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A PLESSNER DECOMPOSITION ALONG TRANSVERSE CURVES

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A classical theorem of Plessner [**6**] asserts that any holomorphic function f on the unit disk partitions the unit circle, modulo a null set, into two disjoint pieces such that at each point of the first piece, f has a non-tangential limit, and at each point of the second piece, the cluster set of f in any Stolz angle is the entire plane. Higher dimensional versions of this result were first obtained by Calderon [**2**], who considered holomorphic functions on Cartesian products of half-planes. In this setting, an exact analogue of the one-dimensional result is obtained, in which the circle is replaced by the distinguished boundary, and the Stolz angles are replaced by products of cones in the coordinate half-planes. The ideas of Calderon were further developed by Rudin [**8**, pp. 79-83], who considered holomorphic and invariant harmonic functions in the ball of $\mathbf{C}^n$. In this case, the circle is replaced by the unit sphere, and the Stolz angles are replaced by the approach regions of Korányi [**4**].

More recently, Ahern and Nagel [**1**] have obtained a Plessner decomposition for H^p functions, with p sufficiently large, on smoothly bounded domains in $\mathbf{C}^n$, with respect to the arc-length measure along a transverse curve. In this case, the relevant approach regions allow tangential approach in the complex tangential directions, but their precise shape depends on p. Their result may be viewed as a natural extension of the theorem of Nagel and Rudin [**5**] concerning boundary behavior of bounded holomorphic functions along transverse curves, since as p approaches ∞, the approach regions fill out a Korányi-Stein approach region.

In this paper, we obtain Plessner decompositions along transverse curves for arbitrary holomorphic functions. A decomposition is obtained for each of a scale of approach regions which includes the approach regions used in [**1**], as well as the Korányi-Stein approach regions. However, in contrast to the results in [**1**] for H^p functions, we must restrict the approach to the boundary in the evaluation of limits in a way which is analogous to the restricted approach used by Nagel and Rudin [**5**] in their analysis of the behavior of bounded holomorphic functions along transverse curves. In fact, the theorem of Nagel and Rudin [**5**] is a special case of our main result (Theorem 1.1).

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The paper is organized as follows. In Section 1, we collect the necessary definitions and formulate our main theorem. The proof of the main result is given in Section 2. In Section 3, we show by example that the restricted approach alluded to in the preceding paragraph cannot be dispensed with.

1. The main theorem. For clarity of exposition, we shall formulate and prove our main result for the unit ball in $\mathbf{C}^n$, although it will be evident from the proof that the result remains valid, with obvious modifications of the relevant definitions, for domains in $\mathbf{C}^n$ with C^2 boundary.

We begin with some notation. We will denote by B and S the unit ball and unit sphere respectively in $\mathbf{C}^n$. In the plane, the unit disk and unit circle will be denoted by Δ and T respectively, and H will denote the upper half-plane. For any η in T and $\alpha > 1$, we denote by $\Gamma_\alpha(\eta)$ the *Stolz angle* defined by

$$\Gamma_\alpha(\eta) = \left\{\lambda \in \Delta: |\lambda - \eta| < \frac{\alpha}{2}(1 - |\lambda|^2)\right\}.$$

The half-plane analogue is defined similarly. For $x_0 \in \mathbf{R}$ and $\alpha > 0$, we define

$$\Gamma_\alpha^H(x_0) = \{ (x, y) \in H: |x - x_0| < \alpha y\}.$$

In the ball, the natural analogue to a Stolz angle is a *Korányi region*, defined by

$$D_\alpha(\zeta) = \left\{z \in B: |1 - \langle z, \zeta\rangle | < \frac{\alpha}{2}(1 - |z|^2)\right\}$$

for any $\zeta \in B$ and any $\alpha > 1$. We will also need approach regions which mediate between the Korányi regions and non-tangential regions. For any $\zeta \in B$, any $\alpha > 1$, and $1 \leqq \gamma \leqq 2$, we define

$$\Gamma_{\alpha;\gamma}(\zeta) = \{z \in D_\alpha(\zeta): |z - \langle z, \zeta\rangle\zeta|^\gamma < \alpha(1 - |\langle z, \zeta\rangle |^2) \},$$

where $\langle\ ,\ \rangle$ denotes the hermitian inner product of $\mathbf{C}^n$. Thus, the region $\Gamma_{\alpha;1}(\zeta)$ is essentially conical, while $\Gamma_{\alpha;2}(\zeta) = D_\alpha(\zeta)$. In general, the region $\Gamma_{\alpha;\gamma}(\zeta)$ is non-tangential in the complex normal direction, but, for $\gamma > 1$, allows tangential approach of order γ in the complex tangential directions.

