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RESEARCH ARTICLE

Semi-infinite orbits in affine flag varieties and homology of affine Springer fibers

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Abstract

Let G be a connected reductive group over an algebraically closed field k, and let Fl be the affine flag variety of G. For every regular semisimple element γ of $G(k((t)))$, the affine Springer fiber Fl_γ can be presented as a union of closed subvarieties $\mathrm{Fl}_\gamma^{\leq w}$, defined as the intersection of Fl_γ with an affine Schubert variety $\mathrm{Fl}^{\leq w}$.

The main result of this paper asserts that if elements $w_1, \ldots, w_n$ are sufficiently regular, then the natural map $H_i(\bigcup_{j=1}^n \mathrm{Fl}_\gamma^{\leq w_j}) \to H_i(\mathrm{Fl}_\gamma)$ is injective for every $i \in \mathbb{Z}$. It plays an important role in our work [BV], where our result is used to construct good filtrations of $H_i(\mathrm{Fl}_\gamma)$. Along the way, we also show that every affine Schubert variety can be written as an intersection of closures of semi-infinite orbits.

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Introduction

Let k be an algebraically closed field, $K := k((t))$ the field of Laurent power series over k, and $\mathcal{O} = \mathcal{O}_K = k[[t]]$ the ring of integers of K. Let G be a connected reductive group over k, and let G^{sc} be the simply-connected covering of the derived group of G. For an algebraic group H over K (resp. $\mathcal{O}$), we denote by LH (resp. $L^+(H)$) the corresponding loop (resp. arc) group.

We fix a maximal torus $T \subseteq G$ and an Iwahori subgroup scheme $I \subseteq L^+(G)$ such that $I \cap LT = L^+(T)$, and let $T_{G^{\mathrm{sc}}} \subseteq G^{\mathrm{sc}}$ and $I^{\mathrm{sc}} \subseteq L^+(G^{\mathrm{sc}})$ be the corresponding maximal torus and the Iwahori subgroups of G^{sc}, respectively. Let $W = W_G$ be the Weyl group of G, let $\Lambda = X_*(T_{G^{\mathrm{sc}}})$ be the group of cocharacters, and let $\widetilde{W} := W \ltimes \Lambda$ be the affine Weyl group of G.

Denote by $\mathrm{Fl} = L(G^{\mathrm{sc}})/I^{\mathrm{sc}}$ the affine flag variety of G^{sc}. Then we have a natural embedding $\widetilde{W} \hookrightarrow \mathrm{Fl}$. For every $w \in \widetilde{W}$, we denote by $\mathrm{Fl}^{\leq w} \subseteq \mathrm{Fl}$ the closure of the I^{sc}-orbit $I^{\mathrm{sc}}w \subseteq \mathrm{Fl}$. Then each $\mathrm{Fl}^{\leq w}$ is a closed projective subscheme of Fl, usually referred to as the affine Schubert variety, while Fl is an inductive limit of the $\mathrm{Fl}^{\leq w}$'s.

For a regular semi-simple element $\gamma \in G(K)$, we denote by $\mathrm{Fl}_\gamma \subseteq \mathrm{Fl}$ the corresponding affine Springer fiber (i.e., the closed ind-subscheme of points $gI^{\mathrm{sc}} \in \mathrm{Fl}$ such that $g^{-1}\gamma g \in I$).

Let G_γ be the centralizer of γ in G. It is a torus defined over K. Let $S_\gamma \subseteq G_\gamma$ be the maximal K-split torus. We will always assume that S_γ is contained in T_K, where T_K denote the extension of scalars of T to K.

For every ind-subscheme $Z \subseteq \mathrm{Fl}_G$, we denote by Z_γ the intersection $Z \cap \mathrm{Fl}_\gamma$. Then Fl_γ is a union of the $\mathrm{Fl}_\gamma^{\leq w}$; hence, each homology group $H_i(\mathrm{Fl}_\gamma)$ is by definition the direct limit of the $H_i(\mathrm{Fl}_\gamma^{\leq w})$'s. The main result of this paper implies that the canonical map $H_i(\mathrm{Fl}_\gamma^{\leq w}) \to H_i(\mathrm{Fl}_\gamma)$ is injective if w is sufficiently regular.

More precisely, let $\pi : \widetilde{W} \to \widetilde{W}/W = \Lambda$ be the projection. For $m \in \mathbb{N}$, we say that $w \in \widetilde{W}$ is *m-regular* if $|\langle \alpha, \pi(w)\rangle| \geq m$ for every root α of (G,T). The main goal of this paper is to prove the following result used in our companion work [BV].

Theorem 0.1. *There exists $m \in \mathbb{N}$ (depending on γ) such that for every finite set $w_1, \ldots, w_n$ of m-regular elements of $\widetilde{W}$, the natural map $H_i(\bigcup_{j=1}^n \mathrm{Fl}_\gamma^{\leq w_j}) \to H_i(\mathrm{Fl}_\gamma)$ is injective for every $i \in \mathbb{Z}$.*

If the group G and element γ are defined over $\mathbb{F}_q$, the expression

$$\left|\left(\bigcup_{j=1}^{n} \mathrm{Fl}_\gamma^{\leq w_j}\right)(\mathbb{F}_q)\right| = \mathrm{Tr}\left(\mathrm{Fr}, H_*\left(\bigcup_{j=1}^{n} \mathrm{Fl}_\gamma^{\leq w_j}\right)\right)$$

appears in computation of truncated orbital integrals.

As explained in [BV], Theorem 0.1 allows one to interpret $H_i(\bigcup_{j=1}^n \mathrm{Fl}_\gamma^{\leq w_j})$ as a term of a filtration on $H_i(\mathrm{Fl}_\gamma)$, which turns out to have favorable properties with respect to the affine Springer action: it is a good filtration compatible with a natural filtration on the group ring of the affine Weyl group. This provides a way to interpret a certain weighted orbital integral (or rather the closely related value of the averaging of a distribution) in terms of $H_*(\mathrm{Fl}_\gamma)$ equipped with an action of Frobenius and affine Springer action.

Theorem 0.1 will be deduced from a more general result. For each Borel subgroup $B \supseteq T$ of G, we denote its unipotent radical by $U_B \subseteq G$. For every $w \in \widetilde{W}$, we denote by $\mathrm{Fl}^{\leq_B w} \subseteq \mathrm{Fl}$ the closure of the $U_B(K)$-orbit $U_B(K)w \subseteq \mathrm{Fl}$, which is called *the semi-infinite orbit*. Then $\mathrm{Fl}^{\leq_B w}$ is a closed ind-subscheme of Fl.

We consider tuples $\overline{w} = \{w_B\}_B$ of elements of $\widetilde{W}$, where B runs over the set of all Borel subgroups $B \supseteq T$ of G. Most of the time will restrict ourselves to tuples, which are *admissible* (see Definition 1.3.1) and *m-regular* (see Notation 1.3.9). In particular, the last assumption implies that each w_B is m-regular.

For each tuple $\overline{w}$, we denote by $\mathrm{Fl}^{\leq \overline{w}}$ the reduced intersection $\bigcap_B \mathrm{Fl}^{\leq_B w_B}$. Each $\mathrm{Fl}^{\leq \overline{w}}$ is a projective scheme (see Corollary 2.1.7(c)).

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Theorem 0.1 follows from the following two results:

Theorem 0.2. *For every $w \in \widetilde{W}$, there exists a unique admissible tuple $\overline{w}$ such that $\mathrm{Fl}^{\leq w} = \mathrm{Fl}^{\leq \overline{w}}$. Moreover, there exists $r \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every $(m+r)$-regular $w \in \widetilde{W}$, the tuple $\overline{w}$ is m-regular.*

Theorem 0.3. *There exists $m \in \mathbb{N}$ (depending on γ) such that for every finite set $\overline{w}_1, \ldots, \overline{w}_n$ of m-regular admissible tuples, the natural map $H_i(\bigcup_{j=1}^n \mathrm{Fl}_\gamma^{\leq \overline{w}_j}) \to H_i(\mathrm{Fl}_\gamma)$ is injective for all i.*

Notice that Theorem 0.3 is vacuous if γ is elliptic. Indeed, in this case, the affine Springer fiber Fl_γ is of finite type, so there exists an integer m such that for every m-regular admissible tuple $\overline{w}$, we have an equality $\mathrm{Fl}_\gamma^{\leq \overline{w}} = \mathrm{Fl}_\gamma$.

To show the assertion in general, we use induction on the semisimple rank of G. Namely, if γ is not elliptic, then Fl_γ is equipped with an action of a nontrivial torus S, and the scheme of fixed points Fl_γ^S is naturally isomorphic to a disjoint union of affine Springer fibers corresponding to a proper Levi subgroup M of G. Thus, an analog of Theorem 0.3 for Fl_γ^S holds by induction hypothesis, and we use finiteness properties of $H_i(\mathrm{Fl}_\gamma)$ and localization theorem in equivariant cohomology to relate homology of Fl_γ with that of Fl_γ^S.

The paper is organized as follows. In Section 1, we study orderings on affine Weyl groups and introduce admissible tuples, which play a central role later. In Section 2, we study semi-infinite orbits in affine flag varieties and their intersections, establish Theorem 0.2, and show technical results needed later. In Section 3, we study geometric properties of the affine Springer fibers and establish a finiteness property of its homology.

Finally, in Section 4, we prove Theorem 0.3 using results of the previous sections. Namely, we review the localization theorem in the equivariant cohomology with compact support in subsection 4.1, give a criterion of an injectivity of the map on homology in subsection 4.2, and complete the proof in subsection 4.3.

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1. Combinatorics of affine Weyl groups

1.1. Preliminaries

1.1.1. Roots

(a) Let V be a finite dimensional vector space over $\mathbb{R}$, V^* the dual space, and let $\Phi \subseteq V^*$ be a (reduced) root system (see, for example, [Be] or [Bo, Section VI]).

(b) We denote by $\mathcal{C} = \mathcal{C}_\Phi$ the set of all Weyl chambers $C \subseteq V$ of Φ. For each $C \in \mathcal{C}$, we denote by $\Phi_C \subseteq \Phi$ the set of C-positive roots, by $\Delta_C \subseteq \Phi_C$ the set of C-simple roots, and by $\Psi_C \subseteq V^*$ the set of C-fundamental weights.

(c) We set $\widetilde{\Phi} := \Phi \times \mathbb{Z}$ and call it the set of *affine roots*. Every $\widetilde{\alpha} = (\alpha, n)$ is identified with an affine function $\widetilde{\alpha} : V \to \mathbb{R}$, given by the rule $\widetilde{\alpha}(x) = \alpha(x) + n$. In particular, we identify each root $\alpha \in \Phi$ with affine root $(\alpha, 0) \in \widetilde{\Phi}$. For a subset $\Phi' \subseteq \Phi$ (resp. a Weyl chamber $C \in \mathcal{C}$), we denote by $\widetilde{\Phi'}$ (resp. $\widetilde{\Phi}_C$), the set of all $\widetilde{\alpha} = (\alpha, n) \in \widetilde{\Phi}$ such that $\alpha \in \Phi'$ (resp. $\alpha \in \Phi_C$).

(d) Let $W = W_\Phi \subseteq \mathrm{Aut}(V)$ be the Weyl group of Φ, let $\Lambda \subseteq V$ be the subgroup generated by coroots $\{\check{\alpha}\}_{\alpha \in \Phi}$, and let $\widetilde{W} := W \ltimes \Lambda$ be the affine Weyl group of Φ. We will denote by π the natural projection $\widetilde{W} \to \widetilde{W}/W = \Lambda$.

(e) The lattice Λ acts on V by translations. Then the group $\widetilde{W}$ acts on V by affine transformations; hence, it acts on $\widetilde{\Phi}$ by the rule $w(\widetilde{\alpha})(x) = \widetilde{\alpha}(w^{-1}(x))$ for all $x \in V$. In particular, for each $\mu \in \Lambda$ and $(\alpha, n) \in \widetilde{\Phi}$, we have $\mu(\alpha, n) = (\alpha, n - \langle \alpha, \mu \rangle)$.

(f) For each $\widetilde{\alpha} \in \widetilde{\Phi}$, the affine reflection $s_{\widetilde{\alpha}}$ satisfies $s_{\widetilde{\alpha}}(x) = x - \widetilde{\alpha}(x)\check{\alpha}$ for all $x \in V$. In particular, for all $(\alpha, n) \in \widetilde{\Phi}$, we have equality $s_{\alpha,n} = (-n\check{\alpha})s_\alpha \in \widetilde{W}$.

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(g) For each $\alpha \in \Phi$, we denote by $\widetilde{W}_\alpha \subseteq \widetilde{W}$ the subgroup generated by reflections $s_{\widetilde{\alpha}}$, with $\widetilde{\alpha} = (\alpha, n), n \in \mathbb{Z}$.

1.1.2. The fundamental Weyl chamber

(a) We fix a Weyl chamber $C_0 \in \mathcal{C}$ and denote by A_0 the fundamental alcove such that $A_0 \subseteq C_0$ and such that $0 \in V$ lies in the closure of A_0.

(b) The choice of C_0 defines the set of positive roots $\Phi_{>0} = \Phi_{C_0} \subseteq \Phi$ and the set of positive affine roots $\widetilde{\Phi}_{>0} \subseteq \widetilde{\Phi}$. Explicitly, $\widetilde{\alpha} = (\alpha, n) \in \widetilde{\Phi}$ is positive if and only if either $n > 0$, or $n = 0$ and $\alpha > 0$.

(c) Then C_0 defines a set of simple reflection $S \subseteq W$, and A_0 defines a set of simple affine reflections $\widetilde{S} \subseteq \widetilde{W}$. In particular, a choice of C_0 defines length functions and Bruhat orders $\leq$ on both W and $\widetilde{W}$.

(d) Using A_0, we identify each $w \in \widetilde{W}$ with the corresponding alcove $w(A_0) \subseteq V$. In particular, we will say that $w \in \widetilde{W}$ belongs to $C \in \mathcal{C}$, or $w \in C$, if $w(A_0) \subseteq C$. Explicitly, this means that $\langle \alpha, w(A_0)\rangle = \langle w^{-1}(\alpha), A_0\rangle \geq 0$ for each $\alpha \in \Phi_C$, or, what is the same, $w^{-1}(\Phi_C) \subseteq \widetilde{\Phi}_{>0}$.

1.1.3. Fundamental weights

(a) We set $\Psi := \bigcup_{C \in \mathcal{C}} \Psi_C \subseteq V^*$. For $\psi \in \Psi$ and $C \in \mathcal{C}$, we write $C \ni \psi$, if $\psi \in \Psi_C$.

(b) Every $\psi \in \Psi$ gives rise to a fundamental coweight $\check{\psi} \in \Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq V$. Namely, $\check{\psi}$ is characterized by condition that for every $C \in \mathcal{C}$ such that $\psi \in \Psi_C$ and every $\alpha \in \Delta_C$, we have $\langle \alpha, \check{\psi}\rangle = \langle \psi, \check{\alpha}\rangle$. In particular, for every $C \in \mathcal{C}$, we have $\psi \in \Psi_C$ if and only if $\check{\psi}$ lies in the closure of C.

(c) For every $\psi \in \Psi$, we denote by $\Phi(\psi)$ (resp. Φ^ψ) the set of $\alpha \in \Phi$ such that $\langle \alpha, \check{\psi}\rangle \geq 0$ (resp. $\langle \alpha, \check{\psi}\rangle = 0$). Notice that Φ^ψ is a root system, and there is a bijection $C \mapsto C^\psi$ between Weyl chambers $C \ni \psi$ of Φ and Weyl chambers of Φ^ψ. This bijection satisfies the property that $(\Phi^\psi)_{C^\psi} = \Phi_C \cap \Phi^\psi$. We denote by $W^\psi \subseteq W$ and $\widetilde{W}^\psi \subseteq \widetilde{W}$ the Weyl group and the affine Weyl group of Φ^ψ, respectively.

(d) For every $\psi \in \Psi$, we fix a Weyl chamber C_0^ψ of Φ^ψ. As in Section 1.1.2(b), this choice defines the set of positive affine roots $\widetilde{\Phi}^\psi_{>0} \subseteq \widetilde{\Phi}^\psi$, and we denote by $\widetilde{W}_\psi \subseteq \widetilde{W}$ the set of all $w \in \widetilde{W}$ such that $w^{-1}(\widetilde{\Phi}^\psi_{>0}) \subseteq \widetilde{\Phi}_{>0}$. Then for every $w \in \widetilde{W}$, there exists a unique decomposition $w = w^\psi w_\psi$, where $w^\psi \in \widetilde{W}^\psi$ and $w_\psi \in \widetilde{W}_\psi$ (compare, for example, [BV, Lemma B.1.7(b)]). In other words, $\widetilde{W}_\psi \subseteq \widetilde{W}$ is a set of representatives of the set of left cosets $\widetilde{W}^\psi \backslash \widetilde{W}$.

1.1.4. Properties of the Bruhat order

(a) Let $w', w'' \in \widetilde{W}$ and $s \in \widetilde{S}$ be such that $w' \leq w''$. Then we have either $w's \leq w''s$ (resp. $sw' \leq sw''$) or $w's \leq w''$ and $w' \leq w''s$ (resp. $sw' \leq w''$ and $w' \leq sw''$) or both (see, for example, [BB, Proposition 2.2.7]).

