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SEMILINEAR ELLIPTIC PROBLEMS WITH PAIRS OF DECAYING POSITIVE SOLUTIONS

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1. Introduction. Our main objective is to prove the existence of a pair of positive, exponentially decaying, classical solutions of the semilinear elliptic eigenvalue problem

$$(1.1)\qquad \begin{cases} Lu = \lambda f(x, u), \ x \in \Omega, \\ u|_{\partial\Omega} = 0, \ \lim_{|x|\to\infty} u(x) = 0 \end{cases}$$

in a smooth unbounded domain $\Omega \subset \mathbf{R}^N$, $N \geqq 2$, where λ is a positive parameter and L is a uniformly elliptic operator in Ω defined by

$$L = -\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j] + b(x), \quad x \in \Omega,$$

$$D_i = \partial/\partial x_i, \quad i = 1, \ldots, N.$$

The nonlinearity $f(x, u)$ is assumed to be bounded, as described in detail below. For some numbers λ_* and λ^*, $0 < \lambda_* \leqq \lambda^*$, the main Theorems 3.2 and 4.1 establish that (1.1) has at least two distinct positive solutions in Ω with exponential decay at ∞ for all $\lambda > \lambda^*$, but no nontrivial nonnegative solutions for $0 \leqq \lambda < \lambda_*$. Parallel results also hold for the analogue of (1.1) in the entire space $\mathbf{R}^N$, i.e.,

$$(1.2)\qquad \begin{cases} Lu = \lambda f(x, u), \quad x \in \mathbf{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

The hypotheses on L are: Each $a_{ij} \in C^{1+\alpha}_{\text{loc}}(\Omega)$, $b \in C^{\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$, each a_{ij}, $D_i(a_{ij})x_j$, and b are bounded in Ω, $b(x) \geqq b_0 > 0$, and L is uniformly elliptic in Ω. The hypotheses on the nonlinearity $f(x, u)$ are listed below:

(f_1) $f:(\Omega \cup \partial\Omega) \times \mathbf{R}_+ \to \mathbf{R}$ is locally Lipschitz continuous and $f(x, 0) = 0$ for all $x \in \Omega$, where $\mathbf{R}_+ = [0, \infty)$.

(f_2) There exists a number $T > 0$ such that $f(x, t) < 0$ for all $t > T$, $x \in \Omega$.

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(f_3) $F(x_0, t_0) > 0$ for some $(x_0, t_0) \in \Omega \times [0, T]$, where

$$F(x, t) = \int_0^t f(x, s)ds.$$

(f_4) f is bounded on $\bar{\Omega} \times [0, T]$ and $f(x, t) = o(t)$ as $t \to 0$ uniformly on $\bar{\Omega}$.

(f_5) $\limsup_{|x| \to \infty} \frac{f(x, t)}{t} = 0$ uniformly on $[0, T]$.

For problem (1.2), it is understood that $\Omega = \mathbf{R}^N$ and $\partial\Omega$ is deleted. For example, all these conditions are satisfied in the case of the equation

$$(1.3) \qquad -\Delta u + b(x)u = \lambda u[p(x)|u|^\gamma - q(x)|u|^\beta], \quad x \in \mathbf{R}^N,$$

where $0 < \gamma < \beta$, and b, p, q are appropriate nonnegative functions in $\mathbf{R}^N$.

Condition (f_2) implies that (1.1) has a "bounded nonlinearity". This is crucial for our method in Sections 2 and 3 for establishing two distinct decaying positive solutions. Condition (f_5) is not needed for the nonexistence results in Section 4, but it is essential in Section 3 for our procedure for proving the multiplicity Theorem 3.2 via Lemma 3.1. The key exponential decay estimates in this lemma, for two distinct sequences of local solutions, are obtained from the fact that these local solutions satisfy linear elliptic inequalities of type (3.2) as a consequence of (f_2) and (f_5). Of course our results neither preclude other possible multiplicity theorems under conditions replacing (f_1)-(f_5), nor generalize known multiplicity theorems for special equations. For example it is well known that the "critical exponent" Lane-Emden (associated with Yang-Mills) equation

$$-\Delta u = u^{(N+2)/(N-2)} \quad \text{in } \mathbf{R}^N, N \geqq 3$$

has an uncountable infinitude of radially symmetric positive solutions $u(|x|) = O(|x|^{2-N})$ in $\mathbf{R}^N$ (but $u \notin L^2(\mathbf{R}^N)$), see, e.g., [**3**, p. 370; **4**]. As far as we are aware, multiplicity results for positive decaying solutions of our type for (1.1) have not been obtained previously, even for the case of constant coefficients. We do not know to what extent our existence results remain true if (f_5) is weakened.

