\frac{\partial u_k}{\partial \bar{y}}, \omega(\bar{x})\right) = F_k(\bar{x}, \bar{y}, u_1, v_1, \cdots, v_{k-1}, u_{k-1}, A_2, B_2, \cdots, A_k), \\ \left(\frac{\partial v_k}{\partial \bar{y}}, \omega(\bar{x})\right) = \Psi_k(\bar{x}, \bar{y}, u_1, v_1, \cdots, v_{k-1}, u_k, A_2, B_2, \cdots, A_k, B_k), \end{cases} \tag{12}$$

$$k = 2, 3, \cdots$$

The system (12) has a remarkable property: it is possible to integrate it in analytical way for any vector-index k if, for functions $\bar{X}$ and $\bar{Y}$, are chosen some averages of the functions X and Y.

Really, let the generators $\bar{X}(\bar{x}, \bar{y}), \bar{Y}(\bar{x}, \bar{y})$ be the partial sums of series (9):

$$\bar{X}(\bar{x}, \bar{y}) = \sum_{||k|| \in I'} X_k(\bar{x}) e^{i(k, \bar{y})}, \qquad \bar{Y}(\bar{x}, \bar{y}) = \sum_{||k|| \in I''} Y_k(\bar{x}) e^{i(k, \bar{y})}, \tag{13}$$

where $I' \in I$ and $I'' \in I$ are "resonance sets". Then

$$\begin{cases} X(\bar{x}, \bar{y}) - \bar{X}(\bar{x}, \bar{y}) = \sum_{||k|| \in I \setminus I'} X_k(\bar{x}) e^{i(k, \bar{y})}, \\ Y(\bar{x}, \bar{y}) - \bar{Y}(\bar{x}, \bar{y}) = \sum_{||k|| \in I \setminus I''} Y_k(\bar{x}) e^{i(k, \bar{y})}, \end{cases} \tag{14}$$

The sets $I \setminus I'$ and $I \setminus I$" are not "resonance sets". It is possible to find the exact solution of the first equations (12):

$$u_1(\bar{x}, \bar{y}) = \sum_{\|k\| \in I \setminus I'} \frac{X_k(\bar{x})}{i(k, \omega(\bar{x}))} e^{i(k, \bar{y})} + \varphi_1(\bar{x}), \tag{15}$$

$$v_1(\bar{x}, \bar{y}) = \sum_{\|k\| \in I \setminus I"} \frac{Y_k(\bar{x})}{i(k, \omega(\bar{x}))} e^{i(k, \bar{y})} + \left(\frac{\partial \omega(\bar{x})}{\partial x}, \sum_{\|k\| \in I \setminus I'} \frac{X_k(\bar{x}) e^{i(k, \bar{y})}}{i^2(k, \omega(\bar{x}))^2} \right)$$

$$+ \left(\left(\frac{\partial u_1}{\partial \bar{x}}, \varphi_1(\bar{x}) \right), \bar{y} \right) + \psi_1(\bar{x}).$$

Here, ψ_1, φ_1 are arbitrary differentiable functions of their arguments $\bar{x}_1, \cdots, \bar{x}_m$.

Integration of equations (12) at $k = 2, 3, \cdots$ is without complicated difficulties. Rather important is the fact that, while determining functions u_2 and v_2 , we can use functions $A_2, B_2, \psi_1, \varphi_1$.

The stated analytical algorithm means that we consequently construct the replacement of variables

$$(x, y) \to (\bar{x}_1, \bar{y}_1) \to (\bar{x}_2, \bar{y}_2) \to \cdots \to (\bar{x}_s, \bar{y}_s) \to \cdots.$$

Naturally, for the final construction of the solution of initial equations (8), one should solve the generating equation (10) with new initial conditions $\bar{x}(0), \bar{y}(0)$. In conclusion we want to note once again that, in formulas (11), the functions u_k, v_k are found by analytical methods. The solution of the generating equation (10) can be found with the combination of numerical and analytical methods.

References

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