(b) Let $w', w'' \in \widetilde{W}$ and $s \in \widetilde{S}$ be such that $sw' < w'$ and $sw'' < w''$. Then, by part (a), we have $w' \leq w''$ if and only if $sw' \leq sw''$.

(c) Let w, w' and w'' be elements of $\widetilde{W}$ such that $l(ww') = l(w) + l(w')$ and $ww' \leq ww''$. Then $w' \leq w''$. Indeed, if $w = s \in \widetilde{S}$, then the assertion follows from part (a). The general case follows by induction on $l(w)$. By a similar argument, if $l(ww'') = l(w) + l(w'')$ and $w' \leq w''$, then $ww' \leq ww''$.

(d) For every $\mu \in \Lambda$ and $u \in W$, we have $l(u\mu u^{-1}) = l(\mu)$. Indeed, it is enough to show the assertion in the case $u = s = s_\alpha$ for a simple root α. In this case, we have $s\mu s = \mu$, if $\langle \alpha, \mu\rangle = 0$; $s\mu > \mu > \mu s$ if $\langle \alpha, \mu\rangle > 0$; and $s\mu < \mu < \mu s$ if $\langle \alpha, \mu\rangle < 0$.

(e) Note that $w \in \widetilde{W}$ belongs to C_0 if and only if $l(sw) > l(w)$ for every $s \in S$. In other words, $\widetilde{W} \cap C_0$ is the set of the shortest representatives of cosets $W \backslash \widetilde{W}$. In particular, for every $w \in \widetilde{W} \cap C_0$ and $u \in W$, we have $l(uw) = l(u) + l(w)$, and for every $u \leq u'$ in W, we have $uw \leq u'w$.

(f) The characterization of C_0 given in part (e) implies that for every $w \in \widetilde{W} \cap C_0$ and $s \in \widetilde{S}$ with $ws < w$, we have $ws \in C_0$.

(g) For every $u \in W$ and every $\mu \in \Lambda \cap C_0$, we have $u \leq \mu$. Indeed, it is enough to show that $u \leq_R \mu$ (see [BB, Definition 3.1.1]). Hence, by [BB, Proposition 3.1.3], it is enough to show that for every affine root $\widetilde{\alpha} > 0$ such that $u(\widetilde{\alpha}) < 0$, we have $\mu(\widetilde{\alpha}) < 0$. If $\widetilde{\alpha} = (\alpha, n) > 0$ satisfies $u(\widetilde{\alpha}) = (u(\alpha), n) < 0$, then $n = 0$, and $\alpha > 0$. Hence, $\mu(\widetilde{\alpha}) = (\alpha, -\langle \alpha, \mu\rangle) < 0$ because $\mu \in C_0$ is regular; thus, $\langle \alpha, \mu\rangle > 0$.

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Lemma 1.1.5. *Assume that $w', w'' \in C \cap \widetilde{W}$ for some $C \in \mathcal{C}$ and $w' < w''$. Then*

(a) *for every $u \in W$, we have $uw' < uw''$;*

(b) *there exists a sequence $w' < w_1 < \ldots < w_n = w''$ such that $w_i \in C$ and $l(w_i) = l(w') + i$ for each i;*

(c) *for every $\mu \in \Lambda \cap C$ and $w \in \widetilde{W} \cap C$, we have $l(\mu w) = l(\mu) + l(w)$.*

Proof. (a) By induction, it is enough to show that for every element $s \in S$, we have $sw' < sw''$. By Section 1.1.4(b), it is enough to show that $w' < sw'$ if and only if $w'' < sw''$. Let $u \in W$ be such that $C = u(C_0)$. Then it follows from Section 1.1.4(e) that each condition $w' < sw'$ and $w'' < sw''$ is equivalent to $u < su$.

(b) Using part (a) and Section 1.1.4(e), we may assume that $C = C_0$. If $l(w'') - l(w') = 1$, there is nothing to prove, so we can assume that $l(w'') - l(w') > 1$. By induction, it is enough to show the existence of $w \in C_0$ such that $w' < w < w''$.

Choose $s \in \widetilde{S}$ such that $w''s < w''$. Then $w''s \in C_0$ by Section 1.1.4(f). If $w' < w''s$, then $w := w''s$ does the job. If not, then by Section 1.1.4(a) we get $w's < w'$ and $w's < w''s$. Then by Section 1.1.4(f), we have $w's \in C_0$, so by induction on $l(w'')$, there exist $w \in C_0$ such that $w's < w < w''s$.

If $ws < w$, then it follows from Section 1.1.4(a) that $w' \leq w < w''s$, contradicting our assumption. Hence, we may assume that $ws > w$, in which case by Section 1.1.4(a) we have $w' < ws < w''$; thus, it is enough to show that $ws \in C_0$.

Assume that $ws \notin C_0$. Since $w \in C_0$, this would imply that there exists a simple root α of C_0 such that $ws = s_\alpha w$. Then we have $w' < s_\alpha w$ and $w' \in C_0$ and therefore by Section 1.1.4(c) that $w' \leq w < w''s$, contradicting the assumption.

(c) Using Sections 1.1.4(d),(e), we can assume that $C = C_0$. Now the proof goes by induction on $l(w)$. Choose $s \in \widetilde{S}$ such that $ws < w$. Then $ws \in C_0$ by Section 1.1.4(f); hence, by the induction hypothesis, we have

$$l(\mu ws) = l(\mu) + l(ws) = l(\mu) + l(w) - 1.$$

Thus, it is enough to show that $\mu ws < \mu w$.

Let α be a simple affine root such that $s = s_\alpha$. Then $\widetilde{\beta} := w(\alpha) < 0$ because $ws < w$, and we want to show that $\mu(\widetilde{\beta}) = \mu w(\alpha) < 0$. Write $\widetilde{\beta}$ in the form (β, n), where $\beta \in \Phi$. Then $\mu(\widetilde{\beta}) = \widetilde{\beta} - \langle \beta, \mu \rangle$, so it remains to show that $\langle \beta, \mu \rangle \geq 0$.

Since $\widetilde{\beta} < 0$, we get $n \leq 0$; therefore, $w^{-1}(\beta) = \alpha - n > 0$. This implies that $\beta \in \Phi_{C_0}$ because $w \in C_0$; hence, $\langle \beta, \mu \rangle \geq 0$ because $\mu \in C_0$. □

1.2. Orderings on affine Weyl groups

Notation 1.2.1. (a) Let $\widetilde{\alpha} \in \widetilde{\Phi}$ and $w \in \widetilde{W}$. We say that $s_{\widetilde{\alpha}} w <_{\widetilde{\alpha}} w$ if $w^{-1}(\widetilde{\alpha}) > 0$.

(b) Let $\Phi' \subseteq \Phi$ be a subset, and $w', w'' \in \widetilde{W}$. We say that $w'' <_{\Phi'} w'$ if there exist affine roots $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n \in \widetilde{\Phi'}$ such that $s_{\widetilde{\alpha}_i} \ldots s_{\widetilde{\alpha}_1} w' <_{\widetilde{\alpha}_i} s_{\widetilde{\alpha}_{i-1}} \ldots s_{\widetilde{\alpha}_1} w'$ for all i, and $w'' = s_{\widetilde{\alpha}_n} \ldots s_{\widetilde{\alpha}_1} w'$. For $\alpha \in \Phi$, we write $w'' <_\alpha w'$ instead of $w'' <_{\{\alpha\}} w'$.

(c) Let $\Phi' \subseteq \Phi$, and $x', x'' \in V$. We say that $x'' <_{\Phi'} x'$ if the difference $x' - x''$ is a positive linear combination of elements $\check{\alpha}$ with $\alpha \in \Phi'$. For $\alpha \in \Phi$, we write $x'' <_\alpha x'$ instead of $x'' <_{\{\alpha\}} x'$.

(d) For each $C \in \mathcal{C}$, $\psi \in \Psi$ (and $\psi \in C$), we write $<_C$ (resp. $<_\psi$, resp. $<_{C^\psi}$) instead of $<_{\Phi_C}$ (resp. $<_{\Phi(\psi)}$, resp. $<_{\Phi^\psi(C^\psi)}$).

Lemma 1.2.2. (a) *For each $\widetilde{\alpha} = (\alpha, n) \in \widetilde{\Phi}$ and $w \in \widetilde{W}$, we have $s_{\widetilde{\alpha}} w <_{\widetilde{\alpha}} w$ (see Section 1.2.1(a)) if and only if $s_{\widetilde{\alpha}} w(x) <_\alpha w(x)$ (see Section 1.2.1 (c)) for all $x \in A_0$.*

(b) *For each $\alpha \in \Phi$ and $w \in \widetilde{W}$, we have $w <_\alpha \check{\alpha} w$ (see Section 1.2.1(b)).*

(c) *For each $x', x'' \in V$ and $\psi \in \Psi$, we have $x' \leq_\psi x''$ (see Section 1.2.1(c)) if and only if $\langle \psi, x' \rangle \leq \langle \psi, x'' \rangle$.*

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Proof. (a) Fix $x \in A_0$. Then $w^{-1}(\widetilde{\alpha}) > 0$ if and only if $w^{-1}(\widetilde{\alpha})(x) = \widetilde{\alpha}(w(x)) > 0$. Thus, $s_{\widetilde{\alpha}}w <_{\widetilde{\alpha}} w$ if and only if $s_{\widetilde{\alpha}}w(x) = w(x) - \widetilde{\alpha}(w(x))\check{\alpha} <_{\alpha} w(x)$.

(b) Let $r \in \mathbb{Z}$ such that the affine root $\widetilde{\alpha} = (\alpha, r)$ satisfies $0 < \widetilde{\alpha}(\check{\alpha}w(x)) < 1$. Using identity $\widetilde{\alpha}(s_{\widetilde{\alpha}}(\check{\alpha}w(x))) = -\widetilde{\alpha}(\check{\alpha}w(x))$, we get $0 < (\widetilde{\alpha}+1)(s_{\widetilde{\alpha}}(\check{\alpha}w(x))) < 1$. Thus, by the observation of part (a), we have $w = s_{\widetilde{\alpha}+1}s_{\widetilde{\alpha}}(\check{\alpha}w) <_{\widetilde{\alpha}+1} s_{\widetilde{\alpha}}(\check{\alpha}w) <_{\widetilde{\alpha}} \check{\alpha}w$; hence, $w <_{\alpha} \check{\alpha}w$.

(c) The 'only if' assertion follows from definitions. To see the 'if' assertion, we choose a Weyl chamber $C \ni \psi$, and let $\alpha_{\psi} \in \Delta_C$ be the simple root, corresponding to ψ. Then the difference $x'' - x'$ can be (uniquely) written in the form $\sum_{\alpha \in \Delta_C} c_{\alpha}\check{\alpha}$ with $c_{\alpha} \in \mathbb{R}$, and the assumption that $\langle \psi, x' \rangle \leq \langle \psi, x'' \rangle$ implies that $c_{\alpha_{\psi}} \geq 0$. Now the assertion follows from the observation that for every $\alpha \in \Delta_C \setminus \{\alpha_{\psi}\}$, we have $\alpha \in \Phi(\psi)$ and $-\alpha \in \Phi(\psi)$. □

Corollary 1.2.3. (a) *For each $w, w' \in \widetilde{W}$ and $\alpha \in \Phi$, we have $w <_{\alpha} w'$ if and only if we have $w \in \widetilde{W}_{\alpha}w'$ and $w(x) <_{\alpha} w'(x)$ for all $x \in A_0$.*

(b) *For each $\Phi' \subseteq \Phi$ and $w, w' \in \widetilde{W}$ with $w <_{\Phi'} w'$, we have $w(x) <_{\Phi'} w'(x)$ for each $x \in A_0$; hence, $\pi(w) \leq_{\Phi'} \pi(w')$ in the sense of Section 1.2.1(c).*

(c) *Let $\Phi' \subseteq \Phi$ have a property that if $\mu \in \Lambda$ is a positive linear combination of elements $\check{\alpha}$ with $\alpha \in \Phi'$, then μ is a finite sum of elements $\check{\alpha}$ with $\alpha \in \Phi'$. Then for every $\mu, \mu' \in \Lambda$, we have $\mu <_{\Phi'} \mu'$ in the sense of Section 1.2.1(b) if and only if $\mu <_{\Phi'} \mu'$ in the sense of Section 1.2.1(c).*

Proof. (a) If $w <_{\alpha} w'$, then $w \in \widetilde{W}_{\alpha}w'$ (by definition), and $w(x) <_{\alpha} w'(x)$ for all $x \in A_0$ (by Lemma 1.2.2(a)). Conversely, assume that $w = uw'$ with $u \in \widetilde{W}_{\alpha}$ such that $w(x) <_{\alpha} w'(x)$ for all $x \in A_0$. Then we have either $u = s_{\widetilde{\alpha}}$ or $u = \check{\alpha}^m$ for some $m \in \mathbb{Z}_{<0}$. In the first case, we have $w <_{\alpha} w'$ by Lemma 1.2.2(a), while in the second one, we have $w <_{\alpha} w'$ by Lemma 1.2.2(b).

(b) By definition, it is enough to assume that $w = s_{\widetilde{\alpha}}w' <_{\widetilde{\alpha}} w'$. In this case, the first assertion follows from Lemma 1.2.2(a). Next, since $0 \in V$ lies in the closure of $A_0 \subseteq V$, the second one follows from the equality $\pi(w) = w(0)$.

(c) Assume that $\mu <_{\Phi'} \mu'$ in the sense of Section 1.2.1(c). By our assumption of Φ', we may assume that $\mu = \mu' - \check{\alpha}$ for some $\alpha \in \Phi'$. In this case, it follows from Lemma 1.2.2(b) that $\mu <_{\Phi'} \mu'$ in the sense of Section 1.2.1(b). The converse assertion follows from part (b). □

Remarks 1.2.4. (a) Let $\Phi' \subseteq \Phi$, let $w \in \widetilde{W}$, and let $w_{\text{fin}} \in W$ be the image of $w \in \widetilde{W}$ under the projection $\widetilde{W} \to W$. Then it follows from definition that for every $w' \leq_{\Phi'} w''$, we have $ww' \leq_{w_{\text{fin}}(\Phi')} ww''$. In particular,

(i) for every $\mu \in \Lambda$, we have $w' \leq_{\Phi'} w''$ if and only if $\mu w' \leq_{\Phi'} \mu w''$;

(ii) for every $u \in W$, we have $w' \leq_{\Phi'} w''$ if and only if $uw' \leq_{u(\Phi')} uw''$.

(b) Note that for each $\alpha \in \Phi$, the subset $\Phi' := \{\alpha\}$ satisfies the assumption of Corollary 1.2.3(c).

(c) Arguing as in Lemma 1.2.2(c), we see that for each $\psi \in \Psi$, the subset $\Phi' := \Phi(\psi)$ satisfies the assumption of Corollary 1.2.3(c).

Proposition 1.2.5. *Let $w', w'' \in \widetilde{W}$, and let C be a Weyl chamber.*

Then $w' \leq_C w''$ if and only if for every sufficiently regular $\mu \in \Lambda \cap C$, we have $\mu w' \leq \mu w''$; that is, there exists $\mu \in \Lambda \cap C$ such that $\mu' \mu w' \leq \mu' \mu w''$ for every $\mu' \in \Lambda \cap C$.

Proof. First, we claim that for every $w', w'' \in \widetilde{W} \cap C$ and $\widetilde{\alpha} \in \widetilde{\Phi}$ such that $w' = s_{\widetilde{\alpha}}w''$, we have $w' <_C w''$ if and only if $w' < w''$.

Replacing $\widetilde{\alpha}$ by $-\widetilde{\alpha}$, if necessary, we may assume that $\widetilde{\alpha} = \alpha + n$ with $\alpha \in \Phi_C$. Then $w' <_C w''$ holds if and only if $w''^{-1}(\widetilde{\alpha}) > 0$. However, since $w', w'' \in C$, we get that $(w'')^{-1}(\alpha) > 0$ and $(w')^{-1}(\alpha) > 0$. Since $s_{\widetilde{\alpha}}(\widetilde{\alpha}) = -\widetilde{\alpha}$, we get $s_{\widetilde{\alpha}}(\alpha) = -\alpha - 2n$; therefore,

$$(w')^{-1}(\alpha) = (w'')^{-1}s_{\widetilde{\alpha}}(\alpha) = -(w'')^{-1}(\alpha) - 2n > 0.$$

This together with $(w'')^{-1}(\alpha) > 0$ implies that $n < 0$; thus, $\widetilde{\alpha} < 0$. Therefore, $w' < w''$ holds if and only if $w''^{-1}(\widetilde{\alpha}) > 0$.

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Now we are ready to show our assertion. Assume that $w' \leq_C w''$, and we are going to show that for each sufficiently regular $\mu \in \Lambda \cap C$, we have $\mu w' \leq \mu w''$. By induction, we can assume that $w' = s_{\widetilde{\alpha}} w''$ for some $\widetilde{\alpha} \in \widetilde{\Phi}$. Choose $\mu \in \Lambda \cap C$ sufficiently regular so that $\mu w', \mu w'' \in C$. Then $\mu w' \leq_C \mu w''$ (by Remark 1.2.4(a)(i)), and $\mu w' = s_{\mu(\widetilde{\alpha})} \mu w''$. Hence, by what is shown above, $\mu w' \leq \mu w''$.