Solutions $u(x)$ of (1.3) of course provide stationary states $e^{i\omega t}u(x)$ of the corresponding wave equation, called the Klein-Gordon equation in nonlinear quantum field theory [**11**]. In the case of constant coefficients b, p, q and in some radially symmetric cases, the existence of a positive decaying solution of (1.3) in $\mathbf{R}^N$ has been proved by Berestycki and Lions [**2**] and Strauss [**11**], in particular. Existence theorems guaranteeing infinitely many distinct radial solutions (not necessarily positive) of equations including (1.3) have been obtained by Berestycki and Lions [**3**], Jones and Küpper [**5**] and Strauss [**11**]. A remarkable and powerful "principle of concentration-compactness" has been developed by Lions [**6**, **7**, etc.] as a method in the Calculus of Variations capable of attacking equations with variable coefficients. Jones and Küpper [**5**] employ a dynamical systems approach to obtain radial solutions of an equation of the type $\Delta u + f(u) = 0$ with exactly m zeros in $[0, \infty]$. Many additional related results can be found in the bibliographies of the References cited, especially [**3**].

On the basis of the variational method of Ambrosetti and Rabinowitz [**1**], distinct local solutions u_n and v_n of $Lu = \lambda f(x, u)$ in bounded domains

$$(1.4) \qquad \Omega_n = \{x \in \Omega: \ |x| < n\}, \quad n = 1, 2, \dots$$

are obtained in Section 2 for sufficiently large λ. It is then shown in Section 3 that $\{u_n\}$ and $\{v_n\}$ have subsequences which converge locally uniformly in $C^2(\Omega)$ to distinct solutions u and v of (1.1) (or (1.2)) with exponential decay at ∞. Section 4 contains theorems guaranteeing nonexistence of nontrivial nonnegative solutions of (1.1) and related problems for all λ in some interval $[0, \lambda_*)$.

2. Existence of local solutions. With T as in (f_2) we define

$$(2.1) \qquad f_T(x, t) = \begin{cases} 0 & \text{if } t < 0, \\ f(x, t) & \text{if } 0 \leqq t \leqq T, \\ f(x, T) & \text{if } t > T; \end{cases}$$

$$(2.2) \qquad F_T(x, t) = \int_0^t f_T(x, s)ds, \quad t \geqq 0 \quad x \in \Omega.$$

Then f_T is locally Lipschitz continuous in $(\Omega \cup \partial\Omega) \times \mathbf{R}_+$ by (f_1), and $F_T(x, t) = F(x, t)$ for all $x \in \Omega$, $0 \leqq t \leqq T$ by (f_3). Consider the eigenvalue problem

$$(2.3) \qquad Lu = \lambda f_T(x, u), \quad x \in \Omega_n, \quad u|_{\partial}\Omega_n = 0,$$

where Ω_n is defined by (1.4), for integers $n \geqq n_0$, $\Omega_{n_0} \neq \emptyset$. We note that a nonnegative solution $u_n(x)$ of (2.3) must satisfy

$$(2.4) \qquad 0 \leqq u_n(x) \leqq T, \quad x \in \Omega_n,$$

for if $u_n(x_0) > T$ is a positive maximum of $u_n(x)$ at $x_0 \in \Omega_n$, then in suitable coordinates

$$\lambda f_T(x_0, u_n(x_0)) = (Lu_n)(x_0) \geqq b_0 u_n(x_0) > 0,$$

contradicting (f_2). It follows from (2.4) that a nonnegative solution of (2.3) also is a solution of

$$(2.5) \qquad Lu = \lambda f(x, u), \quad x \in \Omega_n, \quad u|_{\partial}\Omega_n = 0.$$

Let E_n denote the Sobolev space $W_0^{1,2}(\Omega_n)$, with norm $\| \ \|$. Let H_n, I_n, and J_n be the functionals on E_n defined by