For $\zeta \in S$, a *ζ-curve* is a continuous map $\psi:(0, h) \rightarrow B$ such that $\psi(t) \rightarrow \zeta$ as $t \rightarrow 0^+$. If ψ is any ζ-curve, we shall denote by ψ_ν and ψ_τ the *normal* and *tangential* components of ψ, defined by

$$\psi_\nu(t) = \langle\psi(t), \zeta\rangle\zeta \text{ and } \psi_\tau(t) = \psi(t) - \psi_\nu(t).$$

For $1 \leqq \gamma \leqq 2$, we shall say that a ζ-curve ψ is of *type* γ if for some $\alpha > 1$ we have $\psi(t) \in D_\alpha(\zeta)$ for t sufficiently small and if

$$|\psi_\tau(t)|^\gamma = o(1 - |\psi_\nu(t)|^2).$$

If f is a function on B, and if $\zeta \in S$, we will say that f has a *restricted γ-limit* at ζ if there is an $L \in \mathbf{C} \cup \{\infty\}$ such that $\lim f(\psi(t)) = L$ as $t \to 0^+$ for every ζ-curve ψ of type γ.

A C^1 curve $\varphi:\to S$ is called *transverse* if it has non-zero complex normal component at each point, i.e., if $\langle \varphi'(t), \varphi(t) \rangle \neq 0$ for all $t \in [a, b]$.

Our main result is

1.1. THEOREM. *Let $\varphi:[a, b] \to S$ be a transverse curve of class $C^{3/2+\rho}$ for some $\rho > 0$, and let f be a holomorphic function on B. Then for each $1 < \gamma \leqq 2$, the interval $[a, b]$ can be partitioned into disjoint measurable subsets E, F, and N such that*

(i) *$f(\Gamma_{\alpha;\gamma}(\varphi(x)))$ is dense in $\mathbf{C}$ for every $x \in F$ and every $\alpha > 1$;*
(ii) *f has a restricted γ-limit at $\varphi(x)$ for every $x \in E$;*
(iii) *N is a null set.*

It is natural to ask whether the restricted limit in item (ii) can be replaced by the limit of $f(z)$ as $z \to \varphi(x)$ in $\Gamma_{\alpha;\gamma}(\varphi(x))$. Indeed, Ahern and Nagel [**1**] have shown that this is possible for $f \in H^p$ with $p \geqq 2(n - 1)$ and $\gamma^{-1} = 1/2 + (n - 1)/p$. That this is not the case for arbitrary holomorphic functions follows from an example of Nagel and Rudin [**5**] in the case $\gamma = 2$. In Section 3, we will indicate how the example of [**5**] can be modified to apply to any rational value of γ between 1 and 2.

Before proceeding with the proof, we should note that we do not know whether the limits in item (ii) can be asserted to be finite. If the curve φ bounds a complex analytic curve, then it follows from the uniqueness theorem of Privalov [**7**] that the limit must be finite at almost every point of E. Also, if we impose growth conditions on f and additional smoothness on φ, then we can show that the limit must be finite for almost every $x \in E$. More precisely, if $|f(z)| \leqq (1 - |z|^2)^{-q}$ and φ is of class $C^{q+1/2+\rho}$ for some $\rho > 0$, then the limit in (ii) must be finite for almost every $x \in E$. This last assertion can be proved by realizing the curve as the boundary of an almost analytic disk. A detailed proof will be omitted. In general, we conjecture that a holomorphic function cannot have infinite non-tangential limit along a set of positive linear measure in a transverse curve.

2. Proof of the main theorem. In addition to the regions $D_\alpha(\zeta)$ and $\Gamma_{\alpha;\gamma}(\zeta)$ defined in Section 1, we must introduce another family of approach regions which will play a technical role in the proof, since we will need to control the rate of approach in the complex normal and complex tangential directions separately. For any $\zeta \in S$, any $\alpha > 1$, and any $\beta > 0$, we define

$$\Gamma_{\alpha,\beta;\gamma}(\zeta) = \{z \in D_\alpha(\zeta): |z - \langle z, \zeta\rangle\zeta|^\gamma < \beta(1 - |\langle z, \zeta\rangle|^2)\}.$$

Thus, the size of $\Gamma_{\alpha,\beta;\gamma}(\zeta)$ in the normal direction is controlled by α, while β controls the size in the tangential directions. Note in particular that $\Gamma_{\alpha,\alpha;\gamma}(\zeta) = \Gamma_{\alpha;\gamma}(\zeta)$.

We will also need a notation for small neighborhoods of the boundary. For any $\epsilon > 0$ we define

$$S(\epsilon) = \{z \in B:1 - |z|^2 < \epsilon\}.$$

We begin by collecting a few lemmas summarizing some essentially well-known results in a form which will be convenient for our purposes.