Conversely, assume that for every sufficiently regular element $\mu \in \Lambda \cap C$, we have $\mu w' \leq \mu w''$, and we want to show that $w' \leq_C w''$. Replacing w' and w'' by $\mu w'$ and $\mu w''$, respectively, and using Remark 1.2.4(a)(i), we may assume that $w', w'' \in C$ and $w' \leq w''$. Then using Lemma 1.1.5(b), we may assume in addition that $w' = s_{\widetilde{\alpha}} w''$. Then, by what is shown above, $w' \leq_C w''$. □

Corollary 1.2.6. *Let $w', w'' \in \widetilde{W}$, and let C be a Weyl chamber.*

(a) *If $w' \leq_C w''$ and $w' \in C$, then $w' \leq w''$.*

(b) *If $w' \leq w''$ and $w'' \in C$, then $w' \leq_C w''$.*

(c) *If $w', w'' \in C$, then $w' \leq_C w''$ if and only if $w' \leq w''$.*

Proof. (a) By Proposition 1.2.5, there exists $\mu \in \Lambda \cap C$ such that $\mu w' \leq \mu w''$. Since $\mu, w' \in C$, the assertion follows from Lemma 1.1.5(c) and Section 1.1.4(c).

(b) Using Lemma 1.1.5(c) and Section 1.1.4(c), we conclude that $\mu w' \leq \mu w''$ for every $\mu \in \Lambda \cap C$. Therefore, we get $w' \leq_C w''$ by Proposition 1.2.5.

(c) follows from parts (a) and (b). □

Lemma 1.2.7. *Let $\psi \in \Psi$, $w', w'' \in \widetilde{W}^{\psi}$, $C \ni \psi$ and $w \in \widetilde{W}$.*

(a) *We have $w''w \leq_C w'w$ if and only if $w''w \leq_{C^{\psi}} w'w$.*

(b) *If $w \in \widetilde{W}_{\psi}$, then $w''w \leq_C w'w$ if and only if $w'' \leq_{C^{\psi}} w'$.*

Proof. (a) Since $(\Phi^{\psi})_{C^{\psi}} \subseteq \Phi_C$, the 'if' assertion is obvious. Conversely, assume that $w''w \leq_C w'w$. Then there exist affine roots

$$\widetilde{\beta}_1 = (\beta_1, n_1), \ldots, \widetilde{\beta}_r = (\beta_r, n_r) \in \widetilde{\Phi}_C$$

such that $w''w = s_{\widetilde{\beta}_n} \ldots s_{\widetilde{\beta}_1} w'w$, and $s_{\widetilde{\beta}_i} \ldots s_{\widetilde{\beta}_1} w'w <_{\widetilde{\beta}_i} s_{\widetilde{\beta}_{i-1}} \ldots s_{\widetilde{\beta}_1} w'w$ for all i. Then for every $x \in A_0$, the difference $w'w(x) - w''w(x)$ is a positive linear combination of the $\check{\beta}_i$'s (by Lemma 1.2.2(a)).

Since $w''w \in \widetilde{W}^{\psi} w'w$, we conclude that $w'w(x) - w''w(x)$ is a linear combination of coroots of Φ^{ψ}. Therefore, each β_i is a root of Φ^{ψ}; thus, $\beta_i \in (\Phi^{\psi})_{C^{\psi}}$. But this implies that $w''w \leq_{C^{\psi}} w'w$.

(b) By part (a), we have to show that $w''w \leq_{C^{\psi}} w'w$ if and only if $w'' \leq_{C^{\psi}} w'$. Thus, we can assume that $w'' = s_{\widetilde{\beta}} w'$ for some $\widetilde{\beta} \in \widetilde{\Phi}^{\psi}$. In other words, we have to show that $w'^{-1}(\widetilde{\beta}) \in \widetilde{\Phi}^{\psi}_{>0}$ if and only if $w^{-1}(w'^{-1}(\widetilde{\beta})) \in \widetilde{\Phi}_{>0}$. But this follows from the assumption that $w \in \widetilde{W}_{\psi}$. □

1.3. Admissible tuples

Definition 1.3.1. (a) We say that a tuple $\overline{\mu} = \{\mu_C\}_{C \in \mathcal{C}} \in V^{\mathcal{C}}$ is *admissible* (resp. *quasi-admissible*, resp. *strictly admissible*) if for every $C \in \mathcal{C}$ and $\alpha \in \Delta_C$, the difference $\mu_C - \mu_{s_\alpha(C)}$ belongs to $\mathbb{R}_{\geq 0}\check{\alpha}$ (resp. $\mathbb{R}\check{\alpha}$, resp. $\mathbb{R}_{>0}\check{\alpha}$).

(b) A tuple $\overline{w} = \{w_C\}_C \in \widetilde{W}^{\mathcal{C}}$ is called admissible (resp. *quasi-admissible*, resp. *strictly admissible*) if for every $C \in \mathcal{C}$ and $\alpha \in \Delta_C$, we have $w_{s_\alpha(C)} \leq_\alpha w_C$ (resp. $w_{s_\alpha(C)} \in \widetilde{W}_\alpha w_C$, resp. $w_{s_\alpha(C)} <_\alpha w_C$).

Remarks 1.3.2. (a) It follows from Corollary 1.2.3(b) that if $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is (quasi)-admissible, then the tuple $\pi(\overline{w}) \in \Lambda^{\mathcal{C}} \subseteq V^{\mathcal{C}}$ is (quasi)-admissible as well.

(b) Moreover, it follows from Corollary 1.2.3(c) and Section 1.2.4(b) that a tuple $\overline{\mu} \in \Lambda^{\mathcal{C}}$ is (quasi)-admissible as an element of $\widetilde{W}^{\mathcal{C}}$ if and only if it is such as an element of $V^{\mathcal{C}}$.

(c) The notion of an admissible tuple in $V^{\mathcal{C}}$ is not new. For example, it is called *complementary polyhedron* in [Be, Definition 2.1]. However, we do not know whether admissible tuples in $\widetilde{W}^{\mathcal{C}}$ were studied earlier.

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Notation 1.3.3. (a) For $\overline{\mu}, \overline{\mu}' \in V^{\mathcal{C}}$ (resp. $\overline{w}, \overline{w}' \in \widetilde{W}^{\mathcal{C}}$), we will say that $\overline{\mu} \leq \overline{\mu}'$ (resp. $\overline{w} \leq \overline{w}'$) if $\mu_C \leq_C \mu'_C$ (resp. $w_C \leq_C w'_C$) for all $C \in \mathcal{C}$.

(b) For $\overline{\mu} \in V^{\mathcal{C}}$, we define by $V^{\leq \overline{\mu}}$ the set of all $x \in V$ such that $x \leq_C \mu_C$ for all $C \in \mathcal{C}$.

1.3.4. Quasi-admissible tuples in $V^{\mathcal{C}}$. (a) The set of quasi-admissible tuples in $V^{\mathcal{C}}$ (resp. $\Lambda^{\mathcal{C}}$) can be naturally identified with $\mathbb{R}^{\Psi}$ (resp. $\mathbb{Z}^{\Psi}$).

Indeed, for each quasi-admissible tuple $\overline{\mu} \in V^{\mathcal{C}}$ and every $\psi \in \Psi$, the element $\overline{\mu}(\psi) := \langle \psi, \mu_C \rangle$ does not depend on $C \ni \psi$. To see this, we observe that for every pair of Weyl chambers $C, C' \ni \psi$, there exists $w \in W_{\Phi^\psi}$ such that $C' = w(C)$. Therefore, $\overline{\mu}$ defines a tuple $\{\overline{\mu}(\psi)\}_{\psi \in \Psi} \in \mathbb{R}^{\Psi}$.

Conversely, every tuple $\{\overline{\mu}(\psi)\} \in \mathbb{R}^{\Psi}$ gives rise to a quasi-admissible tuple $\overline{\mu} \in V^{\mathcal{C}}$ defined by the rule $\mu_C := \sum_{\alpha_i \in \Delta_C} \overline{\mu}(\psi_i)\check{\alpha}_i$, where $\psi_i \in \Psi_C$ is the fundamental weight corresponding to $\alpha_i \in \Delta_C$.

(b) The set of quasi-admissible tuples in $V^{\mathcal{C}}$ (resp. $\Lambda^{\mathcal{C}}$) is a group with respect to the coordinatewise addition in V (resp. Λ). Moreover, the identification of part (a) identifies this group with $\mathbb{R}^{\Psi}$ (resp. $\mathbb{Z}^{\Psi}$). Also, the set of admissible tuples in $V^{\mathcal{C}}$ (resp. $\Lambda^{\mathcal{C}}$) is a submonoid.

(c) The identification of part (a) preserves coordinatewise ordering. In other words, for every two quasi-admissible tuples $\overline{\mu}, \overline{\mu}' \in V^{\mathcal{C}}$, we have $\mu_C \leq_C \mu'_C$ for all $C \in \mathcal{C}$ if and only if $\overline{\mu}(\psi) \leq \overline{\mu}'(\psi)$ for all $\psi \in \Psi$. In particular, for every quasi-admissible tuples $\overline{\mu} \in V^{\mathcal{C}}$, the subset $V^{\leq \overline{\mu}} \subseteq V$ (see Section 1.3.3(b)) consists of all $x \in V$ such that $\langle \psi, x \rangle \leq \overline{\mu}(\psi)$ for all $\psi \in \Psi$.

(d) From now on, we will not distinguish between a quasi-admissible tuple $\{\mu_C\}_C$ in $V^{\mathcal{C}}$ (resp. $\Lambda^{\mathcal{C}}$) and the corresponding tuple $\{\overline{\mu}(\psi)\}_{\psi}$ in $\mathbb{R}^{\Psi}$ (resp. $\mathbb{Z}^{\Psi}$). In particular, for every $\psi \in \Psi$, we denote by $\overline{e}_\psi \in \Lambda^{\mathcal{C}}$ the quasi-admissible tuple, corresponding to the standard vector $\overline{e}_\psi \in \mathbb{Z}^{\Psi}$, given by the rule $\overline{e}_\psi(\psi') = \delta_{\psi,\psi'}$.

Examples 1.3.5. (a) Every $\mu \in C_0 \subseteq V$ gives rise to an admissible tuple $\overline{\mu} \in V^{\mathcal{C}}$ defined by the rule $\mu_{u(C_0)} := u(\mu)$ for all $u \in W$.

(b) Consider the tuple $\overline{w}_{\mathrm{st}} \in W^{\mathcal{C}} \subseteq \widetilde{W}^{\mathcal{C}}$, defined by the rule $(w_{\mathrm{st}})_{u(C_0)} = u$. Then $\overline{w}_{\mathrm{st}}$ is admissible. Indeed, by definition, we have to show that for every $u \in W$ and $\alpha \in \Phi_{u(C_0)}$, we have $s_\alpha u <_\alpha u$; that is, $u^{-1}(\alpha) > 0$. Since $u^{-1}(\alpha) \in \Phi_{C_0}$, we are done.

(c) Using Remark 1.2.4(a)(i) and Lemma 1.2.2(b), for every (quasi)-admissible tuples $\overline{\mu} \in \Lambda^{\mathcal{C}}$ and $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, the tuple $\overline{\mu} \cdot \overline{w} := \{\mu_C w_C\}_C \in \widetilde{W}^{\mathcal{C}}$ is (quasi)-admissible as well. In particular, for every $\mu \in \Lambda$ and (quasi)-admissible tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, the tuple $\mu\overline{w} := \{\mu w_C\}_C$ is (quasi)-admissible.

Notation 1.3.6. Arguing as in Section 1.3.4(a), for each quasi-admissible $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ and every $\psi \in \Psi$, the class $[w_C] \in \widetilde{W}^{\psi} \backslash \widetilde{W}$ and hence also element $(w_C)_\psi \in \widetilde{W}_\psi$ (see Section 1.1.3(d)) does not depend on $C \ni \psi$. We will denote this element by $\overline{w}_\psi$.

The following characterization of admissible tuples will be crucial for the rest of the paper.

Lemma 1.3.7. *A tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ (resp. $\overline{\mu} \in V^{\mathcal{C}}$) is admissible if and only if for all $C, C' \in \mathcal{C}$, we have $w_C \leq_{C'} w_{C'}$ (resp. $\mu_C \leq_{C'} \mu_{C'}$).*

Proof. We will only prove the assertion for $\overline{w}$, while the other case is similar, but easier. Assume first that $\overline{w}$ is admissible, and we want to show that for every two Weyl chambers C and C', we have $w_C \leq_{C'} w_{C'}$. Using Remark 1.2.4(a)(ii), we may assume that $C' = C_0$. Let $u \in W$ be such that $C = u(C_0)$, choose a reduced decomposition $u = s_1 \ldots s_n$ of u, and for each $j = 1, \ldots, n$, we set $u_j := s_1 \ldots s_j$ and $C_j := u_j(C_0)$. It is enough to show that $w_{C_{j+1}} \leq_{C_0} w_{C_j}$ for each $j = 0, \ldots, n-1$.

Let $\alpha_{j+1} \in \Delta_{C_0}$ be such that $s_{j+1} = s_{\alpha_{j+1}}$. By construction, we obtain that $u_{j+1} = u_j s_{j+1} > u_j$; hence, $u_j(\alpha_{j+1}) \in \Phi_{C_0}$. Also since $\alpha_{j+1} \in \Delta_{C_0}$, we get that $u_j(\alpha_{j+1}) \in \Delta_{C_j}$. Since $C_{j+1} = s_{u_j(\alpha_{j+1})}(C_j)$, the admissibility assumption implies that $w_{C_{j+1}} \leq_{u_j(\alpha_{j+1})} w_{C_j}$; thus, we have $w_{C_{j+1}} \leq_{C_0} w_{C_j}$ because $u_j(\alpha_{j+1}) \in \Phi_{C_0}$.

Conversely, assume that $w_C \leq_{C'} w_{C'}$ for all $C, C' \in \mathcal{C}$. Choose $C \in \mathcal{C}$, $\alpha \in \Delta_C$, and set $C' = s_\alpha(C)$. Since $w_{C'} \leq_C w_C$, there exist a tuple of affine roots $\widetilde{\beta}_1 = (\beta_i, n_i), \ldots, \widetilde{\beta}_r = (\beta_r, n_r) \in \widetilde{\Phi}_C$ such that $w_{C'} = s_{\widetilde{\beta}_r} \ldots s_{\widetilde{\beta}_1} w_C$, and $s_{\widetilde{\beta}_i} \ldots s_{\widetilde{\beta}_1} w_C <_{\widetilde{\beta}_i} s_{\widetilde{\beta}_{i-1}} \ldots s_{\widetilde{\beta}_1} w_C$ for all i. Therefore, for each $x \in A_0$, the

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difference $w_C(x) - w_{C'}(x)$ is a positive linear combination of the $\check{\beta}_i$'s (by Lemma 1.2.2(a)), and hence a positive linear combination of C-simple coroots.

However, since $w_C \leq_{C'} w_{C'}$, the difference $w_C(x) - w_{C'}(x)$ is also a negative linear combination of C'-simple coroots. Combining these two statements, we conclude that $w_C(x) - w_{C'}(x)$ has to be a positive multiple of $\check{\alpha}$. Hence, all the β_i's have to be α; thus, $w_{C'} \leq_\alpha w_C$. □

The following corollary seems to be known to specialists.

Corollary 1.3.8. *Let $\overline{\mu}$ be an admissible tuple in $V^{\mathcal{C}}$, and let $\psi \in \Psi$.*

(a) *The subset $V^{\leq \overline{\mu}} \subseteq V$ equals the convex hull of $\{\mu_C\}_{C \in \mathcal{C}}$.*

(b) *If $\overline{\mu}$ is strictly admissible, then for every $C \not\ni \psi$, we have $\langle \psi, \mu_C \rangle < \overline{\mu}(\psi)$.*

(c) *If $\overline{\mu}$ is strictly admissible, then the intersection of $V^{\leq \overline{\mu}}$ and the set of $x \in V$ such that $\langle \psi, x \rangle = \overline{\mu}(\psi)$ equals the convex hull of $\{\mu_C\}_{C \ni \psi}$.*

Proof. (a) For every $C \in \mathcal{C}$, we have $\mu_C \leq_{C'} \mu_{C'}$ for every $C' \in \mathcal{C}$ (by Lemma 1.3.7); thus, $\mu_C \in V^{\leq \overline{\mu}}$. Since subset $V^{\leq \overline{\mu}} \subseteq V$ is convex, this implies that the convex hull of $\{\mu_C\}_{C \in \mathcal{C}}$ is contained in $V^{\leq \overline{\mu}}$.

To show the opposite inclusion, it suffices to show that for every affine function l on V such that $l(\mu_{C'}) \leq 0$ for all $C' \in \mathcal{C}$ and every $x \in V^{\leq \overline{\mu}}$, we have $l(x) \leq 0$. Let $\lambda \in V^*$ be the vector part of l and choose $C \in \mathcal{C}$ such that $\alpha \in \overline{C}$. Since $x \leq_C \mu_C$, we get $l(\mu_C) - l(x) = \langle \lambda, \mu_C \rangle - \langle \lambda, x \rangle \geq 0$; therefore, $l(x) \leq l(\mu_C) \leq 0$.