(2.6) $$H_n(u) = \frac{1}{2} \int_{\Omega_n} \left[\sum_{i,j=1}^{N} a_{ij}(x) D_i u D_j u + b(x) u^2(x) \right] dx;$$

(2.7) $$I_n(u) = \int_{\Omega_n} F_T(x, u(x)) dx;$$

(2.8) $$J_n(u, \lambda) = H_n(u) - \lambda I_n(u), \quad u \in E_n.$$

For $\rho > 0$ define

$$B_{n\rho} = \{u \in E_n : \|u\| < \rho\};$$

$$S_{n\rho} = \{u \in E_n : \|u\| = \rho\}.$$

LEMMA 2.1. *If* (f_1)-(f_4) *hold, for arbitrary* $\lambda > 0$ *there exist positive numbers* ρ *and* σ, *independent of* n, *such that*

$$J_n(u, \lambda) > 0 \text{ for all } u \in B_{n\rho} \setminus \{0\};$$

$$J_n(u, \lambda) \geqq \sigma \text{ for all } u \in S_{n\rho}.$$

Proof. For a fixed number $p \in (0, 4/(N - 2))$, condition (f_4) implies that

$$|F_T(x, t)| \leqq \epsilon |t|^2 + C|t|^{2+p}, \quad x \in \Omega_n$$

for all $\epsilon > 0$ and for all real t, where $C = C(\epsilon)$ is a positive constant. By (2.7),

$$I_n(u) \leqq \epsilon \|u\|_2^2 + C\|u\|_{2+p}^{2+p} \quad \text{for all } u \in E_n,$$

where $\| \ \|_q$ denotes the norm in $L^q(\Omega_n)$. Since $2 < 2 + p < 2N/(N - 2)$, the Sobolev embedding $E_n \hookrightarrow L^q(\Omega_n)$ for $2 \leqq q < 2N/(N - 2)$ implies that

(2.9) $$I_n(u) \leqq K(\epsilon \|u\|^2 + C\|u\|^{2+p}), \quad u \in E_n$$

for some positive constant K independent of n. The uniform ellipticity of L and (2.6)-(2.9) show that

$$J_n(u, \lambda) \geqq (K_0 - \lambda \epsilon K - \lambda C K \|u\|^p) \|u\|^2$$

for all $u \in E_n$, where K_0 is another positive constant independent of n. With the choice $\epsilon = K_0/4\lambda K$, the conclusion of Lemma 2.1 follows if

$$\rho = (K_0/4\lambda C K)^{1/p}, \quad \sigma = \frac{1}{2} K_0 \rho^2.$$

LEMMA 2.2. *If* (f_1)-(f_4) *hold, there exists* $\lambda^* > 0$, *a positive integer* M, *and a function* $z \in E_n$ *for all* $n \geqq M$ *such that* $J_n(z, \lambda) < 0$ *for all* $\lambda > \lambda^*$.

Proof. By (f_3) there exists $\phi \in C_0^\infty(\Omega_M)$, for M sufficiently large, such that $I_M(\phi) > 0$. Let λ^* be defined by the condition

$$H_M(\phi) = \lambda^* I_M(\phi),$$

i.e., $J_M(\phi, \lambda^*) = 0$, and let $z(x)$ be the extension of $\phi(x)$ to Ω defined to be 0 outside Ω_M. Since $\Omega_M \subset \Omega_n$ for $n > M$, it follows that $J_n(z, \lambda) < 0$ for all $n \geqq M, \lambda > \lambda^*$.

Let H, I, J denote the functionals on $E = W_0^{1,2}(\Omega)$ corresponding to (2.6), (2.7), (2.8), respectively, i.e.,

$$H(u) = \frac{1}{2} \int_\Omega \left[\sum_{i,j=1}^N a_{ij}(x) D_i u D_j u + b(x) u^2(x) \right] dx;$$

$$I(u) = \int_\Omega F_T(x, u(x)) dx;$$

$$J(u, \lambda) = H(u) - \lambda I(u), \quad u \in E.$$

THEOREM 2.3. *Suppose that* (f_1)-(f_4) *hold, and let* λ^* *and* M *be as in Lemma* 2.2. *Then there exist two distinct sequences* $\{u_n\}$ *and* $\{v_n\}$ *in* E *with the following properties*:

(i) *Each* u_n *and* v_n *has support* Ω_n;

(ii) u_n *and* v_n *are distinct positive solutions of* (2.3) *in* Ω_n *for all* $n \geqq M$, $\lambda > \lambda^*$; *and*

(iii) $J(u_{n+1}, \lambda) \leqq J(u_n, \lambda) \leqq J(z, \lambda) < 0,$

$$J(v_{n+1}, \lambda) \geqq J(v_n, \lambda) \geqq \sigma > 0$$

for all $n \geqq M, \lambda > \lambda^*$.

