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ON THE CONNECTEDNESS OF THE SETS OF LIMIT POINTS OF CERTAIN TRANSFORMS OF BOUNDED SEQUENCES[1]

BY

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1. **Definitions and statements of results.** The transforms discussed here are the quasi-Hausdorff (Theorem 1), the $[J, f(x)]$ (Theorem 2) and the Borel integral means (Theorem 3). We are concerned here with whether or not the limit-points of these transforms of bounded sequences form connected sets. Such a set is one which cannot be decomposed into the union of two disjoint nonempty open sets.

Let $\alpha(v)$ be a function of bounded variation over $[0, 1]$. The quasi-Hausdorff transformation generated by $\alpha(v)$, of the bounded sequence $S=\{s_n\}$ of complex numbers, or, in short, $T=QH(\alpha)S$, is defined by

$$t_m = \int_0^1 \sum_{n=m}^{\infty} \binom{n}{m}(1-v)^{n-m}v^{m+1}s_n \, d\alpha(v), \quad m = 0, 1, 2, \ldots.$$

It is known [7] that $QH(\alpha)$ is conservative and that it is regular if and only if $\alpha(1)-\alpha(0+)=1$. Among the regular quasi-Hausdorff methods are the Taylor (Circle) methods T_r $(0<r<1)$ which are obtained by taking $\alpha(v)=0$, $0 \le v < r$; $\alpha(v)=1$, $r \le v \le 1$.

Let $\beta(u)$ be a function of bounded variation over $[0, \infty)$. The $[J, f(x)]$ transform generated by

$$f(x) = \int_0^\infty e^{-ux} \, d\beta(u), \quad x > 0,$$

of the bounded sequence $S=\{s_n\}$, or, in short $T=J(\beta)S$, is defined by [5]

$$T = t(x) = \int_0^\infty e^{-ux} \sum_{n=0}^{\infty} \frac{(ux)^n}{n!} s_n \, d\beta(u), \quad x > 0.$$

The sequence S is said to be $J(\beta)$-summable to s if $\lim t(x)=s$, as $x \to \infty$. It was proved in [5] that $J(\beta)$ is conservative and that it is regular if and only if $\beta(0) = \beta(0+)$ and $\beta(\infty-)-\beta(0)=1$. Among the regular $[J, f(x)]$ methods are the Borel exponential means and the Abel scale [4] A_γ, $\gamma > -1$, obtained by taking $f(x)=e^{-x}$ and $f(x)=(1+x)^{-\gamma-1}$, respectively, with $\gamma=0$ giving the classic Abel method. Alternatively, the Borel exponential means can be generated by $\beta(u)=0$, $0 \le u < 1$; $\beta(u)=1$, $1 \le u < \infty$, and the A_γ method by

$$\beta(u) = \frac{1}{\Gamma(\gamma+1)} \int_0^u v^\gamma e^{-v} \, dv.$$

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The Borel integral means of the bounded sequence $S=\{s_n\}$, i.e. $T=B'S$, are defined by

$$T = t(x) = \int_0^x e^{-u} \sum_{m=0}^{\infty} \frac{s_m - s_{m-1}}{m!} u^m \, du,$$

with $s_{-1}=0$. The sequence S is said to be B' summable to s if $\lim t(x)=s$ as $x \to \infty$.

Let T denote either the sequence $\{t_m\}$, $m=0, 1, \ldots$, or the function $t(x)$, $x>0$. The set of limit points of T (taken as $m \to \infty$ or as $x \to \infty$, respectively) is denoted by $L\{T\}$.

In this notation we formulate three theorems. The first parallels that obtained for regular Hausdorff means by Wells [9], except that our result, unlike his, does not assume regularity of the method (but see §6, Remark (a)).

THEOREM 1. *For every $QH(\alpha)$ method the following two statements are equivalent*:

(i) $\alpha(1)=\alpha(1-)$;

(ii) *$L\{QH(\alpha)S\}$ is connected for each bounded sequence S.*

The other results are:

THEOREM 2. *For every $J(\beta)$ method, the set $L\{J(\beta)S\}$ is connected for each bounded sequence S.*

THEOREM 3. *The set $L\{B'S\}$ is connected for each bounded sequence S.*

2. **Preliminary lemmas.** The proof of Theorem 1 is based on a result due to Barone [3].