(b) Since $\langle \psi, \mu_C \rangle \leq \overline{\mu}(\psi)$ for every $C \in \mathcal{C}$, it suffices to show that for every $C \in \mathcal{C}$ such that $\langle \psi, \mu_C \rangle = \overline{\mu}(\psi)$, we have $\psi \in \Psi_C$. To show the result, we essentially repeat the first part of the proof of Lemma 1.3.7.

Using Remark 1.2.4(a)(ii), we may assume that $\psi \in \Psi_{C_0}$, and let $u \in W$ be such that $C = u(C_0)$. It suffices to show that $u \in W^\psi$. Let u_j, C_j and α_{j+1} be as in the proof of Lemma 1.3.7. Since $\overline{\mu}$ is strictly admissible, we get $\mu_{C_{j+1}} <_{u_j(\alpha_{j+1})} \mu_{C_j}$ for every $j = 0, \ldots, n-1$. Moreover, since $\psi \in \Psi_{C_0}$ and $u_j(\alpha_{j+1}) \in \Phi_{C_0}$, we conclude that $\langle \psi, u_j(\alpha_{j+1}) \rangle \geq 0$. So the assumption that $\langle \psi, \mu_C \rangle = \overline{\mu}(\psi) = \langle \psi, \mu_{C_0} \rangle$ implies that for every j, we have $\langle \psi, u_j(\alpha_{j+1}) \rangle = 0$; hence, $u_j(\alpha_{j+1}) \in \Phi^\psi$ and thus $s_{u_j(\alpha_{j+1})} \in W^\psi$. Therefore, $u = s_{u_{n-1}(\alpha_n)} \cdot \ldots \cdot s_{\alpha_1} \in W^\psi$, as claimed.

(c) is an immediate consequence of parts (a) and (b). □

Notation 1.3.9. (a) Let $m \in \mathbb{R}$ and $C \in \mathcal{C}$. We say that $\mu \in V$ is (C, m)*-regular* if $\langle \alpha, \mu \rangle \geq m$ for every $\alpha \in \Phi_C$. We say that $w \in \widetilde{W}$ is (C, m)*-regular* if $\pi(w) = w(0) \in \Lambda$ is (C, m)-regular.

(b) Let $m \in \mathbb{R}$. We say that a tuple $\overline{\mu} \in V^{\mathcal{C}}$ is m*-regular* if μ_C is (C, m)-regular for every $C \in \mathcal{C}$. We say that a tuple $\overline{\mu} \in V^{\mathcal{C}}$ is regular if it is m-regular for some $m > 0$. A tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is called m*-regular* (resp. regular) if $\pi(\overline{w}) \in \Lambda^{\mathcal{C}} \subseteq V^{\mathcal{C}}$ is m-regular (resp. regular).

(c) For every $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ and every $\psi \in \Psi$, we define $\overline{w}^\psi \in (\widetilde{W}^\psi)^{\mathcal{C}^\psi}$ by the rule $(w^\psi)_{C^\psi} = (w_C)^\psi$ for each $C \ni \psi$ (see Section 1.1.3).

Lemma 1.3.10. (a) *If $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ (resp. $\overline{\mu} \in V^{\mathcal{C}}$) is quasi-admissible and regular, then it is strictly admissible.*

(b) *If $\overline{\mu} \in V^{\mathcal{C}}$ is quasi-admissible and regular, then for every $\psi \in \Psi$, we have $\overline{\mu}(\psi) > 0$ (see Section 1.3.4(a)).*

(c) *If the tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is admissible, then the tuple $\overline{w}^\psi$ is admissible as well.*

(d) *If $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is $(m+1)$-regular, then $\overline{w}^\psi$ is m-regular.*

Proof. (a) We will only show the assertion for $\overline{w}$. Fix $C \in \mathcal{C}$, let $\alpha \in \Phi_C$, and set $C' = s_\alpha(C)$. We want to show that $w_{C'} \leq_\alpha w_C$. Since $\overline{w}$ is quasi-admissible, we get $w_{C'} \in \widetilde{W}_\alpha w_C$. Therefore, for every $x \in A_0$, we have $w_{C'}(x) = w_C(x) - a\check{\alpha}$ for some $a \in \mathbb{R}$. Since $\overline{w}$ is regular, we conclude $\langle \alpha, w_C(x) \rangle > 0$ and $\langle \alpha, w_{C'}(x) \rangle < 0$. Thus, $a > 0$, and hence, $w_{C'} <_\alpha w_C$ (by Corollary 1.2.3(a)).

(b) Since $\overline{\mu}(\psi) = \langle \psi, \mu_C \rangle$ for every Weyl chamber $C \ni \psi$ (by definition), we have $\langle \alpha, \mu_C \rangle > 0$ for every $\alpha \in \Delta_C$ (since $\overline{\mu}$ is regular), and $\psi = \sum_{\alpha \in \Delta_C} c_\alpha \alpha$ with $c_\alpha \geq 0$ for all $\alpha \in \Delta_C$, the assertion follows.

(c) follows from Lemma 1.2.7(b) and Lemma 1.3.7.

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(d) Let $C \ni \psi$ be a Weyl chamber. We have to show that if w is $(C, m+1)$-regular, and $\psi \in C$, then w^ψ is (C^ψ, m)-regular.

Notice that if w is $(C, m+1)$-regular, then for all $\alpha \in \Phi_C$ and $x \in A_0$, we have $\langle w^{-1}(\alpha), x\rangle = \langle \alpha, w(x)\rangle > m$, or equivalently, $w^{-1}(\alpha) - m \in \widetilde{\Phi}_{>0}$. Conversely, if $\langle \alpha, w(x)\rangle > m$ for all $\alpha \in \Phi_C$, then $\langle \alpha, w(0)\rangle \geq m$ for all $\alpha \in \Phi_C$; thus, w is (C, m)-regular.

Thus, it suffices to show that for every $\alpha \in (\Phi^\psi)_{C^\psi}$, we have $w^{-1}(\alpha) - m \in \widetilde{\Phi}_{>0}$ if and only if $(w^\psi)^{-1}(\alpha) - m \in \widetilde{\Phi}^\psi_{>0}$. Since $w = w^\psi w_\psi$, the assertion follows from the fact that $w_\psi \in \widetilde{W}_\psi$. □

The following lemma will be used in Lemma 3.2.10.

Lemma 1.3.11. *Let $\overline{\mu} \in V^{\mathcal{C}}$ be regular and quasi-admissible. Then for every $x \in V^{\leq \overline{\mu}}$, $\psi \in \Psi$ and $\alpha \in \Phi$ such that $\langle \alpha, \check{\psi}\rangle > 0$ and $\langle \psi, x\rangle = \overline{\mu}(\psi)$, we have $\langle \alpha, x\rangle > 0$.*

Proof. By Lemma 1.3.10(a), the tuple $\overline{\mu}$ is strictly admissible. Then, by Corollary 1.3.8(c), the intersection of $V^{\leq \overline{\mu}}$ with the set of $x \in V$ such that $\langle \psi, x\rangle = \overline{\mu}(\psi)$ is equal to the convex hull of $\{\mu_C\}_{C \ni \psi}$.

Therefore, it is enough to show that for every Weyl chamber $C \ni \psi$, we have $\langle \alpha, \mu_C\rangle > 0$. Since tuple $\overline{\mu}$ is regular, it is enough to show that $\langle \alpha, y\rangle > 0$ for some $y \in C \subseteq V^*$. But this follows from our assumption $\langle \alpha, \check{\psi}\rangle > 0$ together with observation that $\check{\psi} \in \overline{C}$ (see Section 1.1.3(b)). □

The following very important technical result will be used in Proposition 3.1.8.

Lemma 1.3.12. *Let $\overline{u} \in \widetilde{W}^{\mathcal{C}}$ be admissible, and $\psi \in \Psi$. Then there exists $m \in \mathbb{N}$ such that for every m-regular admissible tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ and every $\mu \in \Lambda$ such that $\mu u_C \leq_{C^\psi} w_C$ for each $C \ni \psi$, we have $\mu u_C \leq_C w_C$ for each C.*

Proof. First, we claim that there exists an admissible tuple $\overline{\mu} \in \Lambda^{\mathcal{C}}$ such that $\mu_C^{-1} u_C \leq_C \mu_{C'}^{-1} u_{C'}$ for every $C, C' \in \mathcal{C}$. Indeed, $\overline{u}$ is admissible; hence, for each $C \in \mathcal{C}$, $\alpha \in \Delta_C$ and $x \in A_0$, we have $u_C(x) - u_{s_\alpha(C)}(x) = m_{C,\alpha,x}\check{\alpha}$ for some constant $m_{C,\alpha,x} \geq 0$ (use Lemma 1.2.2(a)). Let m' be the supremum of the $m_{C,\alpha,x}$'s, choose $\mu \in C_0 \cap \Lambda$ such that $\langle \alpha, \mu\rangle \geq m'$ for all $\alpha \in \Delta_{C_0}$, and let $\overline{\mu} \in \Lambda^{\mathcal{C}}$ be the tuple, corresponding to μ as in Section 1.3.5(a).

We claim that $\mu_C^{-1} u_C \leq_C \mu_{C'}^{-1} u_{C'}$ for every $C, C' \in \mathcal{C}$. Indeed, arguing as in Lemma 1.3.7 word-by-word, it is enough to check that $\mu_C^{-1} u_C \leq_\alpha \mu_{C'}^{-1} u_{C'}$ for all $C \in \mathcal{C}, \alpha \in \Delta_C$ and $C' = s_\alpha(C)$. Then by Corollary 1.2.3(a), it is enough to check that $\mu_C^{-1} u_C(x) \leq_\alpha \mu_{C'}^{-1} u_{C'}(x)$ for each $x \in A_0$. By construction, we have $\mu_{C'}^{-1} u_{C'}(x) - \mu_C^{-1} u_C(x) = (\langle \alpha, \mu_C\rangle - m_{C,\alpha,x})\check{\alpha}$, so the assertion follows from the fact that $m_{C,\alpha,x} < m' \leq \langle \alpha, \mu_C\rangle$.

Denote m to be the maximum of the $\langle \alpha, \mu_C\rangle + 1$'s, taken over all $C \in \mathcal{C}$ and $\alpha \in \Delta_C$. We claim that such an m satisfies the required property.

To see this, we choose any m-regular admissible tuple $\overline{w}$, and we claim that tuple $\overline{\mu}^{-1} \cdot \overline{w} = \{\mu_C^{-1} w_C\}_C$ is admissible. By Section 1.3.5(c), it is quasi-admissible, so by Lemma 1.3.10(a), it is enough to show that it is regular. For every $C \in \mathcal{C}$, $\alpha \in \Delta_C$, we have $\langle \alpha, \mu_C^{-1}\pi(w_C)\rangle = \langle \alpha, \pi(w_C)\rangle - \langle \alpha, \mu_C\rangle > 0$ because $\langle \alpha, \pi(w_C)\rangle \geq m$ by m-regularity of $\overline{w}$, and $\langle \alpha, \mu_C\rangle \leq m - 1$, by construction.

Let $\mu \in \Lambda$ be such that $\mu u_C \leq_{C^\psi} w_C$ for each $C \ni \psi$, and let $C' \in \mathcal{C}$ be arbitrary. We want to show that $\mu u_{C'} \leq_{C'} w_{C'}$. Using Remark 1.2.4(a)(ii), it is enough to do it in the case $C' = C_0$, so using Remark 1.2.4(a)(i), we have to show that $\mu_{C_0}^{-1} \mu u_{C_0} \leq_{C_0} \mu_{C_0}^{-1} w_{C_0}$.

Choose $u \in W$ of minimal length such that $\psi_0 := u^{-1}(\psi)$ belongs to Ψ_{C_0}, and set $C := u(C_0)$. Since $\overline{\mu}^{-1} \cdot \overline{w}$ is admissible, we conclude from Lemma 1.3.7 that $\mu_C^{-1} w_C \leq_{C_0} \mu_{C_0}^{-1} w_{C_0}$, while by our construction, we get $\mu_{C_0}^{-1} u_{C_0} \leq_{C_0} \mu_C^{-1} u_C$; hence, $\mu_{C_0}^{-1} \mu u_{C_0} \leq_{C_0} \mu_C^{-1} \mu u_C$ (by Remark 1.2.4(a)(i)). Thus, it is enough to show that $\mu_C^{-1} \mu u_C \leq_{C_0} \mu_C^{-1} w_C$, or, equivalently, that $\mu u_C \leq_{C_0} w_C$.

Since $\psi_0 \in C_0$, we get that $\psi \in C$. Hence, by our assumption, $\mu u_C \leq_{C^\psi} w_C$. Therefore, to show that $\mu u_C \leq_{C_0} w_C$, it suffices to check that $(\Phi^\psi)_{C^\psi} \subseteq \Phi_{C_0}$.

If $\beta \in (\Phi^\psi)_{C^\psi}$, then $u^{-1}(\beta) \in (\Phi^{\psi_0})_{C_0^{\psi_0}}$. Since $u \in W$ is an element of minimal length such that $\psi = u(\psi_0)$, we get that $u((\Phi^{\psi_0})_{C_0^{\psi_0}}) \subseteq \Phi_{C_0}$. In particular, we have $\beta = u(u^{-1}(\beta)) \in \Phi_{C_0}$. □

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2. Semi-infinite orbits in affine flag varieties

2.1. Definitions and basic properties

Notation 2.1.1. (a) Let k be an algebraically closed field, $K := k((t))$ the field of Laurent power series over k, and $\mathcal{O} = \mathcal{O}_K = k[[t]]$ the ring of integers of K. For every affine scheme X over $\mathcal{O}$ (resp. K), we denote by L^+X (resp. LX) the corresponding arc- (resp. loop-) space.

(b) Let G be a semi-simple and simply connected group over k. Fix a maximal torus $T \subsetneq G$, let $\Phi = \Phi(G,T)$ be the root system of (G,T), let $W = W_G$ be the Weyl group of G, and $\widetilde{W} = N_{LG}(LT)/L^+T$ the affine Weyl group of G. Then, in the notation of Section 1.1.1, we have natural isomorphisms $\Lambda \xrightarrow{\sim} X_*(T)$ and $W_\Phi \xrightarrow{\sim} W$. Moreover, the map $\mu \mapsto \mu(t)$ defines an embedding $\Lambda \hookrightarrow LT$, which in turn induces isomorphisms of groups $\Lambda \xrightarrow{\sim} LT/L^+T$ and $\widetilde{W}_\Phi \xrightarrow{\sim} \widetilde{W}$.

Notation 2.1.2. (a) For every $C \in \mathcal{C}$, let $B_C \subsetneq G$ be the Borel subgroup containing T such that $\Phi(B_C,T) = \Phi_C$, and let $U_C \subsetneq B_C$ be the unipotent radical.

(b) Choose $C_0 \in \mathcal{C}_\Phi$ as in Section 1.1.2, let $T \subsetneq B_0 = B_{C_0} \subsetneq G$ be the corresponding Borel subgroup, let $B_0^- \supsetneq T$ be the opposite Borel subgroup, and let $I \subsetneq L^+G$ be the Iwahori subgroup, defined as the preimage of $B_0^- \subsetneq G$ under the projection $L^+G \to G$.

(c) For every $\alpha \in \Phi$, we have a natural isomorphism $\exp_\alpha : \operatorname{Lie} U_\alpha \xrightarrow{\sim} U_\alpha$. For $\widetilde{\alpha} = (\alpha, n) \in \widetilde{\Phi}$, we set $U_{\widetilde{\alpha}} := \exp_\alpha(t^n \operatorname{Lie} U_\alpha) \subsetneq L(U_\alpha)$, and $\widetilde{\alpha}' := (-\alpha, n)$.

(d) In the conventions of parts (b), (c), we get the equality $I = L^+T \cdot \prod_{\widetilde{\alpha}\in\widetilde{\Phi}_{>0}} U_{\widetilde{\alpha}'}$.

2.1.3. Affine flag varieties

(a) Denote by $\mathrm{Fl} = \mathrm{Fl}_G$ the affine flag variety LG/I of G over k, and by $\mathrm{Gr} = \mathrm{Gr}_G$ the affine Grassmannian LG/L^+G. We have a natural projection $\mathrm{pr} : \mathrm{Fl} \to \mathrm{Gr}$. Note that both Fl and Gr are equipped with an action of the ind-group scheme LG, and that projection pr is LG-equivariant.

(b) The embedding $N_{LG}(LT) \hookrightarrow LG$ induces embeddings $\widetilde{W} \hookrightarrow \mathrm{Fl}$ and $\Lambda \hookrightarrow \mathrm{Gr}$, and we identify $\widetilde{W}$ (resp. Λ) with its image in Fl (resp. Gr). Furthermore, both Fl and Gr are equipped with the action of $T \subsetneq L^+(T) \subsetneq LG$, and these identifications identify $\widetilde{W}$ (resp. Λ) with the locus of T-fixed points Fl^T (resp. Gr^T).

(c) Note that Fl decompose as a union $\mathrm{Fl} = \bigcup_{w\in\widetilde{W}} Iw$ of I-orbits, and for every $w \in \widetilde{W}$, we denote by $\mathrm{Fl}^{\leq w} \subsetneq \mathrm{Fl}$ the closure of the I-orbit $Iw \subsetneq \mathrm{Fl}$. Then $\mathrm{Fl}^{\leq w}$ is a reduced projective subscheme of Fl called *the affine Schubert variety*.