Proof. Since $S_{n\rho}$ separates 0 and z by Lemmas 2.1 and 2.2, (f_1)-(f_4) imply by a theorem of Ambrosetti and Rabinowitz [**1**, Theorem 3.35] that (2.3) has at least two distinct positive solutions u_n, v_n in Ω_n for all $n \geqq M$, $\lambda > \lambda^*$ such that

$$(2.10) \quad J(u_n, \lambda) = \inf\{J_n(u, \lambda) : u \in E_n\},$$

$$(2.11) \quad J(v_n, \lambda) = \inf_{g \in \Gamma_n} \max_{0 \leqq t \leqq 1} J_n(g(t), \lambda),$$

where

$$\Gamma_n = \{g \in C([0, 1], E_n): \quad g(0) = 0, g(1) = z\}.$$

Extensions of u_n, v_n to Ω are defined to have support Ω_n, and also are denoted by u_n, v_n. Thus $J_n(u_n, \lambda) = J(u_n, \lambda)$ in (2.10), and similarly in (2.11). We note that assumption (p_7) of [**1**] is not needed here since it was used in [**1**] only to construct $z \in E_n$ satisfying $J_n(z, \lambda) \leqq 0$. The monotony properties (iii) follow from the variational definitions (2.10) and (2.11) since $\Omega_n \subset \Omega_{n+1}$, $\quad n \geqq M$. The property

$$J(v_n, \lambda) \geqq \sigma > 0$$

is a consequence of Lemma 2.1 since $S_{n\rho}$ separates 0 and z.

3. Pairs of positive solutions of (1.1). The following notation will be used:

$$A(x) = |x|^{-2} \sum_{i,j=1}^{N} a_{ij}(x)x_i x_j;$$

$$B(x) = \sum_{i,j=1}^{N} D_i[a_{ij}(x)]x_j;$$

$$\Lambda = \sup_{x \in \Omega} A(x).$$

The next lemma enables us to show that the sequences in Theorem 2.3 have subsequences which converge locally uniformly in $C^2(\Omega)$.

LEMMA 3.1. *Suppose that* (f_1)-(f_5) *hold. Let* $\{u_n\}$ *and* $\{v_n\}$ *be the sequences in Theorem* 2.3, *and let* δ *be any number satisfying*

$$0 < \delta < \sqrt{b_0/\Lambda}.$$

Then there exists a positive constant K, *independent of* n, *such that*

$$u_n(x) \leqq Ke^{-\delta|x|}, \quad v_n(x) \leqq Ke^{-\delta|x|} \tag{3.1}$$

for all $x \in \Omega$, $n \geqq M$.

Proof. Choose $\epsilon > 0$ such that $b(x) - \epsilon > \delta^2\Lambda$ for all $x \in \Omega$. For $R > 0$, $n > R$, define

$$\Omega_{0,R} = \{x \in \Omega : |x| > R\}; \quad \Omega_{n,R} = \{x \in \Omega_n : |x| > R\}.$$

By (f_2) and (f_5), R can be chosen large enough that

$$\lambda f(x, t) \leqq \epsilon t \quad \text{for all } x \in \Omega_{0,R}, t \geqq 0.$$

Theorem 2.3 (ii) shows that u_n is a solution of (2.5) and hence satisfies the inequality

$$(Lu_n - \epsilon u_n)(x) = \lambda f(x, u_n(x)) - \epsilon u_n(x) \leqq 0, \quad x \in \Omega_{n,R}. \tag{3.2}$$

For any positive constant C, a calculation shows that $w(x) = Ce^{-\delta|x|}$ satisfies

$$\frac{(Lw - \epsilon w)(x)}{w(x)} = [b(x) - \epsilon - \delta^2 A(x)] + \frac{\delta}{|x|}\left[\sum_{i=1}^{N} a_{ii}(x) + B(x) - A(x)\right],$$

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and therefore a sufficiently large number R exists for which