LEMMA B. *If T is a bounded sequence such that $t_m - t_{m-1} \to 0$ as $m \to \infty$, then $L\{T\}$ is connected.*

To establish Theorems 2 and 3 (which deal with sequence-to-function transforms) a variant can be employed. Its proof is a straightforward modification of Barone's; the details are left to the reader(s).

LEMMA B′. *If $T=t(x)$ is a bounded function such that there exists an M with the property that, for each sequence $x_m \to \infty$, the condition* $\sup_m |x_{m+1}-x_m| \le M$ *implies $t(x_{m+1})-t(x_m) \to 0$ as $m \to \infty$, then $L\{T\}$ is connected.*

(Obviously, if there is one such M, then any finite M will do.)

3. **Proof of Theorem 1.** First suppose $\alpha(1)=\alpha(1-)$, with S a bounded sequence. Then $T=QH(\alpha)S$ is bounded and

$$|t_{m+1}-t_m| \le K \int_0^1 \sum_{n=m}^{\infty} \left| \binom{n}{m}(1-v)^{n-m}v^{m+1} - \binom{n}{m+1}(1-v)^{n-m-1}v^{m+2} \right| |d\alpha(v)|$$

$$= K \int_{0+}^1 \sum_{n=m}^{\infty} \left| \binom{n}{m}(1-v)^{n-m}v^{m+1} - \binom{n}{m+1}(1-v)^{n-m-1}v^{m+2} \right| |d\alpha(v)|,$$

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where $K=\sup_n |s_n|$ and $\binom{m}{m+1}=0$; the vanishing of the integrand at 0 implies the equality of the two integrals.

Now, keeping in mind that $\binom{m}{m+1}=0$, it is seen readily that

$$\binom{n}{m}(1-v)^{n-m}v^{m+1}-\binom{n}{m+1}(1-v)^{n-m-1}v^{m+2}$$
$$=\binom{n+1}{m+1}(1-v)^{n-m}v^{m+1}-\binom{n}{m+1}(1-v)^{n-m-1}v^{m+1},\quad n\geq m\geq 0.$$

Also, for each fixed $1>v>0$, we have

$$(1)\quad \binom{n+1}{m+1}(1-v)^{n-m}v^{m+1}-\binom{n}{m+1}(1-v)^{n-m-1}v^{m+1}\begin{cases}\geq 0 & \text{for } n<[(m+1)/v]\\ <0 & \text{for } n\geq[(m+1)/v]\end{cases},$$

where $[x]$ is the greatest integer not exceeding x.

Finally,

$$\lim_{n\to\infty}\binom{n+1}{m+1}(1-v)^{n-m}v^{m+1}=0.$$

Therefore, with $b_m(v)=[(m+1)/v]$, it follows that

$$|t_{m+1}-t_m|\leq 2K\int_{0+}^{1}\binom{b_m(v)}{m+1}(1-v)^{b_m(v)-m-1}v^{m+1}\,|d\alpha(v)|.$$

Let $\varepsilon>0$. Since $\alpha(1)=\alpha(1-)$ we can choose $\frac{1}{2}>\delta>0$ such that

$$\int_{0+}^{\delta}|d\alpha(v)|<\varepsilon,\qquad \int_{1-\delta}^{1}|d\alpha(v)|<\varepsilon.$$

For $\delta\leq v\leq 1-\delta$ it follows from Stirling's formula that

$$\binom{b_m(v)}{m+1}(1-v)^{b_m(v)-m-1}v^{m+1}\leq\frac{C}{\sqrt{m+1}},$$

where C is a constant independent of m and v. Since $\alpha(v)$ is of bounded variation, it follows that $|t_{m+1}-t_m|\to 0$ as $m\to\infty$ and so, by Lemma B, $L\{QH(\alpha)S\}$ is connected.

