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MATHEMATICAL NOTES

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THE BOUNDS BASED ON THE FUNCTIONS OF OBSERVATIONS FOR MAXIMUM OF STABLE LAW

BY

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Gnedenko and Kolmogorov [3, pp. 181–182] have shown that if X_n with law $F(x)$ belong to the domain of normal attraction of a stable law of index $0<\alpha<2$, i.e. if partial sum $S_n/an^{1/\alpha}$ converges in distribution to some stable law V_α, $a>0$ then there exist c_1 and c_2 such that

(1) $$1-F(x) \sim c_1a^\alpha x^{-\alpha} \quad \text{as } x\to\infty$$

and

(2) $$F(-x) \sim c_2a^\alpha|x|^{-\alpha} \quad \text{as } x\to\infty.$$

They also proved that for every constant $k>0$

(3) $$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)}\to k^\alpha \quad \text{as } x\to\infty.$$

Now, if X_n are nonnegative random variables then (3) reduces to

(4) $$\frac{1-F(x)}{1-F(kx)}\to k^\alpha \quad \text{as } x\to\infty.$$

Hence, by a theorem of Gnedenko [2],

(5) $$\lim_{n\to\infty} F^n(A_nx) = \begin{cases} 0 & \text{if } x\le 0 \\ \exp(-x^{-\alpha}) & \text{if } x>0 \end{cases}$$

where

$$F(A_n) \simeq 1-\frac{1}{n}.$$

However, equation (5) is difficult to solve as the form of F is not known explicitly. So that it is interesting to construct two functions of n between which the entire probability mass of the maximum lies.

THEOREM. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables belonging to the domain of normal attraction of a stable law with characteristic exponent $\alpha(0<\alpha<2)$.*

If Y_n is the nth order statistic (maximum) of $X_1, X_2, \ldots, X_n$, then, for any fixed $\delta>0$

$$\lim_{n\to\infty} P[n^{1/\alpha-\delta} \le Y_n \le n^{1/\alpha+\delta}] = 1 \quad \text{if } X_1 \ge 0 \text{ a.s.} \tag{6}$$

and

$$\lim_{n\to\infty} P[-n^{1/\alpha+\delta} \le Y_n \le n^{1/\alpha+\delta}] = 1 \quad \text{if } X_1 \text{ is symmetric.} \tag{7}$$

Proof. For (6) it is sufficient to show

$$\lim_{n\to\infty} F^n(n^{1/\alpha-\delta}) = 0 \tag{8}$$

and

$$\lim_{n\to\infty} F^n(n^{1/\alpha+\delta}) = 1, \tag{9}$$

where F is the common d.f. This is equivalent to showing that

$$\lim_{n\to\infty} n \log F(n^{1/\alpha-\delta}) = -\infty$$

and

$$\lim_{n\to\infty} n \log F(n^{1/\alpha+\delta}) = 0.$$

If $X_1 \ge 0$ a.s., then from (1) there exists $d_1, d_2 \ge 0$ such that

$$1-d_2x^{-\alpha} \le F(x) \le 1-d_1x^{-\alpha}$$

for x sufficiently large.

Also

$$\log(1-d_2x^{-\alpha}) \le \log F(x) \le \log(1-d_1x^{-\alpha}).$$

Since F is not degenerate, without loss of generality, we can assume

$$0 < d_1x^{-\alpha} < 1.$$

Therefore by expansion,

$$\log\frac{1}{1-d_1x^{-\alpha}} > d_1x^{-\alpha}.$$

So, if $x=n^{1/\alpha-\delta}$,

$$-\log(1-d_1x^{-\alpha}) > d_1n^{\alpha\delta}n^{-1},$$

which implies

$$\lim_{n\to\infty} n\log(1-d_1x^{-\alpha}) < \lim_{n\to\infty} -n^{\alpha\delta}d_1 = -\infty.$$

Therefore,

$$\lim_{n\to\infty} n\log(F(n^{1/\alpha-\delta})) = -\infty.$$

Now, by usual power series expansion of $\log(1-x)$,

$$\lim_{n\to\infty}\left|n\log\left(\frac{1}{1-d_2 n^{-(1+\alpha\delta)}}\right)\right| = \lim_{n\to\infty}\left|n\log(1-d_2 n^{-(1+\alpha\delta)})\right| = 0.$$

But

$$1-d_2 n^{-(1+\alpha\delta)} \le F(n^{1/\alpha+\delta}) < 1 \quad \text{if } n \text{ is large,}$$

$$|\log(1-d_2 n^{-(1+\alpha\delta)})| \ge |\log F(n^{1/\alpha+\delta})|,$$

$$\lim_{n\to\infty} n\log F(n^{1/\alpha+\delta}) = \lim_{n\to\infty} n\log F(n^{1/\alpha+\delta}) = 0.$$

If X_1 is symmetric it is sufficient to show

$$\lim_{n\to\infty} F^n(n^{1/\alpha+\delta}) = 1,$$

where F satisfies (1).

The rest of the proof follows, same as (9).

REMARK. This covers the result of Doubleday and Wasan [1], which is a particular case of this result if we put $\alpha=\frac{1}{2}$.

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