(3.3) $\quad (Lw - \epsilon w)(x) \geqq 0, \quad x \in \Omega_{0,R}.$

Define

(3.4) $\quad C = Te^{\delta R}.$

Then by (2.4) and (3.2)-(3.4),

$$\begin{cases} L(w - u_n) - \epsilon(w - u_n) \geqq 0 & \text{in } \Omega_{n,R} \\ w - u_n \geqq 0 & \text{on } \partial\Omega_{n,R}, \end{cases}$$

implying that $w - u_n \geqq 0$ throughout $\bar{\Omega}_{n,R}$ by the maximum principle. Thus

$$u_n(x) \leqq Ce^{-\delta|x|} \quad \text{for all } x \in \bar{\Omega}_{0,R}$$

since u_n has support Ω_n, establishing the uniform estimate (3.1) in $\Omega_{0,R}$. The same estimate is obvious if $|x| \leqq R$ by (2.4). The conclusion for $v_n(x)$ is proved in exactly the same way.

THEOREM 3.2. *If* (f_1)-(f_5) *hold, there exists a number* $\lambda^* > 0$ *such that* (1.1) *has at least two positive solutions u and v in* Ω *satisfying*

$$J(u, \lambda) < 0 < \sigma \leqq J(v, \lambda) \quad \textit{for all } \lambda > \lambda^*.$$

These solutions are both bounded above by a constant multiple of $\exp(-\delta|x|)$ *in* Ω *for some* $\delta > 0$. *Parallel statements apply to problem* (1.2).

Proof. Since the sequence $\{u_n(x)\}$ is uniformly bounded in Ω by Lemma 3.1, a standard argument via L^p-estimates, Schauder estimates, and Sobolev embedding theorems (see, e.g. [**9**, pp. 124-126]) establishes that $\{u_n\}$ has a subsequence $\{u_n^*\}$ which converges locally uniformly in $C^2(\Omega)$ to a nonnegative solution $u(x)$ of

$$Lu = \lambda f(x, u), \quad u|_{\partial\Omega} = 0.$$

An interior Schauder estimate for this equation in any bounded subdomain implies that

$$u \in C^{2+\alpha}_{\text{loc}}(\Omega) \quad \text{for some } \alpha \in (0, 1).$$

In view of property (iii) of Theorem 2.3,

$$J(u_n^*, \lambda) \leqq J(z, \lambda) < 0 \quad \text{for all } n \geqq M, \lambda > \lambda^*.$$

Since u_n^* satisfies (2.3) and has support Ω_n, integration by parts yields

$$J(u_n^*, \lambda) = \int_\Omega \left[\frac{1}{2}u_n^*(x)f_T(x, u_n^*(x)) - \lambda F_T(x, u_n^*(x))\right]dx$$
$$\leqq J(z, \lambda) < 0.$$

The uniform estimate (3.1) and the pointwise convergence of $\{u_n^*(x)\}$ to $u(x)$ guarantee an estimate

$$u(x) \leqq Ke^{-\delta|x|}$$

in Ω for a constant $K > 0$. Then the dominated convergence theorem implies that

$$\lim_{n\to\infty} J(u_n^*, \lambda) = J(u, \lambda) \leqq J(z, \lambda) < 0,$$

showing that $u(x)$ is not identically zero.

To prove that $u(x) > 0$ in Ω, define

$$(3.5) \quad g(x, t) = \begin{cases} f(x, t)/t & \text{if } t > 0 \\ 0 & \text{if } t \leqq 0; \end{cases}$$

$$g^+(x) = \max\{g(x, u(x)), 0\};$$

$$g^-(x) = \min\{g(x, u(x)), 0\}, \quad x \in \Omega.$$

Then

$$f(x, u(x)) = [g^+(x) + g^-(x)]u(x), \quad x \in \Omega,$$

and consequently $u(x)$ is a nontrivial nonnegative solution of the linear elliptic inequality

$$Lu - \lambda g^-(x)u = \lambda g^+(x)u \geqq 0, \quad x \in \Omega.$$

Since $g^-(x) \leqq 0$ in Ω, the strong maximum principle implies that $u(x) > 0$ throughout Ω.

The corresponding properties of $v(x)$ are proved in the same way. Since $J(v, \lambda) > 0$, it follows that $u(x)$ and $v(x)$ are distinct positive solutions of (1.1) in Ω. The proof for (1.2) is essentially the same.