Conversely, suppose $\alpha(1)\neq\alpha(1-)$. Let

$$\alpha_1(v)=\begin{cases}\alpha(v) & 0\leq v<1\\ \alpha(1-) & v=1\end{cases}$$

and $\alpha_2(v)=\alpha(v)-\alpha_1(v)$. First we prove that

$$(2)\qquad \max_n\left|\int_0^1\binom{n}{m}(1-v)^{n-m}v^{m+1}\,d\alpha_1(v)\right|\to 0\quad\text{as } m\to\infty.$$

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Now,

$$\int_0^1 \binom{n}{m}(1-v)^{n-m}v^{m+1}\,d\alpha_1(v) = \int_{0+}^1 \binom{n}{m}(1-v)^{n-m}v^{m+1}\,d\alpha_1(v)$$

and, since $\alpha_1(1)=\alpha_1(1-)$, if $\varepsilon>0$ is prescribed, we can choose $\frac{1}{2}>\delta>0$ such that

$$\int_{0+}^{\delta} |d\alpha_1(v)| < \varepsilon, \qquad \int_{1-\delta}^{1} |d\alpha_1(v)| < \varepsilon.$$

By (1)

$$\max_n \left|\int_\delta^{1-\delta} \binom{n}{m}(1-v)^{n-m}v^{m+1}\,d\alpha_1(v)\right|$$
$$\le \int_\delta^{1-\delta} \binom{b_{m-1}(v)}{m}(1-v)^{b_{m-1}(v)-m}v^{m+1}\,|d\alpha_1(v)| \to 0 \quad \text{as } m\to\infty,$$

as we have shown above. This proves (2).

A result of Agnew [1, Theorem 3] shows that, since $QH(\alpha_1)$ is conservative, (2) guarantees the existence of a divergent sequence S of zeros and ones such that $QH(\alpha_1)S\to 0$. On the other hand, $QH(\alpha_2)S = [\alpha(1)-\alpha(1-)]S$ and $T=QH(\alpha_1)S + QH(\alpha_2)S$. Therefore $L\{QH(\alpha)S\}$ is not connected. This completes the proof.

4. **Proof of Theorem 2.** Let S be a bounded sequence. Then $t(x)$ is a bounded function. Let $x_n\to\infty$ be an arbitrary sequence such that $\sup_n |x_{n+1}-x_n|\le 1$. By Lemma B′ our proof is complete once we show that $|t(x_{n+1})-t(x_n)|\to 0$ as $n\to\infty$.

Now let $\varepsilon>0$. We can choose $\delta>0$ such that $\int_\delta^\infty |d\beta(u)|<\varepsilon$. Thus

$$\left|\int_\delta^\infty e^{-ux_{n+1}}\sum_{k=0}^\infty \frac{(ux_{n+1})^k}{k!}s_k\,d\beta(u) - \int_\delta^\infty e^{-ux_n}\sum_{k=0}^\infty \frac{(ux_n)^k}{k!}s_k\,d\beta(u)\right|$$
$$\le 2K\int_\delta^\infty |d\beta(u)| < 2K\varepsilon,$$

where $K=\sup_n |s_n|$.

For $x>0$, put

$$t_1(x) = \int_0^\delta e^{-ux}\sum_{k=0}^\infty \frac{(ux)^k}{k!}s_k\,d\beta(u)$$
$$= \int_0^\delta e^{-ux}\sum_{k=0}^\infty \frac{(ux)^k}{k!}\sigma_k^{(1)}\,d\beta(u) + i\int_0^\delta e^{-ux}\sum_{k=0}^\infty \frac{(ux)^k}{k!}\sigma_k^{(2)}\,d\beta(u)$$
$$= t_2(x)+it_3(x).$$

Here $\sigma_k^{(1)}$ and $\sigma_k^{(2)}$ are the respective real and imaginary parts of s_k so that $t_2(x)$ and $t_3(x)$ are both real functions. Let $\tau(x)$ denote either $t_2(x)$ or $t_3(x)$ and σ_k represent the corresponding $\sigma_k^{(1)}$ or $\sigma_k^{(2)}$, as the case may be. Then the mean-value theorem asserts that

$$\tau(x_{n+1})-\tau(x_n) = \tau'(y_n)(x_{n+1}-x_n),$$

where y_n is between x_n and x_{n+1}.