(d) Fix any $C \in \mathcal{C}$. Then we have decompositions $\mathrm{Fl} = \bigcup_{w\in\widetilde{W}} L(U_C)w$ and $\mathrm{Gr} = \bigcup_{\mu\in\Lambda} L(U_C)\mu$ by $L(U_C)$-orbits. For every $w \in \widetilde{W}$ (resp. $\mu \in \Lambda$), we denote by $\mathrm{Fl}^{\leq_C w} \subsetneq \mathrm{Fl}$ (resp. $\mathrm{Gr}^{\leq_C \mu} \subsetneq \mathrm{Gr}$) the closure of the $L(U_C)$-orbit $L(U_C)w \subsetneq \mathrm{Fl}$ (resp. $L(U_C)\mu \subsetneq \mathrm{Gr}$). We also set $\mathrm{Fl}^{\leq'_C \mu} := \mathrm{pr}^{-1}(\mathrm{Gr}^{\leq_C \mu}) \subsetneq \mathrm{Fl}$. Notice that $\mathrm{Fl}^{\leq_C w}$, $\mathrm{Gr}^{\leq_C \mu}$ and $\mathrm{Fl}^{\leq'_C w}$ are closed reduced ind-subschemes.

(e) For every tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ (resp. $\overline{\mu} \in \Lambda^{\mathcal{C}}$), we denote by $\mathrm{Fl}^{\leq \overline{w}}$ (resp. $\mathrm{Gr}^{\leq \overline{\mu}}$) the reduced intersection $\bigcap_C \mathrm{Fl}^{\leq_C w_C}$ (resp. $\bigcap_C \mathrm{Gr}^{\leq_C \mu_C}$), and set $\mathrm{Fl}^{\leq' \overline{\mu}} := \mathrm{pr}^{-1}(\mathrm{Gr}^{\leq \overline{\mu}})$.

The following simple lemma will play a central role later.

Lemma 2.1.4. *Let $Z \subsetneq \mathrm{Fl}$ (resp. $Z \subsetneq \mathrm{Gr}$) be a closed reduced T-invariant ind-subscheme, C a Weyl chamber, and $w \in \widetilde{W}$ (resp. $\mu \in \Lambda$). Then $Z \cap L(U_C)w \neq \emptyset$ if and only if $w \in Z$ (resp. $\mu \in Z$).*

Proof. We will show the assertion for $Z \subsetneq \mathrm{Fl}$ and $w \in \widetilde{W}$, while the proof of the second assertion is identical.

Clearly, if $w \in Z$, then $Z \cap L(U_C)w \neq \emptyset$. Conversely, let z be an element of $Z \cap L(U_C)w$, and pick $u \in L(U_C)$ such that $z = uw$. For any $\nu \in \Lambda = \operatorname{Hom}(\mathbb{G}_m, T)$ and $a \in \mathbb{G}_m$, we have $\nu(a)(z) = (\nu(a)u\nu(a)^{-1})(w)$ because $w \in \mathrm{Fl}$ is T-invariant; hence, $(\nu(a)u\nu(a)^{-1})(w) \in Z$ because Z is T-invariant. Next, for $\nu \in \Lambda \cap C$, the morphism $a \mapsto (\nu(a)u\nu(a)^{-1})(w) : \mathbb{G}_m \to Z \subsetneq \mathrm{Fl}$ extends to the morphism $\mathbb{A}^1 \to \mathrm{Fl}$, which sends 0 to w. Since $Z \subsetneq \mathrm{Fl}$ is closed, we conclude that $w \in Z$. □

Lemma 2.1.5. *Let $w, w' \in \widetilde{W}$, and let C be a Weyl chamber.*

(a) *We have* $w' \in \mathrm{Fl}^{\leq w}$ *if and only if* $w' \leq w$.
(b) *If* $w \in C$*, then* $Iw \subseteq \mathrm{Fl}$ *is contained in* $L(U_C)w \subseteq \mathrm{Fl}$.
(c) *If* $Iw \cap L(U_C)w' \neq \emptyset$*, then* $w' \leq w$.

Proof. (a) is a standard.

(b) In the notation of Section 2.1.2(c), for every $\widetilde{\alpha} \in \widetilde{\Phi}$ and $w \in \widetilde{W}$, we have $wU_{\widetilde{\alpha}}w^{-1} = U_{w(\widetilde{\alpha})}$ and $w(\widetilde{\alpha}') = w(\widetilde{\alpha})'$. Combining this with Section 2.1.2(d), we see that for every $w \in \widetilde{W}$, we have

$$Iw = \left(\prod_{\widetilde{\alpha}>0,\, w^{-1}(\widetilde{\alpha})<0} U_{\widetilde{\alpha}'}\right)w. \tag{2.1}$$

Using formula (2.1), it remains to check that every $\widetilde{\alpha} = (\alpha, n) > 0$ such that $w^{-1}(\widetilde{\alpha}) < 0$ satisfies $U_{\widetilde{\alpha}'} \subseteq L(U_C)$; that is, $-\alpha \in \Phi_C$. However, $n \geq 0$ because $\widetilde{\alpha} > 0$. Therefore, $w^{-1}(\alpha) = w^{-1}(\widetilde{\alpha}) - n < 0$. Thus, $w^{-1}(-\alpha) > 0$; hence, $-\alpha \in \Phi_C$ because $w \in C$.

(c) If $Iw \cap L(U_C)w' \neq \emptyset$, then $\mathrm{Fl}^{\leq w} \cap L(U_C)w' \neq \emptyset$. Since $\mathrm{Fl}^{\leq w} \subseteq \mathrm{Fl}$ is closed and T-invariant, we get $w' \in \mathrm{Fl}^{\leq w}$ (by Lemma 2.1.4); thus, $w' \leq w$ (by part (a)). □

The following proposition gives a geometric interpretation of the ordering $\leq_C$, generalizing the well-known result (see, for example, [MV, Proposition 3.1]) for the affine Grassmannian.

Proposition 2.1.6. *For each* $w', w'' \in \widetilde{W}$ *and every Weyl chamber* $C \in \mathcal{C}$*, we have* $w' \leq_C w''$ *if and only if* $w' \in \mathrm{Fl}^{\leq_C w''}$.

Proof. Assume that $w' \leq_C w''$. Then by Proposition 1.2.5, there exists $\mu \in \Lambda \cap C$ such that $\mu w' \leq \mu w''$ and $\mu w'' \in C$. Then $\mu w' \in \mathrm{Fl}$ lies in the closure of $I\mu w'' \subseteq \mathrm{Fl}$ (by Lemma 2.1.5(a)), and thus in the closure of $L(U_C)\mu w'' \subseteq \mathrm{Fl}$ (by Lemma 2.1.5(b)). Since U_C is normalized by T, this implies that $w' \in \mathrm{Fl}$ lies in the closure of $\mu^{-1}L(U_C)\mu w'' = L(U_C)w'' \subseteq \mathrm{Fl}$; that is, $w' \in \mathrm{Fl}^{\leq_C w''}$.

Conversely, assume that $w' \in \mathrm{Fl}$ lies in the closure of $L(U_C)w'' \subseteq \mathrm{Fl}$. Then there exists a closed subgroup scheme $U' \subseteq L(U_C)$ such that $w' \in \mathrm{Fl}$ lies in the closure of $U'w'' \subseteq \mathrm{Fl}$. Then $\mu w' \in \mathrm{Fl}$ lies in the closure of $\mu U' w'' = (\mu U' \mu^{-1})\mu w''$ for every $\mu \in \Lambda$. However, if $\mu \in \Lambda \cap C$ is sufficiently regular, then $\mu U' \mu^{-1} \subseteq I$; thus, $\mu w' \in \mathrm{Fl}$ lies in the closure of $I\mu w'' \subseteq \mathrm{Fl}$. This implies that $\mu w' \leq \mu w''$ (by Lemma 2.1.5(a)); thus, $w' \leq_C w''$ (by Proposition 1.2.5). □

Corollary 2.1.7. (a) *A tuple* $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ *is admissible if and only if for every* $C \in \mathcal{C}$ *the intersection* $L(U_C)w_C \cap \mathrm{Fl}^{\leq \overline{w}}$ *is nonempty.*

(b) *For a tuple* $\overline{u}$ *and an admissible tuple* $\overline{w}$*, we have* $\mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}^{\leq \overline{u}}$ *if and only if* $\overline{w} \leq \overline{u}$.

(c) *For a tuple* $\overline{w}$*, we have an inclusion* $\mathrm{Fl}^{\leq \overline{w}} \subseteq \bigcup_C \mathrm{Fl}^{\leq w_C}$. *In particular, each* $\mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}$ *is a closed subscheme of finite type.*

(d) *Let* $Z \subseteq \mathrm{Fl}$ *be a closed* T*-invariant ind-subscheme. For every* $z \in Z$*, consider tuple* $\overline{u} = \overline{u}(z) \in \widetilde{W}^{\mathcal{C}}$ *defined by the rule that* $z \in L(U_C)u_C$ *for all* $C \in \mathcal{C}$*. Then the tuple* $\overline{u}$ *is admissible, and* $u_C \in \widetilde{W} \cap Z$ *for all* $C \in \mathcal{C}$.

(e) *In the situation of part (d), we have an inclusion* $Z \subseteq \bigcap_C (\bigcup_{w \in \widetilde{W} \cap Z} \mathrm{Fl}^{\leq_C w})$.

(f) *For every tuple* $\overline{\mu} \in \Lambda^{\mathcal{C}}$*, we have an equality* $\mathrm{Fl}^{\leq' \overline{\mu}} = \mathrm{Fl}^{\leq \overline{\mu} \cdot \overline{w}_{\mathrm{st}}}$ *(compare Sections 1.3.5(b),(c)).*

Proof. (a) By Lemma 1.3.7 and Proposition 2.1.6, a tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is admissible if and only if $w_C \in \mathrm{Fl}^{\leq \overline{w}} = \bigcap_{C' \in \mathcal{C}} \mathrm{Fl}^{\leq_{C'} w_{C'}}$ for each $C \in \mathcal{C}$. Since $\mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}$ is closed and T-invariant, the assertion now follows from Lemma 2.1.4.

(b) The 'if' assertion follows from Proposition 2.1.6. Conversely, if $\mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}^{\leq \overline{u}}$, then $w_C \in \mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}^{\leq \overline{u}}$ (as in part (a)); hence, $w_C \in \mathrm{Fl}^{\leq_C u_C}$ for every C. Therefore, $w_C \leq_C u_C$ by Proposition 2.1.6.

(c) Let z be any element of $\mathrm{Fl}^{\leq \overline{w}}$, let $u \in \widetilde{W}$ be such that $z \in Iu$, and let $C \in \mathcal{C}$ be such that $u \in C$. We want to show that $u \leq w_C$, and thus $z \in \mathrm{Fl}^{\leq w_C}$.

By Lemma 2.1.5(b), we get $z \in Iu \subseteq L(U_C)u$. However, we have $z \in \mathrm{Fl}^{\leq \overline{w}} \subseteq \mathrm{Fl}^{\leq_C w_C}$. Therefore, by Proposition 2.1.6, we get that $u \leq_C w_C$, which by Corollary 1.2.6(a) implies that $u \leq w_C$.

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(d) By construction, $z \in \mathrm{Fl}^{\leq \overline{u}} \cap L(U_C)u_C$ for all $C \in \mathcal{C}$; hence, $\overline{u}$ is admissible by part (a). Since $z \in L(U_C)u_C \cap Z$, we get $u_C \in Z$ by Lemma 2.1.4.

(e) follows immediately from part (d).

(f) It is enough to show that for every $C \in \mathcal{C}$, the preimage $\mathrm{pr}^{-1}(\mathrm{Gr}^{\leq_C \mu_C})$ equals $\mathrm{Fl}^{\leq_C (\mu_C (w_{\mathrm{st}})_C)}$. Using Proposition 2.1.6, we have to check that for every $\mu \in \Lambda$ and $u \in W$, we have $\mu \leq_C \mu_C$ if and only if $\mu u \leq_C \mu_C (w_{\mathrm{st}})_C$.

The 'only if' assertion follows from Corollary 1.2.3(b). Conversely, if $\mu \leq_C \mu_C$, then $\mu u \leq_C \mu_C u$ by Lemma 1.2.2(b). So by Remark 1.2.4(a)(i), it is enough to show that $u \leq_C (w_{\mathrm{st}})_C$. Since $\overline{w}_{\mathrm{st}}$ is admissible and $u = (w_{\mathrm{st}})_{u(C_0)}$, the assertion follows from Lemma 1.3.7. □

2.2. *Proof of Theorem 0.2*

2.2.1. Let $m \in \mathbb{N}$. Recall that $w \in \widetilde{W}$ is called *m-regular* if $\pi(w) \in \Lambda$ is m-regular; that is, we have $|\langle \alpha, \pi(w)\rangle| \geq m$ for all $\alpha \in \Phi$. For each $w \in \widetilde{W}$, we denote by $\widetilde{W}^{\leq w}$ the set of $w' \in \widetilde{W}$ such that $w' \leq w$.

The following result is a more precise version of Theorem 0.2.

Theorem 2.2.2. (a) *For each $w \in \widetilde{W}$, there exists a unique admissible tuple $\overline{w}$ such that the Schubert variety* $\mathrm{Fl}^{\leq w}$ *equals* $\mathrm{Fl}^{\leq \overline{w}}$.

Moreover, $\overline{w} = \{w_C\}_C$ is characterized by the condition that w_C is a unique maximal element of $\widetilde{W}^{\leq w}$ with respect to the ordering $\leq_C$.

(b) *Furthermore, there exists $r \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every $(m+r)$-regular $w \in \widetilde{W}$, the tuple $\overline{w}$ is m-regular.*

Proof. (a) Denote by $X(w)$ the closed ind-subscheme $\bigcap_C (\bigcup_{w' \leq w} \mathrm{Fl}^{\leq_C w'}) \subseteq \mathrm{Fl}$ (compare Corollary 2.1.7(c)), and we claim that $X(w)$ equals $\mathrm{Fl}^{\leq w}$. Indeed, the inclusion $\mathrm{Fl}^{\leq w} \subseteq X(w)$ follows from Corollary 2.1.7(e), while the opposite inclusion $X(w) \subseteq \mathrm{Fl}^{\leq w}$ follows from identity $X(w) = \bigcup_{\overline{w}' \in (\widetilde{W}^{\leq w})^{\mathcal{C}}} \mathrm{Fl}^{\leq \overline{w}'}$ and Corollary 2.1.7(c).

Next, since $\mathrm{Fl}^{\leq w} = \bigcup_{\overline{w}' \in (\widetilde{W}^{\leq w})^{\mathcal{C}}} \mathrm{Fl}^{\leq \overline{w}'}$ is irreducible, there exists a tuple $\overline{w} = \{w_C\}_C \in (\widetilde{W}^{\leq w})^{\mathcal{C}}$ such that $\mathrm{Fl}^{\leq w} = \mathrm{Fl}^{\leq \overline{w}}$.

Then for each $w' \leq w$ and $C \in \mathcal{C}$, we have $w' \in \mathrm{Fl}^{\leq w} \subseteq \mathrm{Fl}^{\leq_C w_C}$. Thus, by Proposition 2.1.6, we have $w' \leq_C w_C$; that is, w_C is the biggest element of $\widetilde{W}^{\leq w}$ with respect to ordering $\leq_C$. In particular, for every other Weyl chamber C', we have $w_{C'} \leq_C w_C$. Thus, by Lemma 1.3.7, we conclude that $\overline{w}$ is admissible.

The uniqueness of $\overline{w}$ follows immediately from Corollary 2.1.7(b).

(b) Choose any $\mu \in \Lambda \cap C_0$, and let r be the maximum of the $2\langle \psi, \mu\rangle$'s, taken over $\psi \in \Psi_{C_0}$. We claim this r satisfies the required property; that is, for every $m \in \mathbb{N}$ and every $(m+r)$-regular $w \in \widetilde{W}$, the tuple $\overline{w}$ is m-regular. In other words, we claim that $w_{u(C_0)}$ is $(u(C_0), m)$-regular or, equivalently, that $u^{-1} w_{u(C_0)}$ is (C_0, m)-regular for all $u \in W$.

Claim 2.2.3. Let $w \in \widetilde{W}$, and let $\overline{w} := \{w_C\}_{C \in \mathcal{C}}$ be the tuple from Theorem 2.2.2(a).

(a) If $w = w_0 w_+$, where $w_0 \in W$ is the longest element, and $w_+ \in \widetilde{W} \cap C_0$, then $w_{u(C_0)} = u w_+$ for all $u \in W$.

(b) If $w \in W w_+$ with $w_+ \in \widetilde{W} \cap C_0$, then for all $u \in W$, we have inequalities

$$\mu^{-1} w_+ \leq_{C_0} u^{-1} w_{u(C_0)} \leq_{C_0} w_+.$$

Proof. (a) Fix $u \in W$. We will show that $w \leq w_0 u^{-1} w_{u(C_0)} \leq w$, which will imply that $w_0 u^{-1} w_{u(C_0)} = w_0 w_+$, and thus $w_{u(C_0)} = u w_+$.