COROLLARY 3.3. *If* (f_1)-(f_5) *hold and* $0 < \delta < \sqrt{b_0/\Lambda}$, *both solutions* u *and* v *in Theorem* 3.2, *and in addition* $|\nabla u|$ *and* $|\nabla v|$, *are bounded above by constant multiples of* $\exp(-\delta|x|)$ *in* Ω.

The estimates for u, v follow from Theorem 3.2, and these imply the estimates for the gradients via standard interior Schauder estimates.

Remarks 3.4. Suppose that hypothesis (f_4) is strengthened to (f_4') below: (f_4') f is bounded on $\bar{\Omega} \times [0, T]$ and $f(x, t) = O(t^\nu)$ as $t \to 0$ uniformly on $\bar{\Omega}$, for some $\nu > 1$.

Also, assume that $a_* > 0$, where

$$a_* = \liminf_{|x|\to\infty} \frac{1}{A(x)}\left[\sum_{i=1}^N a_{ii}(x) + B(x) - A(x)\right].$$

Then the estimate in Corollary 3.3 can be sharpened to

$$(3.6) \qquad u(x) \leqq K|x|^{-d} \exp(-\sqrt{b_0/\Lambda}|x|), \quad x \in \Omega$$

for any $d < a_*/2$ and some constant $K > 0$. The same type estimates (3.6) hold for $v(x)$, $|(\nabla u)(x)|$, and $|(\nabla v)(x)|$. The proof is a slight elaboration of that given in Lemma 3.1.

In the special case $L = -\Delta + b(|x|)$, $x \in \mathbf{R}^N$, so $\Lambda = 1$, it can be shown that (3.6) holds with $d = (N - 1)/2$. This is accomplished by use of estimates for the Green's function for $-\Delta + 1$ in $\mathbf{R}^N$ together with arguments of Gidas, Ni, and Nirenberg [**4**].

4. Nonexistence theorems. The number λ^* in Theorem 3.2 cannot be replaced by 0. In fact, Theorem 4.1 below shows that a nonnegative solution of (1.1) of (1.2) must be identically zero for all λ in some interval $[0, \lambda_*)$. Theorem 4.3 contains, in particular, the stronger statement that a nonnegative solution of $Lu = \lambda f(x, u)$ in $\mathbf{R}^N$ is identically zero for all such λ. Condition (f_5) is not needed for any of the nonexistence theorems.

THEOREM 4.1. *If* (f_1)-(f_4) *hold, there exists a number* $\lambda_* > 0$ *such that neither* (1.1) *nor* (1.2) *has any nontrivial nonnegative solution* u *for any* $\lambda \in [0, \lambda_*)$.

Proof. Let $g(x, t)$ be as in (3.5), and define

$$\mu_* = \sup_{x \in \bar{\Omega}} \sup_{t \geqq 0} g(x, t); \quad \lambda_* = b_0 \mu_*^{-1}.$$

Conditions (f_2)-(f_4) show that $0 < \lambda_* < \infty$. A nonnegative solution of (1.1) satisfies the linear equation

$$(4.1) \qquad -\sum_{i,j=1}^{N} D_i[a_{ij}(x) D_j u] + Q(x, \lambda) u = 0, \quad x \in \Omega,$$

where

$$Q(x, \lambda) = b(x) - \lambda g(x, u(x)), \quad x \in \Omega.$$

If $\lambda < \lambda_*$, then

$$Q(x, \lambda) \geqq b_0 - \lambda \mu_* = (\lambda_* - \lambda)\mu_* > 0$$

in Ω. Since

$$u|_{\partial}\Omega = 0 \quad \text{and} \quad \lim_{|x| \to \infty} u(x) = 0,$$

the maximum principle for (4.1) implies that $u(x)$ is identically zero in Ω. The proof for (1.2) is similar.

A theorem of this type was obtained in [**10**, p. 186] in the special case $L = -\Delta + b(x)$.