Differentiating term by term under the integral sign (easily justified), we get

$$y_n\tau'(y_n) = \int_0^\delta e^{-uy_n} \sum_{k=0}^\infty \left[\frac{(uy_n)^k}{(k-1)!} - \frac{(uy_n)^{k+1}}{k!}\right] \sigma_k \, d\beta(u),$$

where $(uy_n)^k/(k-1)!=0$ when $k=0$. Therefore, since $|x_{n+1}-x_n|\le 1$,

$$|\tau(x_{n+1})-\tau(x_n)| \le |\tau'(y_n)|$$
$$\le Ky_n^{-1} \int_0^\delta e^{-uy_n} \sum_{k=0}^\infty \left|\frac{(uy_n)^k}{(k-1)!} - \frac{(uy_n)^{k+1}}{k!}\right| |d\beta(u)|.$$

Now

$$\frac{(uy_n)^{k+1}}{k!} - \frac{(uy_n)^k}{(k-1)!} \begin{cases} >0 & \text{for } k < [uy_n] \\ \le 0 & \text{for } k \ge [uy_n]. \end{cases}$$

Thus if we put now $b_n(u)=[uy_n]$, it follows that

$$|\tau(x_{n+1})-\tau(x_n)| \le 2Ky_n^{-1} \int_0^\delta e^{-uy_n} \frac{(uy_n)^{b_n(u)+1}}{\{b_n(u)\}!} |d\beta(u)|,$$

since $(uy_n)^{k+1}/k! \to 0$, as $k\to\infty$.

Stirling's formula implies

$$e^{-uy_n} \frac{(uy_n)^{b_n(u)+1}}{\{b_n(u)\}!} \le C\sqrt{uy_n},$$

for some constant $C>0$. Hence

$$|\tau(x_{n+1})-\tau(x_n)| \le 2KC\delta^{1/2}y_n^{-1/2} \int_0^\delta |d\beta(u)| \to 0, \quad \text{as } n\to\infty.$$

Consequently

$$|t(x_{n+1})-t(x_n)| \to 0, \quad \text{as } n\to\infty,$$

and this completes the proof.

5. **Proof of Theorem 3.** Again the argument is based on Lemma B′. Thus, it it will be shown, for any sequence $\{x_m\}$ with $x_m\to\infty$ and $\sup_m |x_{m+1}-x_m|\le 1$, that $t(x_{m+1})-t(x_m)\to 0$ as $m\to\infty$, where now

$$t(x) = \int_0^x e^{-u} \sum_{m=0}^\infty \frac{s_m-s_{m-1}}{m!} u^m \, du, \quad s_{-1} = 0,$$

with $\{s_m\}$ a bounded sequence with, say, $|s_m|\le K$, $m=0, 1, \ldots$.

The transform may be rewritten as

$$t(x) = \int_0^x e^{-u} \left[\sum_{m=0}^\infty s_m \frac{u^m}{m!}\left(1-\frac{u}{m+1}\right) - s_0\right] du,$$

so that

$$|t(x_{n+1})-t(x_n)| \le K\left|\int_{x_n}^{x_{n+1}} e^{-u}\left[\sum_{m=0}^\infty \frac{u^m}{m!}\left|1-\frac{u}{m+1}\right|+1\right] du\right|$$
$$\equiv K|\tau(x_{n+1})-\tau(x_n)|,$$

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where

$$\tau(x) = \int_0^x e^{-u}\left[\sum_{m=0}^{\infty} \frac{u^m}{m!}\left|1-\frac{u}{m+1}\right|+1\right] du.$$