2.1. LEMMA. *Let E be a measurable subset of T with positive arc-length measure, and let $\eta_0 \in T$ be a point of density for E. Then for any $\alpha, \alpha_0 > 1$ there is an $\epsilon > 0$ such that*

$$\Gamma_\alpha(\eta_0) \cap \{\lambda \in \Delta:1 - |\lambda|^2 < \epsilon\} \subset \bigcup_{\eta \in E} \Gamma_{\alpha_0}(\eta).$$

A proof of this result may be found in [**9**, pp. 201-202].

The next result is essentially the local Fatou theorem of Privalov [**7**].

2.2. LEMMA. *Let E be a measurable subset of* **R** *and let $h, \alpha_0 > 0$. Let u be a bounded harmonic function on the open set*

$$D = \bigcup_{x \in E} \Gamma^H_{\alpha_0}(x) \cap \{x + iy \in H: y < h\}.$$

Then for almost every $x_0 \in E$, the limit of $u(x + iy)$ exists as $x + iy$ approaches x_0 in $\Gamma^H_\alpha(x_0)$ for every $\alpha > 0$.

This result is usually formulated for functions which are harmonic throughout H and bounded on D (see e.g., [**9**, p. 201]), but the proof remains valid for functions which are defined only on D.

We will also need the lemma of Nagel and Rudin [**5**] on attaching almost analytic disks to transverse curves.

2.3. LEMMA. *Let $\varphi:[a, b] \to S$ be a transverse curve of class $C^{1+\epsilon}$ for some $\epsilon > 0$. There is an $h > 0$, and, letting $Q = [a, b] \times [0, h]$, there is a $C^{1+\epsilon}$ map $\Phi:Q \to B$ such that Φ is of class C^∞ on $[a, b] \times (0, h]$, and there is a positive constant M such that for all $x, y \in Q$,*

(i) $\Phi(x, 0) = \varphi(x)$;

(ii) $1 - |\Phi(x, y)|^2 \geqq My$;

(iii) $|\bar{\partial}\Phi(x, y)| \leqq My^\epsilon$;

(iv) *For any fixed $x \in [a, b]$, the curve $y \mapsto \Phi(x, y)$ approaches $\varphi(x)$ non-tangentially as $y \to 0^+$.*

Our next result is an analogue of Lemma 2.1 for certain singular measures on the sphere.

2.4. LEMMA. *Let $\varphi:[a, b] \to S$ be a C^1 transverse curve, and let E be a measurable subset of $[a, b]$. Let x_0 be a point of density for E, and let*

$\zeta_0 = \varphi(x_0)$. *For any* α, $\alpha_0 > 1$, *any* $1 < \gamma \leqq 2$, *and any* $\beta > 0$ *satisfying* $\beta < \alpha_0$ *if* $\gamma \neq 2$, *or* $\beta < 1$ *if* $\gamma = 2$, *there is an* $\epsilon > 0$ *such that*

$$\Gamma_{\alpha,\beta;\gamma}(\zeta_0) \cap S(\epsilon) \subset \bigcup_{x \in E} \Gamma_{\alpha_0;\gamma}(\varphi(x)).$$

Proof. We assume without loss of generality that $x_0 = 0$ and $\zeta_0 = \varphi(0) = (1, 0, \dots, 0)$. For $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, we will use the notation $z' = \langle z, \zeta_0 \rangle \zeta_0 = (z_1, 0, \dots, 0)$ and $z'' = z - z' = (0, z_2, \dots, z_n)$. For $x \in \mathbf{R}$ near 0, we define $\theta = \theta(x) \in (-\pi, \pi]$ by

$$e^{i\theta(x)} = \frac{\langle \varphi(x), \zeta_0 \rangle}{|\langle \varphi(x), \zeta_0 \rangle|}.$$

Clearly θ is a C^1 function of x for x near 0, and moreover,

$$\theta'(0) = -i\langle \varphi'(0), \zeta_0 \rangle.$$

Since φ is transverse, the right hand side is non-zero, so by the Inverse Function Theorem, the map $x \mapsto e^{i\theta(x)}$ is a C^1 diffeomorphism of a neighborhood of 0 in $\mathbf{R}$ to a neighborhood of 1 in the unit circle. Letting $\tilde{E}$ denote the image of E under this map, it follows that $\tilde{E}$ is a set of positive measure in the unit circle, and that 1 is a point of density for $\tilde{E}$. In addition, since the map is a diffeomorphism, there is a constant C_1 such that for x near 0,

$$(2.5) \qquad |\varphi(x) - \zeta_0| \leqq C_1 |e^{i\theta(x)} - 1|.$$

Let $\alpha_1 = (1 - \beta)\alpha_0/2$ if $\gamma = 2$, or $\alpha_1 = \alpha_0/4$ if $\gamma < 2$. By Lemma 2.1, there is an $\epsilon_1 > 0$ such that

$$(2.6) \qquad \Gamma_\alpha(1) \cap \{\lambda \in \Delta : 1 - |\lambda|^2 < \epsilon_1\} \subset \bigcup_{\eta \in \tilde{E}} \Gamma_{\alpha_1}(\eta).$$