For $r > 0$, define

$$G_r = \{x \in \mathbf{R}^N : |x| > r\}.$$

Let Ω be an *exterior* domain in $\mathbf{R}^N$, i.e., $G_\rho \subset \Omega$ for some $\rho > 0$. Following **[8]**, we define an *antibarrier* for L at ∞ to be a solution $h \in C^2(G_R)$ of $Lh \geqq 0$ in G_R, for some $R \geqq \rho$, such that

$$\lim_{|x|\to\infty} h(x) = +\infty.$$

THEOREM 4.2. *Suppose that L has an antibarrier $h(x)$ at ∞. If $u \in C^2(\Omega)$ is any solution of $Lu \leqq 0$ in an exterior domain $\Omega \subset \mathbf{R}^N$ such that $u = o(h)$ uniformly at ∞, then*

$$\limsup_{|x|\to\infty} u(x) \leqq \max(0, M), \tag{4.2}$$

where

$$M = \max_{x\in\partial\Omega} u(x).$$

Results of this type appear in the literature (see, e.g., **[8]**), but we could not locate the exact statement of Theorem 4.2 for the operator L under consideration here. A sketch of the proof is given below.

Proof. Choose $R \geqq \rho$ large enough that $h(x) > 0$ for all $|x| \geqq R$. For arbitrary $\eta > 0$ define

$$v_\eta(x) = u(x) - M_R - \eta h(x), \quad x \in G_R,$$

where

$$M_R = \max_{|x|=R} u^+(x).$$

Then

$$\begin{cases} Lv_\eta \leqq 0 & \text{in } G_R, \\ v_\eta(x) < 0 & \text{on } |x| = R, \\ \limsup_{|x|\to\infty} v_\eta(x) < 0, \end{cases}$$

implying that $v_\eta(x) < 0$ in G_R by the maximum principle. Since η is arbitrary, $u(x) \leqq M_R$ for all $|x| \geqq R$. Now consider $u(x)$ in the annulus Ω_{2R}. Since $Lu \leqq 0$ in Ω_{2R}, $u(x) \leqq M_R$ on $|x| = 2R$, and $u(x) \leqq M$ on $\partial\Omega$, it follows from the maximum principle that

$$\max_{|x|=R} u(x) \leqq \max\{M_R, M\}.$$

This implies that $M_R \leqq \max(0, M)$, from which

$$u(x) \leqq \max(0, M) \quad \text{for all } |x| \geqq R,$$

proving (4.2).

In the case of an exterior domain Ω, consider the following variant of (1.1), without the decay condition at ∞:

$$(4.3) \qquad \begin{cases} Lu = \lambda f(x, u), & x \in \Omega, \\ u|_{\partial}\Omega = 0. \end{cases}$$

The corresponding problem in $\mathbf{R}^N$ is

$$(4.4) \qquad Lu = \lambda f(x, u), \quad x \in \mathbf{R}^N.$$

THEOREM 4.3. *If* (f_1)-(f_4) *hold, there exists* $\lambda_* > 0$ *such that neither* (4.3) *nor* (4.4) *has any nontrivial nonnegative solution for any* $\lambda \in [0, \lambda_*)$.

Proof. Let ϵ be as in the proof of Lemma 3.1. By (f_2) and (f_4) there exists $\lambda_* > 0$ such that

$$\lambda f(x, t) \leqq \epsilon t \quad \text{for all } x \in \Omega, t \geqq 0, 0 \leqq \lambda < \lambda_*.$$

Then a nonnegative solution $u(x)$ of (4.3) satisfies

$$(4.5) \qquad (L - \epsilon)u \equiv -\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j u] + [b(x) - \epsilon]u \leqq 0$$

in Ω, where $b(x) - \epsilon > 0$ in Ω. As in Lemma 3.1 it is easily seen from (4.5) that $u(x)$ must be bounded in Ω. The calculation preceding (3.3), with δ replacing $-\delta$, shows that $h(x) = e^{\delta|x|}$ is an antibarrier for $L - \epsilon$ at ∞. Theorem 4.2 applied to (4.5) then gives

$$\limsup_{|x|\to\infty} u(x) \leqq 0.$$

Since $u|_{\partial}\Omega = 0$, the maximum principle for (4.5) implies that $u(x) \leqq 0$ in Ω, proving the theorem in the case (4.3).

If $u(x)$ is a nonnegative solution of (4.4), $u(x)$ satisfies (4.5) in G_R for any $R > 0$, and Theorem 4.2 gives

$$\limsup_{|x|\to\infty} u(x) \leqq \max_{|x|=R} u(x).$$

By the maximum principle, $u(x)$ must be a nonnegative constant, which can only be 0 by (4.5).

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