Since $u \leq w_0$, we get $u w_+ \leq w_0 w_+ = w$ (use Section 1.1.4(e)). Therefore, by the characterization of $\overline{w}$, given Theorem 2.2.2(a), we get $u w_+ \leq_{u(C_0)} w_{u(C_0)}$. Hence, $w = w_0 w_+ \leq_{w_0(C_0)} w_0 u^{-1} w_{u(C_0)}$ (by Remark 1.2.4(a)(ii)); thus, $w \leq w_0 u^{-1} w_{u(C_0)}$ (by Corollary 1.2.6(a)). However, $w_{u(C_0)} \leq w = w_0 w_+$; thus, using Section 1.1.4(e), we conclude that $w_0 u^{-1} w_{u(C_0)} \leq w$.

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(b) By Remark 1.2.4(a)(ii), it is enough to show that

$$u\mu^{-1}w_+ \leq_{u(C_0)} w_{u(C_0)} \leq_{u(C_0)} uw_+.$$

Consider element $w' := w_0w_+$. Then $w \leq w'$ (use Section 1.1.4(e)), and thus, we have $w_{u(C_0)} \leq w'$. Hence, $w_{u(C_0)} \leq_{u(C_0)} w'_{u(C_0)}$ (by the characterization of $w'_{u(C_0)}$, given in Theorem 2.2.2(a)); thus, $w_{u(C_0)} \leq_{u(C_0)} uw_+$ (by part (a)). To show the other inequality, it is enough to show that $u\mu^{-1}w_+ \leq w$. Since w and hence also w_+ is $(m+r)$-regular, our definition of r implies that $\mu^{-1}w_+ \in \widetilde{W} \cap C_0$. Since $u \leq \mu$ (by Section 1.1.4(g)), and $l(w_+) = l(\mu) + l(\mu^{-1}w_+)$ (by Lemma 1.1.5(c)), we conclude from Section 1.1.4(c) and part (e) that $u(\mu^{-1}w_+) \leq w_+ \leq w$. □

Let us come back to the proof of the Theorem. By Claim 2.2.3(b) and Corollary 1.2.3(b), we have

$$\mu^{-1}\pi(w_+) \leq_{C_0} \pi(u^{-1}w_{u(C_0)}) \leq_{C_0} \pi(w_+).$$

Hence, we have $\pi(u^{-1}w_{u(C_0)}) = \pi(w_+) - \sum_{\alpha\in\Delta_{C_0}} m_\alpha\check{\alpha}$, such that $0 \leq m_\alpha \leq \langle\psi_\alpha, \mu\rangle$, where $\psi_\alpha \in \Psi_{C_0}$ is the fundamental weight corresponding to α for each $\alpha \in \Delta_{C_0}$. In particular, for each $\alpha \in \Delta_{C_0}$, we have

$$\langle\alpha, \pi(u^{-1}w_{u(C_0)})\rangle \geq \langle\alpha, \pi(w_+)\rangle - 2m_\alpha \geq (m+r) - r = m$$

because w_+ is $(m+r)$-regular, and $2m_\alpha \leq 2\langle\psi_\alpha, \mu\rangle \leq r$. □

2.3. Technical lemmas

Notation 2.3.1. Fix $\psi \in \Psi$.

(a) Denote by $P_\psi \supseteq T$ the parabolic subgroup of G such that $\Phi(P_\psi, T) = \Phi(\psi)$ (see Section 1.1.3(c)), by $M_\psi \supseteq T$ the Levi subgroup of P_ψ, by $U_\psi \subseteq P_\psi$ the unipotent radical, by M_ψ^{sc} the simply connected covering of the derived (=commutator) group of M_ψ. Let $P_\psi \to M_\psi$ be the natural projection, and set $P_\psi^{\mathrm{sc}} := P_\psi \times_{M_\psi} M_\psi^{\mathrm{sc}}$.

(b) Note that we have a natural homomorphism $P_\psi^{\mathrm{sc}} \to P_\psi \subseteq G$; thus, the loop group $L(P_\psi^{\mathrm{sc}})$ acts on Fl. For every $w \in \widetilde{W}$, we denote by $\mathrm{Fl}^{\leq_\psi w} \subseteq \mathrm{Fl}$ the closure of the $L(P_\psi^{\mathrm{sc}})$-orbit $L(P_\psi^{\mathrm{sc}})w \subseteq \mathrm{Fl}$. For every $\mu \in \Lambda$, we denote by $\mathrm{Gr}^{\leq_\psi \mu} \subseteq \mathrm{Gr}$ the closure of the $L(P_\psi^{\mathrm{sc}})$-orbit $L(P_\psi^{\mathrm{sc}})\mu \subseteq \mathrm{Gr}$.

Lemma 2.3.2. (a) *For $w', w'' \in \widetilde{W}$ and $\psi \in \Psi$, we have $w' \leq_\psi w''$ if and only if $w' \in \mathrm{Fl}^{\leq_\psi w''}$.*

(b) *For $u \in \widetilde{W}$, $\psi \in \Psi$ and an admissible tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, we have $\mathrm{Fl}^{\leq\overline{w}} \subseteq \mathrm{Fl}^{\leq_\psi u}$ if and only if $\overline{w}_\psi \leq_\psi u$ (compare Section 1.3.6).*

Proof. (a) Assume first that $w' \leq_\psi w''$, and we want to prove that $L(P_\psi^{\mathrm{sc}})w' \subseteq \mathrm{Fl}$ is contained in the closure of $L(P_\psi^{\mathrm{sc}})w'' \subseteq \mathrm{Fl}$. By definition, we can assume that $w' = s_{\widetilde{\beta}}w'' <_{\widetilde{\beta}} w''$, where $\widetilde{\beta} = (\beta, m)$, and $\langle\beta, \psi\rangle \geq 0$. Then there exists a Weyl chamber $C \ni \psi$ such that $\beta \in \Phi_C$. Then $w' <_C w''$; hence, by Proposition 2.1.6, w' lies in the closure of $L(U_C)w'' \subseteq \mathrm{Fl}$. Since $L(U_C) \subseteq L(P_\psi^{\mathrm{sc}})$, the assertion follows.

Conversely, assume that w' belongs to the closure of $L(P_\psi^{\mathrm{sc}})w'' \subseteq \mathrm{Fl}$. Choose any Weyl chamber $C \ni \psi$. Then $L(P_\psi^{\mathrm{sc}})w''$ is a union of orbits $\bigcup_{w\in\widetilde{W}^\psi} L(U_C)ww''$. Therefore, w' belongs to the closure of $L(U_C)ww'' \subseteq \mathrm{Fl}$ for some $w \in \widetilde{W}^\psi$. Hence, by Proposition 2.1.6, we get $w' \leq_C ww''$, and thus $w' \leq_\psi ww''$. However, since $w \in \widetilde{W}^\psi$, we also get $ww'' \leq_\psi w''$.

(b) Choose any Weyl chamber $C \ni \psi$. Then $w_C \in \widetilde{W}^\psi\overline{w}_\psi$; hence, we have $\overline{w}_\psi \leq_\psi u$ if and only if $w_C \leq_\psi u$.

Assume first that $w_C \leq_\psi u$. Then by part (a) we have $\mathrm{Fl}^{\leq_\psi w_C} \subseteq \mathrm{Fl}^{\leq_\psi u}$. However, we always have inclusions $\mathrm{Fl}^{\leq\overline{w}} \subseteq \mathrm{Fl}^{\leq_C w_C} \subseteq \mathrm{Fl}^{\leq_\psi w_C}$, which imply that $\mathrm{Fl}^{\leq\overline{w}} \subseteq \mathrm{Fl}^{\leq_\psi u}$. Conversely, since $\overline{w}$ is admissible, we get $w_C \in \mathrm{Fl}^{\leq\overline{w}}$ by Lemma 1.3.7. Therefore, if $\mathrm{Fl}^{\leq\overline{w}} \subseteq \mathrm{Fl}^{\leq_\psi u}$, we get $w_C \in \mathrm{Fl}^{\leq_\psi u}$. Hence, by part (a), we have $w_C \leq_\psi u$. □

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The remaining results of this subsection will be only used in Section 4.3.

Corollary 2.3.3. (a) *For $\mu', \mu'' \in \Lambda$ and $\psi \in \Psi$, we have $\mu' \in \mathrm{Gr}^{\leq_\psi \mu''}$ if and only if $\langle \psi, \mu' \rangle \leq \langle \psi, \mu'' \rangle$.*

(b) *For $m \in \mathbb{Z}$ and $\psi \in \Psi$, there exists a unique closed reduced ind-subscheme $\mathrm{Gr}^{\leq_\psi m} \subseteq \mathrm{Gr}$ such that $\mathrm{Gr}^{\leq_\psi m} = \mathrm{Gr}^{\leq_\psi \mu}$ for every $\mu \in \Lambda$ and $\psi \in \Psi$ such that $\langle \psi, \mu \rangle = m$.*

(c) *For every admissible tuple $\overline{\mu} \in \Lambda^{\mathcal{C}}$, we have an equality of reduced subschemes $\mathrm{Gr}^{\leq \overline{\mu}} = \bigcap_{\psi \in \Psi} \mathrm{Gr}^{\leq_\psi \overline{\mu}(\psi)} \subseteq \mathrm{Gr}$ (compare Section 1.3.4(a)).*

Proof. (a) Using equality $\mathrm{pr}^{-1}(\mathrm{Gr}^{\leq_\psi \mu''}) = \bigcup_{w \in W} \mathrm{Fl}^{\leq_\psi \mu'' w}$, we see that $\mu' \in \mathrm{Gr}^{\leq_\psi \mu''}$ if and only if $\mu' \in \mathrm{Fl}^{\leq_\psi \mu'' w}$ for some $w \in W$. Hence, by Lemma 2.3.2(a), this happens if and only if we have $\mu' \leq_\psi \mu'' w$ for some $w \in W$. Since $\pi(\mu'' w) = \mu''$, it thus follows from Corollary 1.2.3(b),(c) and Section 1.2.4(c) that this happens if and only if $\mu' \leq_\psi \mu''$ in the sense of Section 1.2.1(c). Now the assertion follows from Lemma 1.2.2(c).

(b) follows immediately from part (a).

(c) Notice that for every $C \in \mathcal{C}$ and $\psi \in \Psi_C$, the inclusion $U_C \subseteq P^{\mathrm{sc}}_\psi$ implies the inclusion $\mathrm{Gr}^{\leq_C \mu_C} \subseteq \mathrm{Gr}^{\leq_\psi \mu_C} = \mathrm{Gr}^{\leq_\psi \overline{\mu}(\psi)}$, from which the inclusion '$\subseteq$' follows.

Conversely, for every $y \in \bigcap_{\psi \in \Psi} \mathrm{Gr}^{\leq_\psi \overline{\mu}(\psi)}$ and $C \in \mathcal{C}$, let $\nu \in \Lambda$ be such that $y \in L(U_C)\nu$, and we want to show that $\nu \leq_C \mu_C$. Since the ind-subscheme $\bigcap_{\psi \in \Psi} \mathrm{Gr}^{\leq_\psi \overline{\mu}(\psi)} \subseteq \mathrm{Gr}$ is closed and T-invariant, it follows from Lemma 2.1.4 that $\nu \in \bigcap_{\psi \in \Psi} \mathrm{Gr}^{\leq_\psi \overline{\mu}(\psi)}$. Hence, by part (a), we have $\langle \psi, \nu \rangle \leq \overline{\mu}(\psi)$ for each $\psi \in \Psi$, from which inequality $\nu \leq_C \mu_C$ follows from Section 1.3.4(c). □

Lemma 2.3.4. (a) *For all $\overline{w}', \overline{w}'' \in \widetilde{W}^{\mathcal{C}}$, there exist admissible tuples $\overline{w}_1 \ldots, \overline{w}_n$ from $\widetilde{W}^{\mathcal{C}}$ such that the reduced intersection $\mathrm{Fl}^{\leq \overline{w}'} \cap \mathrm{Fl}^{\leq \overline{w}''}$ equals $\bigcup_{t=1}^n \mathrm{Fl}^{\leq \overline{w}_t}$.*

(b) *For all $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, $\psi \in \Psi$ and $u \in \widetilde{W}$, there exist admissible tuples $\overline{w}_1 \ldots, \overline{w}_n$ from $\widetilde{W}^{\mathcal{C}}$ such that the reduced intersection $\mathrm{Fl}^{\leq \overline{w}} \cap \mathrm{Fl}^{\leq_\psi u}$ equals $\bigcup_{t=1}^n \mathrm{Fl}^{\leq \overline{w}_t}$.*

Proof. We denote by Z the reduced intersection $\mathrm{Fl}^{\leq \overline{w}'} \cap \mathrm{Fl}^{\leq \overline{w}''}$ in the case (a), and $\mathrm{Fl}^{\leq \overline{w}} \cap \mathrm{Fl}^{\leq_\psi u}$ in the case (b). Then, by Corollary 2.1.7(c), in both cases, Z is a closed T-invariant subscheme of Fl of finite type; thus, the intersection $\widetilde{W} \cap Z$ is finite.

By Corollary 2.1.7(d), each $z \in Z$ defines an admissible tuple $\overline{u} = \overline{u}(z) \in \widetilde{W}^{\mathcal{C}}$ satisfying $u_C \in \widetilde{W} \cap Z$ for each $C \in \mathcal{C}$. It follows that the set of tuples $\{\overline{u}(z)\}_{z \in Z}$ is finite, so it will suffice to show the equality

$$Z = \bigcup_{z \in Z} \mathrm{Fl}^{\leq \overline{u}(z)}.$$

One inclusion follows from the fact that $z \in \mathrm{Fl}^{\leq \overline{u}(z)}$ for every $z \in Z$. To show the converse, it is enough to show that if $\overline{u} \in \widetilde{W}^{\mathcal{C}}$ satisfies $u_C \in Z$ for all $C \in \mathcal{C}$, then $\mathrm{Fl}^{\leq \overline{u}} \subseteq Z$. Using definition of Z, it remains to show the corresponding assertion in the cases $Z = \mathrm{Fl}^{\leq \overline{w}}$ and $Z = \mathrm{Fl}^{\leq_\psi u}$. In the first case, we have $u_C \in \mathrm{Fl}^{\leq_C w_C}$; hence, $\mathrm{Fl}^{\leq_C u_C} \subseteq \mathrm{Fl}^{\leq_C w_C}$ for all $C \in \mathcal{C}$, and thus, $\mathrm{Fl}^{\leq \overline{u}} \subseteq \mathrm{Fl}^{\leq \overline{w}}$. In the second case, the assertion follows from Lemma 2.3.2(b). □

We will need the following 'effective' version of Lemma 2.3.4.

Lemma 2.3.5. (a) *There exists $r' \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every two $(m + r')$-regular admissible tuples $\overline{w}', \overline{w}'' \in \widetilde{W}^{\mathcal{C}}$, there exist m-regular admissible tuples $\overline{w}_1 \ldots, \overline{w}_n \in \widetilde{W}^{\mathcal{C}}$ such that $\mathrm{Fl}^{\leq \overline{w}'} \cap \mathrm{Fl}^{\leq \overline{w}''} = \bigcup_{t=1}^n \mathrm{Fl}^{\leq \overline{w}_t}$.*

(b) *There exists $r' \in \mathbb{N}$ such that for every $m, d \in \mathbb{N}$, every $(m + 2d + r')$-regular admissible tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ and every $u \in \widetilde{W}$, satisfying $\langle \psi, \pi(u) \rangle = \pi(\overline{w})(\psi) - d$, there exist m-regular admissible tuples $\overline{w}_1 \ldots, \overline{w}_n \in \widetilde{W}^{\mathcal{C}}$ such that*

$$\mathrm{Fl}^{\leq \overline{w}} \cap \mathrm{Fl}^{\leq_\psi u} = \bigcup_{t=1}^n \mathrm{Fl}^{\leq \overline{w}_t}.$$

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The proof is based on the following two claims:

Claim 2.3.6. (a) For every two quasi-admissible tuples $\overline{\mu}',\overline{\mu}''\in\Lambda^{\mathcal{C}}$, there exists a unique maximal quasi-admissible tuple $\overline{\mu}\in\Lambda^{\mathcal{C}}$ such that $\overline{\mu}\leq\overline{\mu}'$ and $\overline{\mu}\leq\overline{\mu}''$. Moreover, $\overline{\mu}$ is m-regular if both $\overline{\mu}'$ and $\overline{\mu}''$ are m-regular.

(b) For every $\psi\in\Psi$, $m,d\in\mathbb{N}$ and every $(m+2d)$-regular quasi-admissible tuple $\overline{\mu}\in\Lambda^{\mathcal{C}}$, the tuple $\overline{\nu}:=\overline{\mu}-d\overline{e}_{\psi}$ is m-regular.

Proof. (a) Notice that $\overline{\mu}\leq\overline{\mu}'$ if and only if $\overline{\mu}(\psi)\leq\overline{\mu}'(\psi)$ for all $\psi\in\Psi$. Thus, a maximal $\overline{\mu}$ satisfies $\overline{\mu}(\psi)=\min\{\overline{\mu}'(\psi),\overline{\mu}''(\psi)\}$ for all $\psi\in\Psi$. This shows the first assertion.