Let $z \in \Gamma_{\alpha,\beta;\gamma}(\zeta_0) \cap S(\epsilon)$, where $\epsilon > 0$ will be chosen below, depending on the curve φ, the set E, and on the parameters α, α_0, γ, and β. Then

$$\begin{aligned} 1 - |z_1|^2 &= 1 - |z|^2 + |z''|^2 \\ &< 1 - |z|^2 + (\beta(1 - |z_1|^2))^{2/\gamma}. \end{aligned}$$

In the case $\gamma = 2$, we have assumed that $\beta < 1$, so we obtain

$$(2.7a) \quad 1 - |z_1|^2 < \frac{1}{1 - \beta}(1 - |z|^2) \quad \text{if } \gamma = 2.$$

On the other hand, since $(1 - |z_1|^2)^2 \leqq \alpha(1 - |z|^2)$ for $z \in D_\alpha(\zeta_0)$, we have

$$1 - |z_1|^2 = 1 - |z|^2 + \beta^{2/\gamma}(\alpha\epsilon)^{1/\gamma - 1/2}(1 - |z_1|^2),$$

so if $\gamma < 2$, then for any $\rho > 1$ we can choose ϵ sufficiently small, depending on α, γ, and ρ, that

(2.7b) $1 - |z_1|^2 < \rho(1 - |z|^2)$ if $\gamma < 2$,

and in particular we may choose $\rho < \min\{\alpha_0/\beta, 2\}$. In either case, it follows for suitably small ϵ, depending on α, γ, and ϵ_1, that $1 - |z_1|^2 < \epsilon_1$. Moreover, the fact that $z \in D_\alpha(\zeta_0)$ implies trivially that $z_1 \in \Gamma_\alpha(1)$, so it follows from (2.6) that there is an $\eta \in \widetilde{E}$ such that $z_1 \in \Gamma_{\alpha_1}(\eta)$. Note also by (2.7a, b) that z_1, and hence η, may be made as near as we please to 1 by choosing ϵ sufficiently small. Thus, if ϵ is sufficiently small, there is a unique $x \in (a, b)$ near 0 such that $\eta = e^{i\theta(x)}$. Letting $\zeta = \varphi(x)$, we claim that

(2.8) $z \in \Gamma_{\alpha_0;\gamma}(\zeta)$.

To prove (2.8), we first show that $z \in D_\alpha(\zeta)$. We have

$$
\begin{aligned}
(2.9)\quad |1 - \langle z, \zeta\rangle| &\leqq |1 - z_1\bar{\zeta}_1| + |\langle z'', \zeta''\rangle| \\
&\leqq |1 - z_1\bar{\eta}| + |\zeta_1 - \eta| + |z''|\,|\zeta''| \\
&= |z_1 - \eta| + (1 - |\zeta_1|) + |z''|\,|\zeta''| \\
&\leqq |z_1 - \eta| + (1 - |\zeta_1|^2) + |z''|\,|\zeta''| \\
&= |z_1 - \eta| + |\zeta''|^2 + |z''|\,|\zeta''|.
\end{aligned}
$$

By (2.5), we have

$$|\zeta''| \leqq |\zeta - \zeta_0| \leqq C_1|\eta - 1|.$$

But since $z_1 \in \Gamma_\alpha(1) \cap \Gamma_{\alpha_1}(\eta)$, it follows from the triangle inequality that

$$(2.10)\quad |\eta - 1| < \frac{\alpha + \alpha_1}{2}(1 - |z_1|^2)$$

and so we have

$$|\zeta''| \leqq \frac{\alpha + \alpha_1}{2} C_1(1 - |z_1|^2).$$

Thus, in view of the fact that $z \in \Gamma_{\alpha,\beta;\gamma}(\zeta_0)$ and $z_1 \in \Gamma_{\alpha_1}(\eta)$, it follows from (2.9) that

$$
\begin{aligned}
|1 - \langle z, \zeta\rangle| &< \frac{\alpha_1}{2}(1 - |z_1|^2) \\
&\quad + C_2(1 - |z_1|^2)^2 + C_3(1 - |z_1|^2)^{1+1/\gamma},
\end{aligned}
$$

where C_2 and C_3 are constants depending on the curve φ and on the parameters α, α_0, β, and γ. By appealing to (2.7a) or (2.7b), depending on whether $\gamma = 2$ or $\gamma < 2$, we obtain

$$|1 - \langle z, \zeta\rangle| < C_4\left(\frac{\alpha_1}{2} + C_2\epsilon^2 + C_3\epsilon^{1/\gamma}\right)(1 - |z|^2),$$

where $C_4 = 1/(1 - \beta)$ if $\gamma = 2$, and $C_4 = 2$ if $\gamma < 2$. Thus, if ϵ is suitably small, it follows from our choice of α_1 that $|1 - \langle z, \zeta \rangle| < (\alpha_0/2)(1 - |z|^2)$, so $z \in D_{\alpha_0}(\zeta)$.