To $\tau(x)$, a real function, may be applied the mean-value theorem of differential calculus which asserts the existence of y_n between x_n and x_{n+1} such that

$$\tau(x_{n+1})-\tau(x_n) = (x_{n+1}-x_n)\tau'(y_n).$$

Hence (taking n large enough so that all x_n, x_{n+1}, and so also y_n, exceed 1),

$$\begin{aligned}|\tau(x_{n+1})-\tau(x_n)| &\leq |\tau'(y_n)| \\ &= e^{-y_n}\left[\sum_{m=0}^{\infty} \frac{y_n{}^m}{m!}\left|1-\frac{y_n}{m+1}\right|+1\right] \\ &= e^{-y_n}\left[\sum_{m=0}^{[y_n]-1} \frac{y_n{}^m}{m!}\left(\frac{y_n}{m+1}-1\right)+\sum_{m=[y_n]}^{\infty} \frac{y_n{}^m}{m!}\left(1-\frac{y_n}{m+1}\right)+1\right] \\ &= 2e^{-y_n}\frac{y_n{}^{[y_n]}}{[y_n]!} = O(y_n{}^{-1/2}) = o(1),\end{aligned}$$

as $n \to \infty$, from Stirling's formula, since $y_n \to \infty$ as $n \to \infty$.

This completes the proof.

6. **Remarks.** (a) Wells's theorem concerning Hausdorff means [9] can be extended, at least partially, beyond the regular methods to which he restricted it. In the implication (i) ⇒ (ii) in his theorem (corresponding to the same implication in Theorem 1 here) the assumption of regularity can be deleted entirely. In the reverse implication, the regularity hypothesis can be weakened to the requirement that for the Hausdorff generating function $\varphi(t)$, $\varphi(0)=\varphi(0+)$.

This follows from making a small alteration in his proof: The integral on line 2 of [9, p. 85] is equal to that taken between the limits $0+$ and 1, since the integrand vanishes at zero. Then, on line 8, the first integral can be taken with limits $0+$ and δ.

(b) The result for $[J, f(x)]$ methods contrasts with the situation obtaining in the Hausdorff and quasi-Hausdorff cases. In the latter instances, connectedness is essentially equivalent to the condition $\alpha(1)=\alpha(1-)$, which eliminates convergence and methods equivalent to it. No such condition is needed for the $[J, f(x)]$ methods, since no such method is equivalent to convergence [6, Corollary 2].

Alternatively, Theorem 2 itself shows that convergence cannot be among the $[J, f(x)]$ methods; this, however, gives somewhat less than [6] since it does not necessarily exclude methods equivalent to convergence. No theorem has been established which asserts that the limit-points of equivalent transforms of bounded sequences have, or do not have, the same connectedness properties.

(c) Another open question would be to decide if the summability method T_1 is stronger than method T_2, and if the T_2-transforms of all bounded sequences are such that the sets of limit-points are connected, then do the T_1-transforms have the same property? ([2])

(d) Methods stronger than convergence need not possess the connectedness property. The transform $t_n = s_{2n}$ is (strictly) stronger than convergence, but the limit points of the transform of the bounded sequence $\{0, 1, 2, 3, 0, 1, 2, 3, \ldots\}$ do not form a connected set. P. Erdös remarked that the Voronoi–Nörlund method $t_n = \frac{1}{2}(s_n + s_{n+1})$ is another such example.

(e) The Barone result, and the analogous Lemma B′, constitute sufficient conditions only. Their strict converses are false. However, at least in Barone's case, a partial converse has been established by Schaeffer [8] (cf. also M. D. Ašić and D. D. Adamović [2]).

(f) The Referee has suggested the following further analogue of Lemma B: *If* $T = t(x)$ *is a bounded function such that for* $0 < \lambda < 1$ *we have* $\lim \{t(x+\lambda) - t(x)\} = 0$, *as* $x \to \infty$, *uniformly in* λ, *then* $L\{T\}$ *is connected.* The proofs of Theorems 2 and 3 could be modified so as to utilize this lemma instead of Lemma B′.

(g) The proofs of Theorems 1 and 2 have been presented for the case in which the respective generating functions $\alpha(v)$ and $\beta(u)$ are real. Only obvious modifications are required to cover also the case in which they are complex functions of a real variable.

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([2]) The answer to (c) is "not necessarily". This has been exemplified by Jean Tzimbalario by a method T_1 stronger than $T_2 = (C,1)$. This also strengthens remark (d), replacing in it convergence by $(C,1)$. (Added in proof, February 23, 1971.)