For the second one, choose $C\in\mathcal{C}$, let $\alpha_1,\ldots,\alpha_r$ be the simple roots of C, and let $\psi_1,\ldots,\psi_r$ be the corresponding fundamental weights. We want to show that $\langle\alpha_j,\mu_C\rangle\geq m$ for all j. Without loss of generality, we may assume that $\overline{\mu}(\psi_j)=\overline{\mu}'(\psi_j)$. Recall that we have $\mu_C=\sum_{i=1}^r\overline{\mu}(\psi_i)\check{\alpha}_i$ and $\mu'_C=\sum_{i=1}^r\overline{\mu}'(\psi_i)\check{\alpha}_i$. Since $\langle\alpha_j,\check{\alpha}_j\rangle=2>0,\overline{\mu}(\psi_j)=\overline{\mu}'(\psi_j)$ and $\langle\alpha_j,\check{\alpha}_i\rangle\leq 0,\overline{\mu}(\psi_i)\leq\overline{\mu}'(\psi_i)$ for all $i\neq j$, we conclude that $\langle\alpha_j,\mu_C\rangle\geq\langle\alpha_j,\mu'_C\rangle\geq m$.

(b) Let C,α_i and ψ_i be as in the proof of part (a). Then for every j, the pairing $\langle\alpha_j,\mu_C\rangle$ equals $\langle\alpha_j,\mu_C\rangle-d\langle\alpha_j,\check{\alpha}_i\rangle\geq(m+2d)-2d=m$, if $\psi=\psi_i$, and equals $\langle\alpha_j,\mu_C\rangle\geq m+2d\geq m$, otherwise. □

Claim 2.3.7. (a) There exists $r\in\mathbb{N}$ such that for every $C\in\mathcal{C}$, every root $\alpha\in\Phi_C$ with corresponding fundamental weight $\psi\in\Psi_C$, and every elements $w,w'\in\widetilde{W}$ with $w\leq_C w'$, we have either $\check{\alpha}w\leq_C w'$ or $\langle\psi,\pi(w')-\pi(w)\rangle\leq r$.

(b) There exists $r\in\mathbb{N}$ such that for every $\psi\in\Psi$, $\alpha\in\Phi$ and $w,w'\in\widetilde{W}$ such that $w\leq_{\psi}w'$ and $\langle\psi,\alpha\rangle=1$, we have either $\check{\alpha}w\leq_{\psi}w'$ or $\langle\psi,\pi(w')-\pi(w)\rangle\leq r$.

Proof. Since W is finite, in both cases (a) and (b), it will be enough to find r to satisfy the condition for $w\in\Lambda u$ and $w'\in\Lambda u'$, where $u,u'\in W$ are fixed. Moreover, using Remark 1.2.4(a)(ii), we may assume that $w'=u'$. Similarly, we fix $C\in\mathcal{C}$, and $\alpha\in\Phi_C$ with corresponding $\psi\in\Psi_C$.

In the case (a), we consider the set S_C of all $\mu\in\Lambda$ such that $\mu u\leq_C u'$. Then, by Corollary 1.2.3(b), every $\mu\in S_C$ satisfies $\mu=\pi(\mu u)\leq_C\pi(u')=0$; hence, the set $S_C^{\max}$ of all maximal elements of S_C with respect to the ordering $\leq_C$ is finite and nonempty. We take $r\in\mathbb{N}$ to be the maximum of all $-\langle\psi,\mu\rangle$ taken over all $\mu\in S_C^{\max}$.

In the case (b), we consider the set S_{ψ} of all $\mu'\in\Lambda$ such that $\mu'u\leq_{\psi}u'$. Then every $\mu'\in S_{\psi}$ satisfies $\mu'\leq_{\psi}0$; hence, the set $S_{\psi}^{\max}$ of all maximal elements of S_{ψ} with respect to the ordering $\leq_{\psi}$ is a finite and nonempty union of cosets of $\Lambda^{\psi}:=\{\mu\in\Lambda\,|\,\langle\psi,\mu\rangle=0\}$. We take $r\in\mathbb{N}$ to be the maximum of all $-\langle\psi,\mu'\rangle$, taken over all $\mu'\in S_{\psi}^{\max}$.

Then in both cases, r satisfies the required property. Indeed, assume that $\mu\in S_C$ (resp. $\mu\in S_{\psi}$) while $\check{\alpha}\mu\notin S_C$, (resp. $\check{\alpha}\mu\notin S_C$), and we want to check that $\langle\psi,\mu\rangle\geq -r$. Choose any $\mu'\in S_C^{\max}$ (resp. $\mu'\in S_{\psi}^{\max}$) to be such that $\mu'\geq_C\mu$ (resp. $\mu'\geq_{\psi}\mu$.) Then $\mu'-\mu=\sum_{\beta\in\Delta_C}m_{\beta}\check{\beta}$ and $m_{\alpha}\geq 0$. Since $\check{\alpha}\mu\notin S_C$ (resp. $\check{\alpha}\mu\notin S_{\psi}$), we have $m_{\alpha}=0$; thus, $\langle\psi,\mu\rangle=\langle\psi,\mu'\rangle\geq -r$. □

Now we are ready to prove Lemma 2.3.5.

2.3.8. *Proof of Lemma 2.3.5.* (a) Let $r\in\mathbb{N}$ be as in Claim 2.3.7(a). We will show that $r':=2r$ satisfies the required property. Let $\overline{w}',\overline{w}''\in\widetilde{W}^{\mathcal{C}}$ be $(m+r')$-regular admissible tuples. Then, by Lemma 2.3.4, there exist admissible tuples $\overline{w}_1\ldots,\overline{w}_n\in\widetilde{W}^{\mathcal{C}}$ such that $\mathrm{Fl}^{\leq\overline{w}'}\cap\mathrm{Fl}^{\leq\overline{w}''}=\bigcup_{t=1}^n\mathrm{Fl}^{\leq\overline{w}_t}$.

Using Corollary 2.1.7(b), one can assume that each $\overline{w}_t$ is a maximal admissible tuple, satisfying $\overline{w}_t\leq\overline{w}',\overline{w}''$, and we have to show that each $\overline{w}_t$ is m-regular.

Let $\overline{w}\in\widetilde{W}^{\mathcal{C}}$ be a maximal quasi-admissible tuple, satisfying $\overline{w}\leq\overline{w}',\overline{w}''$. It is enough to show that such a $\overline{w}$ is m-regular. Indeed, Lemma 1.3.10(a) then would imply that $\overline{w}$ is admissible.

Set $\overline{\mu}':=\pi(\overline{w}')$, and $\overline{\mu}'':=\pi(\overline{w}'')$, and let $\overline{\mu}$ be the maximal tuple such that $\overline{\mu}\leq\overline{\mu}'$ and $\overline{\mu}\leq\overline{\mu}''$. Then $\overline{\mu}$ is $(m+r')$-regular by Claim 2.3.6(a), and $\pi(\overline{w})\leq\overline{\mu}$ by Corollary 1.2.3(b).

It is enough to show that $\pi(\overline{w})(\psi)\geq\overline{\mu}(\psi)-r$ for every $\psi\in\Psi$. Indeed, if this is shown, then for every $C\in\mathcal{C}$ with simple roots $\alpha_1,\ldots,\alpha_r$, we have $\pi(w_C)=\mu_C-\sum_i r_i\check{\alpha}_i$ and $0\leq r_i\leq r$ for all i. Then $\langle\alpha_i,\pi(w_C)\rangle\geq\langle\alpha_i,\mu_C\rangle-2r_i\geq(m+2r)-2r=m$. Thus, $\overline{w}$ is m-regular.

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Assume that there exists $\psi \in \Psi$ such that $\pi(\overline{w})(\psi) < \overline{\mu}(\psi) - r$. Consider the quasi-admissible tuple $\overline{e}_\psi$ defined by $\overline{e}_\psi(\psi') := \delta_{\psi,\psi'}$ (see Section 1.3.4(d)). Then the quasi-admissible tuple $\overline{e}_\psi\overline{w}$ (see Section 1.3.5(c)) satisfies identities $(\overline{e}_\psi\overline{w})_C = w_C$ if $\psi \notin \Psi_C$, and $(\overline{e}_\psi\overline{w})_C = \check{\alpha}w_C$ if $\psi \in \Psi_C$ and $\alpha \in \Delta_C$ corresponds to ψ.

Since $\overline{w} \leq \overline{w}'$ and $\overline{w} \leq \overline{w}''$, the assumption $\pi(\overline{w})(\psi) < \overline{\mu}(\psi) - r$ together with Claim 2.3.7(a) implies that $\overline{e}_\psi\overline{w} \leq \overline{w}'$ and $\overline{e}_\psi\overline{w} \leq \overline{w}''$. Since $\overline{w} < \overline{e}_\psi\overline{w}$, this contradicts the maximality of $\overline{w}$.

(b) The proof is similar to that of part (a). Let $r \in \mathbb{N}$ to satisfy both Claim 2.3.7(a),(b), and set $r' := 2r$. Assume that $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ is $(m + 2d + r')$-regular, $u \in \widetilde{W}$ satisfies $\langle \psi, \pi(u)\rangle = \pi(\overline{w})(\psi) - d$, and let $\overline{w}'$ be a maximal quasi-admissible tuple satisfying $\overline{w}' \leq \overline{w}$ and $\overline{w}'_\psi \leq_\psi u$.

Using Lemma 2.3.4(b) and Lemma 2.3.2(b), and arguing as in part (a), it is enough to show that $\pi(\overline{w})(\psi') \geq \overline{\mu}(\psi') - r$ for every $\psi' \in \Psi$.

Assume that there exists $\psi' \in \Psi$ such that $\pi(\overline{w})(\psi') < \overline{\mu}(\psi') - r$, and let $\overline{e}_{\psi'}\overline{w}$ be as in part (a). Again, to get a contradiction, it is enough to show that $\overline{e}_{\psi'}\overline{w} \leq \overline{w}'$ and $(\overline{e}_{\psi'}\overline{w})_\psi \leq_\psi u$. The proof of the first inequality is identical to that of part (a). Next, if $\psi' \neq \psi$, then $(\overline{e}_{\psi'}\overline{w})_\psi = \overline{w}_\psi \leq_\psi u$ by assumption. Finally, if $\psi' = \psi$, the inequality $(\overline{e}_{\psi'}\overline{w})_\psi \leq_\psi u$ follows from Claim 2.3.7(b). □

Lemma 2.3.9. *There exists $r \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every $(m + r)$-regular $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, there exists an m-regular $\overline{x} \in \Lambda^{\mathcal{C}}$ such that $\mathrm{Fl}^{\leq'\overline{x}} \subseteq \mathrm{Fl}^{\leq\overline{w}}$.*

Proof. Choose any $\mu \in \Lambda \cap C_0$, let $\overline{\mu} \in \Lambda^{\mathcal{C}}$ be the admissible tuple defined by $\mu_{u(C_0)} := u(\mu)$ (see Section 1.3.5(a)), and let r be the maximum of the $\langle \alpha, \mu\rangle$'s, where α runs over all of Δ_{C_0}. We claim that this r satisfies the required property.

Namely, to every $(m + r)$-regular admissible tuple $\overline{w} \in \widetilde{W}^{\mathcal{C}}$, we associate a quasi-admissible tuple $\overline{x} := \overline{\mu}^{-1}\pi(\overline{w})$ (see Section 1.3.4(b)). We claim that $\overline{x}$ is m-regular, and $\mathrm{Fl}^{\leq'\overline{x}} \subseteq \mathrm{Fl}^{\leq\overline{w}}$.

To show that $\overline{x}$ is m-regular, we note that for every $u \in W$, $C = u(C_0) \in \mathcal{C}$ and $\alpha \in \Delta_C$, we have $\langle \alpha, x_C\rangle = \langle \alpha, \pi(w_C)\rangle - \langle \alpha, u(\mu)\rangle \geq (m + r) - r = m$.

Next, we observe that $\mathrm{Fl}^{\leq'\overline{x}} = \mathrm{Fl}^{\leq\overline{x}\cdot\overline{w}_{\mathrm{st}}}$ (use Corollary 2.1.7(f)). So it remains to show that $\overline{x}\cdot\overline{w}_{\mathrm{st}} \leq \overline{w}$ or, what is the same, $x_C u \leq_C w_C$ for each $C = u(C_0) \in \mathcal{C}$. Unwinding the definitions and using Section 1.2.4(a), it is enough to show that for every $u \in W$, we have $1 \leq_{C_0} \mu u$. By Corollary 1.2.6, it remains to show that $\mu u \in \widetilde{W} \cap C_0$; that is, for every $\alpha \in \Phi_{C_0}$ we have $(\mu u)^{-1}(\alpha) > 0$. But $(\mu u)^{-1}(\alpha) = u^{-1}(\mu^{-1}(\alpha)) = (u^{-1}(\alpha), \langle \alpha, \mu\rangle) > 0$ because $\langle \alpha, \mu\rangle > 0$. □

Lemma 2.3.10. *There exists $r \in \mathbb{N}$ such that for every $m \in \mathbb{Z}$ and every $(m + r)$-regular tuple $\overline{x} \in \Lambda^{\mathcal{C}}$, there exists a sequence $\overline{x} = \overline{x}_0 \leq \overline{x}_1 \leq \ldots$ in $\Lambda^{\mathcal{C}}$ such that sequence $\{\overline{x}_i(\psi)\}_i$ tends to infinity for all $\psi \in \Psi$, each $\overline{x}_i$ is m-regular, and $\overline{x}_i = \overline{x}_{i-1} + \overline{e}_{\psi_i}$ for some $\psi_i \in \Psi$ and all i.*

Proof. Choose $\mu \in \Lambda \cap C_0$, and let $\overline{\mu} \in \Lambda^{\mathcal{C}}$ be the tuple $\mu_{u(C_0)} := u(\mu)$ from Section 1.3.5(a). Then $\overline{\mu}$ is regular and admissible. Let $\overline{y} \in \mathbb{N}^\Psi$ be the corresponding tuple (see Section 1.3.4(a) and Lemma 1.3.10(b)). Choose a sequence $\overline{y}_0 = 0, \overline{y}_1, \ldots, \overline{y}_n = \overline{y}$ in $\mathbb{N}^\Psi$ such that $\overline{y}_i - \overline{y}_{i-1} = \overline{e}_{\psi_i}$ for all i and some $\psi_i \in \Psi$, and continue it to all i by the rule $\overline{y}_{i+n} := \overline{y}_i + \overline{y}$.

Define r to be the maximum of the $-\langle (y_i)_C, \alpha\rangle$'s, taken over $i = 1, \ldots, n$, $C \in \mathcal{C}$ and $\alpha \in \Delta_C$. Then the sequence $\overline{x}_i := \overline{x} + \overline{y}_i$ satisfies the required property. □

2.4. *Stratification of the affine flag variety*

Notation 2.4.1. (a) Let k, K and $\mathcal{O}$ be as in Section 2.1.1, let G be a connected reductive group over k, and let $T \subseteq G$ be a maximal torus.

(b) Let G^{sc} be the simply connected covering of the derived group of G, and let $T_{G^{\mathrm{sc}}} \subseteq G^{\mathrm{sc}}$ be the corresponding maximal torus – that is, the pullback of $T \subseteq G$. Let Φ be the root system $\Phi(G, T) = \Phi(G^{\mathrm{sc}}, T_{G^{\mathrm{sc}}})$ of G^{sc}, let Ψ be the set of fundamental weights of G^{sc}, and let $\widetilde{W}$ be the affine Weyl group of G^{sc}.

(c) Choose an Iwahori subgroup scheme $I \subseteq LG$ as in Section 2.1.2, set $I^{\mathrm{sc}} := I \cap L(G^{\mathrm{sc}}) \subseteq L(G^{\mathrm{sc}})$, and let $\mathrm{Fl} = \mathrm{Fl}_{G^{\mathrm{sc}}} := L(G^{\mathrm{sc}})/I^{\mathrm{sc}}$ be the affine flag variety of G^{sc}.

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Notation 2.4.2. In the situation of Section 2.4.1, fix $\psi \in \Psi \subseteq X^*(T_{G^{sc}})$.

(a) Let $P_\psi, M_\psi, U_\psi, M_\psi^{sc}$ and P_ψ^{sc} be as in Section 2.3.1(b). Notice that groups M_ψ^{sc}, U_ψ and P_ψ^{sc} would not change if we replace group G by G^{sc}.

(b) Note that $I_{M_\psi} := I \cap L(M_\psi) \subseteq L(M_\psi)$ is an Iwahori subgroup scheme, let $I_{M_\psi^{sc}} \subseteq L(M_\psi^{sc})$ be the preimage of $I_{M_\psi} \subseteq L(M_\psi)$, and set $\mathrm{Fl}_{M_\psi^{sc}} := L(M_\psi^{sc})/I_{M_\psi^{sc}}$.

(c) As in Section 2.3.1(b), we have a natural homomorphism $P_\psi^{sc} \to G^{sc}$; thus, the loop group $L(P_\psi^{sc})$ acts on Fl. For every $w \in \widetilde{W}$, we denote by $\mathrm{Fl}^{\leq_\psi w} \subseteq \mathrm{Fl}$ the closure of the $L(P_\psi^{sc})$-orbit $L(P_\psi^{sc})w \subseteq \mathrm{Fl}$.

(d) As in Section 2.1.1, we have an equality $\Lambda = X_*(T_{G^{sc}})$. As in Section 1.1.3(b), the coweight $\check{\psi}$ belongs to $\Lambda_{\mathbb{Q}}$. We denote by $T_\psi \subseteq T_{G^{sc}}$ the one-dimensional subtorus such that $X_*(T_\psi) \subseteq \Lambda$ equals $(\mathbb{Z}\check{\psi}) \cap \Lambda \subseteq \Lambda_{\mathbb{Q}}$.