In the case $\gamma = 2$, the proof of (2.8) is complete, since $\Gamma_{\alpha_0;2}(\zeta) = D_{\alpha_0}(\zeta)$. To complete the proof of (2.8) in the case $1 < \gamma < 2$, we must estimate $|z - \langle z, \zeta \rangle \zeta|$. By (2.5), (2.10), and the definition of $\Gamma_{\alpha,\beta;\gamma}(\zeta_0)$,

$$\begin{aligned} |z - \langle z, \zeta \rangle \zeta| &\leqq |z - \langle z, \zeta_0 \rangle \zeta_0| + 2\,|\zeta - \zeta_0| \\ &< (\beta(1 - |z_1|^2))^{1/\gamma} + 2C_1|\eta - 1| \\ &\leqq (\beta(1 - |z_1|^2))^{1/\gamma} + C_1(\alpha + \alpha_1)(1 - |z_1|^2). \end{aligned}$$

Thus, by (2.7b), we obtain

$$|z - \langle z, \zeta \rangle \zeta| < ((\rho\beta)^{1/\gamma} + \rho C_1(\alpha + \alpha_1)\epsilon^{1-1/\gamma})(1 - |z|^2)^{1/\gamma}.$$

Since ρ was chosen so that $\rho\beta < \alpha_0$, it follows that, for suitably small ϵ,

$$\begin{aligned} |z - \langle z, \zeta \rangle \zeta| &< (\alpha_0(1 - |z|^2))^{1/\gamma} \\ &\leqq (\alpha_0(1 - |\langle z, \zeta \rangle|^2))^{1/\gamma}, \end{aligned}$$

and so (2.8), and hence Lemma 2.4, is proved.

2.11. LEMMA. *Let $\varphi:[a, b] \to S$ be a transverse curve of class $C^{1+\rho}$ for some $\rho > 0$, and let $\Phi:Q \to \bar{B}$ be as in Lemma 2.3. There are an $h_1 > 0$ and an $\alpha > 1$ such that, letting $Q_1 = [a, b] \times [0, h_1]$, we have*

$$\Phi[\Gamma_1^H(x) \cap Q_1] \subset \Gamma_{\alpha;1}(\varphi(x))$$

for all $x \in [a, b]$.

Proof. Writing $\lambda = x + iy$, we have that $(\partial\Phi/\partial\bar{\lambda})(x + 0i) = 0$ for all $x \in [a, b]$, so it follows that

$$\Phi(x + iy) = \varphi(x_0) + \varphi'(x_0)(x - x_0 + iy) + o(|(x - x_0) + iy|)$$

uniformly for $x_0 \in [a, b]$. From this it follows immediately that

$$|1 - \langle \Phi(x + iy), \varphi(x_0) \rangle| \leqq C|x - x_0 + iy|$$

and

$$|\Phi(x + iy) - \langle \Phi(x + iy), \varphi(x_0) \rangle \varphi(x_0)| \leqq C|x - x_0 + iy|,$$

where C is a constant depending on φ. Thus, if $x + iy$ is constrained to lie in $\Gamma_1^H(x_0)$, we obtain

$$|1 - \langle \Phi(x + iy), \varphi(x_0) \rangle| \leqq 2Cy$$

and

$$|\Phi(x + iy) - \langle \Phi(x + iy), \varphi(x_0) \rangle \varphi(x_0)| \leqq 2Cy,$$

and the lemma follows from item (ii) of Lemma 2.3.

We will need an elementary gradient estimate.

2.12. LEMMA. *Let* $1 < \alpha_1 < \alpha_2$, *and let* $0 < \epsilon_1 < \epsilon_2$. *There is a constant* $C = C(\alpha_1, \alpha_2, \epsilon_1, \epsilon_2)$ *such that if* f *is a holomorphic function on* $\Gamma_{\alpha_2;1}(\zeta) \cap S(\epsilon_2)$ *satisfying* $\sup|f| \leqq M$, *then*

$$|\nabla f(z)| \leqq CM(1 - |z|^2)^{-1} \quad on \quad \Gamma_{\alpha_1;1}(\zeta) \cap S(\epsilon_1).$$

Proof. It follows easily from the definitions that there is a constant C', depending on α_1, α_2, ϵ_1, and ϵ_2, such that for each $z \in \Gamma_{\alpha_1;1}(\zeta) \cap S(\epsilon_1)$, the ball about z of radius $C'(1 - |z|^2)$ is contained in $\Gamma_{\alpha_1;1}(\zeta) \cap S(\epsilon_2)$. Thus, the required inequality follows from Cauchy Estimates for f.

We are now prepared to begin the proof of Theorem 1.1. Let $\{\lambda_j\}$ be a sequence containing all complex numbers with rational coordinates, and let Δ_j be the disk about λ_j of radius $1/j$. For each j, let E_j be the set consisting of all $x \in [a, b]$ such that for some $\alpha > 1$ and some $\epsilon > 0$, the image of f on $\Gamma_{\alpha;\gamma}(\varphi(x)) \cap S(\epsilon)$ does not meet Δ_j. Since f is continuous on B, each E_j is Borel measurable, and the set $F = [a, b] \setminus \cup E_j$ clearly satisfies condition (i) of Theorem 1.1. Thus, to complete the proof, it suffices to show that for every j, the function f has a restricted γ-limit at $\varphi(x)$ for almost every $x \in E_j$. We therefore fix j, and let $E' = E_j$, $\lambda' = \lambda_j$, and $\Delta' = \Delta_j$.

We can further decompose E' into a countable union of sets E'_k, where E'_k consists of all $x \in E'$ such that f omits Δ' on $\Gamma_{1/k;\gamma}(\varphi(x)) \cap S(1/k)$. Then, once again, it follows that each E'_k is measurable, and it suffices to show that, for each k, the required limit exists at $\varphi(x)$ for almost every $x \in E'_k$. Thus, Theorem 1.1 follows from

2.13. PROPOSITION. *Let* $\varphi:[a, b] \rightarrow S$ *be a transverse curve of class* $C^{3/2+\rho}$ *for some* $\rho > 0$, *let* $\epsilon_0 > 0$, $\alpha_0 > 1$, $1 < \gamma \leqq 2$, *and let* E *be a measurable subset of* $[a, b]$. *Let* f *be a holomorphic function on the open set*

$$D = \bigcup_{x \in E} \Gamma_{\alpha_0;\gamma}(\varphi(x)) \cap S(\epsilon_0)$$

such that $f(D) \cap \Delta_0 = \emptyset$ *for some non-empty open set* Δ_0 *in* $\mathbf{C}$. *Then* f *has a restricted* γ*-limit at* $\varphi(x)$ *for almost every* $x \in E$.

Proof. First, note that, by replacing f by $1/(f - \lambda_0)$, with $\lambda_0 \in \Delta_0$, we may assume that f is bounded on D.

For each positive integer j and $\alpha > 1$, let E_j^α consist of those points $x \in E$ such that

$$\Gamma_{\alpha,1/2;\gamma}(\varphi(x)) \cap S(1/j) \subset D.$$

It follows from Lemma 2.4 that $E \setminus \cup_j E_j^\alpha$ is a null set for every $\alpha > 1$, since almost every point of E is a point of density. Thus, it is enough to show that there is an $\alpha > 1$ such that for every positive integer j, the function f has a restricted γ-limit for almost every $x \in E_j^\alpha$. Let α be so large that Lemma 2.11 applies. Since $\gamma > 1$, it is clear that there is an $\epsilon > 0$ such that

$$\Gamma_{2\alpha;1}(\zeta) \cap S(\epsilon) \subset \Gamma_{2\alpha,1/2;\gamma}(\zeta) \quad \text{for all } \zeta \in S,$$

so, by Lemma 2.12, there is, for each j, an $\epsilon > 0$ such that

$$|\nabla f(z)| \leqq \text{const.}(1 - |z|^2)^{-1}$$

for

$$z \in \cup\{\Gamma_{\alpha;1}(\varphi(x)): x \in E_j^{2\alpha}\} \cap S(\epsilon).$$

Let Φ denote the mapping of Lemma 2.3. By Lemmas 2.3 and 2.11, it follows that, for sufficiently small $h > 0$,

$$(2.14) \quad \begin{aligned} |\bar{\partial}(f \circ \Phi)(x + iy)| &\leqq |\nabla f(\Phi(x + iy))| \, |\bar{\partial}\Phi(x + iy)| \\ &\leqq \text{const.}(1 - |\Phi(x + iy)|^2)^{-1} y^{1/2+\rho} \\ &\leqq \text{const. } y^{-1/2+\rho} \end{aligned}$$

for

$$x + iy \in D' = (a, b) \times (0, h) \cap \cup\{\Gamma_1^H(x): x \in E_j^{2\alpha}\}.$$

(It is here that the hypothesis that φ is of class $C^{3/2+\rho}$ is used.) Let g be the function on $\mathbf{C}$ defined by

$$g(\lambda) = (\partial/\partial\bar{\lambda})(f \circ \Phi(\lambda)) \quad \text{if } \lambda \in D',$$

and $g(\lambda) = 0$ otherwise. It follows from (2.14) that $g \in L^p(\mathbf{C})$ for $p < 2/(1 - 2\rho)$, and so the function

$$u(x + iy) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{g(\lambda)}{\lambda - (x + iy)} d\lambda \wedge d\bar{\lambda}$$

is Lipschitz of order δ for $0 < \delta < 2\rho$. Moreover, the function $f \circ \Phi - u$ is holomorphic on D', and since u is continuous on the entire plane, the boundary behavior of $f \circ \Phi$ along $E_j^{2\alpha}$ is determined by that of $f \circ \Phi - u$. But by applying Lemma 2.2 to the intersection of $E_j^{2\alpha}$ with relatively compact subintervals of (a, b), we conclude that $f \circ \Phi - u$ has a non-tangential limit at almost every point of $E_j^{2\alpha}$, and hence the same is true of $f \circ \Phi$. In particular, it follows that

$$\lim_{y \to 0^+} f(\Phi(x, y))$$

exists for almost every $x \in E_j^{2\alpha}$. But since $E \backslash \cup_j E_j^{2\alpha}$ is a null set, it follows that the above limit exists for almost every $x \in E$.

The proof can now be completed using an argument of Čirka [**3**]. Let $x \in E$ be a point of density such that $\lim_{y \to 0^+} f(\Phi(x + iy))$ exists.

Let $\zeta = \varphi(x)$, and let $\psi:(0, h) \to B$ be any ζ-curve of type γ. Then there is a non-negative function σ such that $\sigma(t) \to 0$ as $t \to 0^+$ and such that

$$|\psi_\tau(t)| \leqq \sigma(t)(1 - |\psi_\nu|^2)^{1/\gamma}.$$

Let $\alpha > 2$ be sufficiently large that $\psi(t) \in D_{\alpha/2}(\zeta)$ when t is near 0. By Lemma 2.4, there is an $\epsilon > 0$ such that

$$\Gamma_{\alpha,1/2;\gamma}(\zeta) \cap S(\epsilon) \subset D.$$

We also choose ϵ sufficiently small that $\sigma(t) < 1/2$ whenever $\psi_\nu(t) \in S(\epsilon)$. If u is any unit vector orthogonal to ζ, then for any $\lambda \in \Gamma_{\alpha/2}(1)$, it follows that

$$\lambda\zeta + wu \in \Gamma_{\alpha,1/2;\gamma}(\zeta)$$

for every complex number w with

$$|w|^\gamma < (1/2)(1 - |\lambda|^2).$$

Applying the Schwarz Lemma to the function

$$w \mapsto f(\lambda\zeta - wu) - f(\lambda\zeta)$$

yields

$$|f(\lambda\zeta - wu) - f(\lambda\zeta)| \leqq 2^{1+1/\gamma}M(1 - |\lambda|^2)^{-1/\gamma}|w|$$

where M is an upper bound for $|f|$ on $\Gamma_{\alpha,1/2;\gamma}(\zeta)$. In particular, taking $\lambda = \langle\psi(t), \zeta\rangle$ and $wu = \psi_\tau(t)$, we obtain

$$(2.15) \quad |f(\psi(t)) - f(\psi_\nu(t))| \leqq 2^{1+1/\gamma}M\sigma(t)$$

and so $\lim_{t\to0^+} f(\psi(t))$ exists if and only if $\lim_{t\to0^+} f(\psi_\nu(t))$ exists, in which case the two limits have the same value. In particular, applying (2.15) to the curve $\psi(t) = \Phi(x + it)$ gives that

$$\lim_{y\to0^+} f(\langle\Phi(x + iy), \zeta\rangle\zeta)$$

exists. Let $\tilde{D} = \{\lambda \in \Delta: \lambda\zeta \in D\}$ and let g be the bounded holomorphic function on $\tilde{D}$ defined by $g(\lambda) = f(\lambda\zeta)$. Then

$$\Gamma_\alpha(1) \cap \{1 - |\lambda|^2 < \epsilon\} \subset \tilde{D},$$

and the curve $\tilde{\psi}(t) = \langle\Phi(x, t), \zeta\rangle$ approaches 1 in $\Gamma_{\alpha/2}(1) \cap \tilde{D}$. It thus follows from a theorem of Lindelöf that g has a limit at 1 in $\Gamma_{\alpha'}(1) \cap \tilde{D}$ for every $1 < \alpha' < \alpha$. But since we are free to choose α as large as we please, it follows that g has a non-tangential limit L at 1. Thus, if $\psi(t)$ is any ζ-curve such that $\psi_\nu(t)$ approaches ζ non-tangentially, then

$$\lim_{t\to0^+} f(\psi_\nu(t)) = \lim_{t\to0^+} f(\langle\psi(t), \zeta\rangle\zeta) = L.$$

In particular, if ψ is of type γ, then it follows from (2.15) that

$$\lim_{t \to 0^+} f(\psi(t)) = L,$$

and the proof is complete.

3. Counterexamples. In this section, we show by example that, at least when γ is rational, Theorem 1.1 does not admit a sharper formulation in which restricted γ-limits are replaced by limits in $\Gamma_{\alpha;\gamma}(\zeta)$. In the case $\gamma = 2$, this follows from an example of Nagel and Rudin [**5**] of a bounded holomorphic function on the ball in $\mathbf{C}^2$ which does not have admissible limits at any point along the transverse curve

$$x \mapsto \varphi(x) = (e^{ix}, 0).$$

Here we show how the example of Nagel and Rudin can be modified to produce, for any rational number γ with $1 \leqq \gamma \leqq 2$, a holomorphic function $f(z, w)$ on the ball B in $\mathbf{C}^2$ which is bounded on $\Gamma_{\alpha;\gamma}(\varphi(x))$ for every x and every $\alpha > 0$, but which does not have a limit as $(z, w) \to \varphi(x)$ in $\Gamma_{\alpha;\gamma}(\varphi(x))$ for any $x \in [0, 2\pi)$ and any $\alpha > 1$.

For each positive integer k, let $n_k = (k!)^2$, and for $z \in \Delta$, let

$$g(z) = \sum_{k=2}^{\infty} (n_k - n_{k-1}) z^{n_k}.$$

For $k \geqq 2$, we have

$$(n_k - n_{k-1})|z|^{n_k} < \sum_{j=n_{k-1}+1}^{n_k} |z|^j,$$

so

$$(3.1) \qquad |g(z)| \leqq \sum_{j=1}^{\infty} |z|^j = \frac{|z|}{1 - |z|}.$$

Let γ be any rational number in $[0, 1]$, and write $\gamma = p/q$ with p and q positive integers. We define a holomorphic function on B by $f(z, w) = w^p g(z)^q$. For each positive integer k, let $r_k = 1 - 1/n_k$, and let

$$h_k(z) = (n_k - n_{k-1}) z^{n_k}.$$

Then $r_k^{n_k}$ increases to $1/e$ as $k \to \infty$, so for $k \geqq 2$

$$\begin{aligned} |h_k(r_k e^{ix})| &\geqq (n_k - n_{k-1}) r_2^{n_2} \\ &= n_k\left(1 - \frac{1}{k^2}\right)\left(\frac{3}{4}\right)^4 \\ &\geqq \left(\frac{3}{4}\right)^5 n_k. \end{aligned}$$

Also,

$$\sum_{j=2}^{k-1} h_j(r_k e^{ix}) < \sum_{j=2}^{k-1} (n_j - n_{j-1}) = n_{k-1} - n_1 < n_{k+1}$$

and

$$\begin{aligned}\sum_{j=k+1}^{\infty} |h_j(r_k e^{ix})| &= \sum_{j=k+1}^{\infty} n_j\left(1 - \frac{1}{j^2}\right)\left(1 - \frac{1}{n_k}\right)^{n_j} \\ &< \sum_{j=k+1}^{\infty} n_j\left(1 - \frac{1}{n_{j-1}}\right)^{n_j} \\ &< \sum_{j=k+1}^{\infty} (j!)^2 e^{-j^2},\end{aligned}$$

so it follows that

$$\begin{aligned}|g(r_k e^{ix})| &\geqq \left(\frac{3}{4}\right)^5 n_k - \left(n_{k-1} + \sum_{j=k+1}^{\infty} (j!)^2 e^{-j^2}\right) \\ &= \left(\left(\frac{3}{4}\right)^5 - \frac{1}{k^2} - \frac{1}{n_k}\sum_{j=k+1}^{\infty} (j!)^2 e^{-j^2}\right) n_k.\end{aligned}$$

Since the series $\sum (j!)^2 e^{-j^2}$ converges, it follows that, for sufficiently large k,

$$|g(r_k e^{ix})| \geqq \frac{1}{2}\left(\frac{3}{4}\right)^5 n_k = \frac{1}{2}\left(\frac{3}{4}\right)^5 \frac{1}{1 - r_k}. \tag{3.2}$$

Now let $\alpha > 1$ be arbitrary, and fix $c \in (0, \alpha)$. Letting

$$z_k = r_k e^{ix} \quad \text{and} \quad w_k = (c(1 - r_k^2))^{1/\gamma},$$

one easily checks that

$$(z_k, w_k) \in \Gamma_{\alpha;\gamma}(\varphi(x)) \quad \text{for every } x \in [0, 2\pi),$$

provided that k is sufficiently large. Moreover, it follows from (3.2) that

$$|f(z_k, w_k)| \geqq \left(\frac{c}{2}\left(\frac{3}{4}\right)^5\right)^q$$

so, in view of the fact that $f(z, 0) = 0$ for all $z \in \Delta$, it follows that $f(z, w)$ does not have a limit as $(z, w) \to \varphi(x)$ in $\Gamma_{\alpha;\gamma}(\varphi(x))$. On the other hand, it follows from (3.1) that f is bounded on $\Gamma_{\alpha;\gamma}(\varphi(x))$ for every $x \in [0, 2\pi)$ and every $\alpha > 1$, so the advertised properties of f are established.

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