(e) Alternatively, T_ψ can be defined as the connected center of the Levi subgroup $(M_\psi)_{G^{sc}}$ of G^{sc}, where $(M_\psi)_{G^{sc}} \subseteq G^{sc}$ is the pullback of $M_\psi \subseteq G$.

2.4.3. Stratification

(a) For each $\nu \in \widetilde{W}_\psi$, we set $Z_\nu := \mathrm{Fl}^{\leq_\psi \nu} \setminus \bigcup_{\nu' <_\psi \nu} \mathrm{Fl}^{\leq_\psi \nu'}$. Then each $Z_\nu \subseteq \mathrm{Fl}$ is a reduced locally closed $L(P_\psi^{sc})$-invariant ind-subscheme. Moreover, since $\widetilde{W}_\psi$ is a set of representatives of the set of cosets $\widetilde{W}^\psi \backslash \widetilde{W}$ (see Section 1.1.3(d)), the set $\{Z_\nu\}_{\nu \in \widetilde{W}_\psi}$ forms a stratification of Fl.

(b) For each $\nu \in \widetilde{W}_\psi$, we consider $I_\nu := \nu I \nu^{-1} \subseteq LG$, $I_{P_\psi,\nu} := I_\nu \cap L(P_\psi) \subseteq L(P_\psi)$ and $I_{U_\psi,\nu} := I_\nu \cap L(U_\psi) \subseteq L(U_\psi)$. Let $I_{P_\psi^{sc},\nu} \subseteq L(P_\psi^{sc})$ be the preimage of $I_{P_\psi,\nu} \subseteq L(P_\psi)$, and set $\mathrm{Fl}_{P_\psi^{sc},\nu} := L(P_\psi^{sc})/I_{P_\psi^{sc},\nu}$.

(c) Note that for each $\nu \in \widetilde{W}_\psi$, we have an equality $I_{M_\psi} = I_\nu \cap L(M_\psi) \subseteq L(M_\psi)$. Therefore, isomorphism $U_\psi \times M_\psi \xrightarrow{\sim} P_\psi : (u,m) \mapsto um$ induces isomorphisms

$$I_{U_\psi,\nu} \times I_{M_\psi} \xrightarrow{\sim} I_{P_\psi,\nu}, U_\psi \times M_\psi^{sc} \xrightarrow{\sim} P_\psi^{sc} \text{ and } I_{U_\psi,\nu} \times I_{M_\psi^{sc}} \xrightarrow{\sim} I_{P_\psi^{sc},\nu}.$$

Moreover, the embedding and the projection $M_\psi^{sc} \to P_\psi^{sc} \to M_\psi^{sc}$ induce morphisms

$$\mathrm{Fl}_{M_\psi^{sc}} \xrightarrow{i_{\psi,\nu}} \mathrm{Fl}_{P_\psi^{sc},\nu} \xrightarrow{p_{\psi,\nu}} \mathrm{Fl}_{M_\psi^{sc}}.$$

(d) By Lemma 2.3.2(a), each $Z_\nu \subseteq \mathrm{Fl}$ is an $L(P_\psi^{sc})$-orbit of $\nu \in \mathrm{Fl}$. Moreover, the group ind-scheme $L(P_\psi^{sc}) \simeq L(M_\psi^{sc}) \times L(U_\psi)$ is reduced (see [BD] if k is of characteristic zero, and [Fa] in general), so the morphism $[h] \mapsto h\nu$ induces an isomorphism $\iota_\nu : \mathrm{Fl}_{P_\psi^{sc},\nu} \xrightarrow{\sim} Z_\nu$.

(e) Since $T_{G^{sc}}$ normalizes P_ψ^{sc} and fixes $\nu \in \mathrm{Fl}$, the orbit $Z_\nu \subseteq \mathrm{Fl}$ is $T_{G^{sc}}$-invariant; hence, Z_ν is T_ψ-equivariant. Furthermore, the isomorphism ι_ν of part (d) identifies the T_ψ-action on Z_ν with the T_ψ-action on $\mathrm{Fl}_{P_\psi^{sc},\nu}$ given by the formula $t[um] = [tut^{-1}m]$ for $u \in L(U_\psi)$ and $m \in L(M_\psi^{sc})$. In particular, the isomorphism ι_ν induces an isomorphism $\iota_\nu^{T_\psi} : \mathrm{Fl}_{M_\psi^{sc}} \xrightarrow{\sim} Z_\nu^{T_\psi} : [m] \mapsto m\nu$, where $Z_\nu^{T_\psi}$ denotes the locus of T_ψ-fixed points.

(f) Since $\mathrm{Fl}_{M_\psi^{sc}}$ is ind-proper, we conclude that $\mathrm{Fl}_{M_\psi^{sc}} \subseteq \mathrm{Fl}$ is closed. So it follows from part (e) that each ind-subscheme $Z_\nu^{T_\psi} \subseteq \mathrm{Fl}^{T_\psi}$ is closed. Moreover, Fl^{T_ψ} is reduced because Fl is such. Since set-theoretically Fl^{T_ψ} decomposes as a disjoint union $\bigsqcup_{\nu \in \widetilde{W}_\psi} Z_\nu^{T_\psi}$, we conclude that each $Z_\nu^{T_\psi} \subseteq \mathrm{Fl}^{T_\psi}$ is open and closed.

2.4.4. Retraction. Let Y be an ind-scheme, and let $Z \subseteq Y$ be a locally closed ind-subscheme. A morphism $p : Y \to Z$ is called a *retraction* if the restriction $p|_Z$ is the identity.

Lemma 2.4.5. *For every $\nu \in \widetilde{W}_\psi$, there is a unique T_ψ-equivariant retraction $p_\nu : Z_\nu \to Z_\nu^{T_\psi}$. Moreover, under an isomorphisms of Sections 2.4.3(d)–(f), the retraction p_ν corresponds to the projection $p_{\psi,\nu} : \mathrm{Fl}_{P_\psi^{sc},\nu} \to \mathrm{Fl}_{M_\psi^{sc}} : [um] \mapsto [m]$.*

Proof. To see the existence of a retraction and its relation to $p_{\psi,\nu}$, we note that ι_ν induces an isomorphism $\mathrm{Fl}_{M^{\mathrm{sc}}_\psi,\nu}=\mathrm{Fl}^{T_\psi}_{P^{\mathrm{sc}}_\psi,\nu}\xrightarrow{\sim}Z^{T_\psi}_\nu$, and that $p_{\psi,\nu}:\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu}\to\mathrm{Fl}_{M^{\mathrm{sc}}_\psi,\nu}$ is a T_ψ-equivariant retract. To see the uniqueness, we note that for every S-point $\eta:S\to\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu}$, the morphism $\eta_{\mathbb{G}_m}:\mathbb{G}_m\times S\to\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu}$, defined by $(a,x)\mapsto\psi(a)\eta(s)$, extends uniquely to the morphism $\eta_{\mathbb{A}^1}:\mathbb{A}^1\times S\to\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu}$, and we have an equality $p_{\psi,\nu}(\eta)=\eta_{\mathbb{A}^1}|_{\{0\}\times S}$. □

3. Affine Springer fibers

3.1. Geometric properties

Assume that we are in the situation Section 2.4.1.

3.1.1. Set-up. (a) Let $\gamma\in G(K)$ be a compact regular semi-simple element, and let $G^{\mathrm{sc}}_\gamma\subseteq G^{\mathrm{sc}}$ be the centralizer of γ inside G^{sc}. In particular, $(G^{\mathrm{sc}}_\gamma)^0\subseteq G^{\mathrm{sc}}$ is a maximal torus defined over K.

(b) Let $S_\gamma\subseteq G^{\mathrm{sc}}_\gamma$ be the maximal K-split torus of G^{sc}_γ, and let $\Lambda_\gamma:=X_*(S_\gamma)$ be the group of cocharacters. The map $\mu\mapsto\mu(t)$ identifies Λ_γ with a subgroup of $S_\gamma(K)$.

(c) Let $\mathrm{Fl}_\gamma\subseteq\mathrm{Fl}$ be the affine Springer fiber. Explicitly, Fl_γ consists of cosets $gI^{\mathrm{sc}}\in L(G^{\mathrm{sc}})/I^{\mathrm{sc}}$ such that $g^{-1}\gamma g\in I$. Then the group Λ_γ acts on Fl_γ. Moreover, it is known that the reduced ind-scheme $\mathrm{Fl}_{\gamma,\mathrm{red}}$ is a scheme of finite type over k, and there exists a closed reduced subscheme $Y\subseteq\mathrm{Fl}_\gamma$ of finite type over k such that $\mathrm{Fl}_{\gamma,\mathrm{red}}=\Lambda_\gamma(Y)$.

(d) For every ind-subscheme $Z\subseteq\mathrm{Fl}$, we set $Z_\gamma:=Z\cap\mathrm{Fl}_\gamma$.

(e) **Main assumption:** We always assume that we have an inclusion $S_\gamma\subseteq T_{G^{\mathrm{sc}}}$, and hence an inclusion $\Lambda_\gamma\subseteq\Lambda=X_*(T_{G^{\mathrm{sc}}})$.

Remark 3.1.2. Note that it follows from [St, Theorem 8.2] (or its particular case [St, Corollary 8.5]) that the centralizer G^{sc}_γ is connected. However, we do not need this fact.

Lemma 3.1.3. *Suppose that we are in the situation of Section 3.1.1. Then the centralizer $G^{\mathrm{sc}}_{S_\gamma}\subseteq G^{\mathrm{sc}}$ is a Levi subgroup, and S_γ is the connected center of $G^{\mathrm{sc}}_{S_\gamma}$.*

Proof. Indeed, the centralizer $G^{\mathrm{sc}}_{S_\gamma}$ is split over K because G^{sc} and S_γ are split over K; therefore, the connected center $Z(G^{\mathrm{sc}}_{S_\gamma})^0$ of $G^{\mathrm{sc}}_{S_\gamma}$ is split over K as well. Moreover, since $(G^{\mathrm{sc}}_\gamma)^0$ is a maximal torus of G^{sc}, it is a maximal torus of $G^{\mathrm{sc}}_{S_\gamma}$, and hence contains $Z(G^{\mathrm{sc}}_{S_\gamma})^0$. Therefore, the assertion follows from the assumption that $S_\gamma\subseteq(G^{\mathrm{sc}}_\gamma)^0$ is the maximal K-split torus. □

3.1.4. Observations. Fix $\psi\in\Psi$.

(a) An inclusion $(G_\gamma)^0\subseteq M_\psi$ is equivalent to the inclusion $(G^{\mathrm{sc}}_\gamma)^0\subseteq(M_\psi)_{G^{\mathrm{sc}}}$, hence to the inclusion $T_\psi\subseteq(G^{\mathrm{sc}}_\gamma)^0$ (by Section 2.4.2(e)), and thus to the inclusion $T_\psi\subseteq S_\gamma$.

(b) Set $(\Lambda_\gamma)_{\mathbb{Q}}:=\Lambda_\gamma\otimes_{\mathbb{Z}}\mathbb{Q}$. By Section 2.4.2(d), an inclusion $T_\psi\subseteq S_\gamma$ holds if and only if $\check{\psi}\in(\Lambda_\gamma)_{\mathbb{Q}}$.

(c) It follows from parts (a) and (b) that if $\check{\psi}\in(\Lambda_\gamma)_{\mathbb{Q}}$, then element γ belongs to

$$(G_\gamma)^0(K)\subseteq M_\psi(K)\subseteq P_\psi(K).$$

(d) It follows from Lemma 3.1.3 that if $\check{\psi}\notin(\Lambda_\gamma)_{\mathbb{Q}}$, then there exists a root $\alpha\in\Phi$ such that $\alpha\in(\Lambda_\gamma)^\perp$, but $\langle\alpha,\check{\psi}\rangle\neq 0$.

Notation 3.1.5. Assume that $\psi\in\Psi$ satisfies $\check{\psi}\in(\Lambda_\gamma)_{\mathbb{Q}}$.

(a) By Section 3.1.4(c), we have $\gamma\in M_\psi(K)\subseteq P_\psi(K)$, and thus, we can consider the affine Springer fibers $\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu,\gamma}\subseteq\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu}$ and $\mathrm{Fl}_{M^{\mathrm{sc}}_\psi,\gamma}\subseteq\mathrm{Fl}_{M^{\mathrm{sc}}_\psi}$. Explicitly, $\mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu,\gamma}$ (resp. $\mathrm{Fl}_{M^{\mathrm{sc}}_\psi,\gamma}$) consists of all elements $gI_{P^{\mathrm{sc}}_\psi,\nu}\in L(P^{\mathrm{sc}}_\psi)/I_{P^{\mathrm{sc}}_\psi,\nu}$ (resp. $gI_{M^{\mathrm{sc}}_\psi}\in L(M^{\mathrm{sc}}_\psi)/I_{M^{\mathrm{sc}}_\psi}$) such that $g^{-1}\gamma g\in I_{P_\psi,\nu}$ (resp. $g^{-1}\gamma g\in I_{M_\psi}$).

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(b) By construction, the isomorphism $\iota_\nu : \mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu} \xrightarrow{\sim} Z_\nu$ from Section 2.4.3(e) restricts to isomorphisms $\iota_{\nu,\gamma} : \mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu,\gamma} \xrightarrow{\sim} Z_{\nu,\gamma}$ and $\iota^{T_\psi}_{\nu,\gamma} : \mathrm{Fl}_{M^{\mathrm{sc}}_\psi} \xrightarrow{\sim} Z^{T_\psi}_{\nu,\gamma}$.

3.1.6. Affine bundle. A morphism $f : X \to Y$ of (ind-)schemes is called an *affine bundle* if locally étale on Y it is isomorphic to the projection $Y \times \mathbb{A}^n \to Y$ and all transition maps are affine.

Proposition 3.1.7. *Assume that we are in the situation of Section 3.1.5. Then for every $\nu \in \widetilde{W}_\psi$, the T_ψ-equivariant retraction $p_\nu : Z_\nu \to Z^{T_\psi}_\nu$ of Lemma 2.4.5 induces a retraction $p_{\nu,\gamma} : Z_{\nu,\gamma} \to Z^{T_\psi}_{\nu,\gamma}$. Furthermore, $p_{\nu,\gamma}$ is a composition of affine bundles.*

Proof. To make the argument more structural, we will divide it into steps.

Step 1. By Lemma 2.4.5 and the observations of Section 3.1.5, it suffices to show that the projection $p_{\psi,\nu} : \mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu} \to \mathrm{Fl}_{M^{\mathrm{sc}}_\psi}$ restricts to the projection

$$p_{\psi,\nu,\gamma} : \mathrm{Fl}_{P^{\mathrm{sc}}_\psi,\nu,\gamma} \to \mathrm{Fl}_{M^{\mathrm{sc}}_\psi,\gamma},$$

and that $p_{\psi,\nu,\gamma}$ is a composition of affine bundles.

Step 2. Let $U_\psi = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_{n-1} \supseteq U_n = \{1\}$ be the lower central series of U_ψ. Then each U_i is a normal subgroup of P_ψ, and we set $P_i := P_\psi/U_i$ and $P^{\mathrm{sc}}_i := P^{\mathrm{sc}}_\psi/U_i$. In particular, $P_0 = M_\psi$ and $P_n = P_\psi$.

For every $i = 0, \ldots, n$, let $\gamma_i \in L(P_i)$ be the image of $\gamma \in L(P_\psi)$, and denote by $I_{P_i,\nu} \subseteq L(P_i)$ (resp. $I_{P^{\mathrm{sc}}_i,\nu} \subseteq L(P^{\mathrm{sc}}_i)$) the image of $I_{P_\psi,\nu}$ (resp. $I_{P^{\mathrm{sc}}_\psi,\nu}$). We set $\mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu} := L(P^{\mathrm{sc}}_i)/I_{P^{\mathrm{sc}}_i,\nu}$, and denote by $\mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu,\gamma} \subseteq \mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu}$ the corresponding affine Springer fiber – that is, the collection of all $g \in L(P^{\mathrm{sc}}_i)/I_{P^{\mathrm{sc}}_i,\nu}$ such that $g^{-1}\gamma_i g \in I_{P_i,\nu}$.

For every $i = 0, \ldots, n-1$, we have a natural projection

$$p_{i,\gamma} : \mathrm{Fl}_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}} \to \mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu,\gamma_i},$$

and it remains to show that each $p_{i,\gamma}$ is an affine bundle.

Step 3. Let $\widetilde{\mathrm{Fl}}_{P^{\mathrm{sc}}_i,\nu,\gamma_i} \subseteq L(P^{\mathrm{sc}}_i)$ be the preimage of $\mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu,\gamma_i} \subseteq \mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu}$ under the natural projection $L(P^{\mathrm{sc}}_i) \to L(P^{\mathrm{sc}}_i)/I_{P^{\mathrm{sc}}_i,\nu}$, and set

$$\widetilde{\mathrm{Fl}}'_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}} := \widetilde{\mathrm{Fl}}_{P^{\mathrm{sc}}_i,\nu,\gamma_i} \times_{\mathrm{Fl}_{P^{\mathrm{sc}}_i,\nu,\gamma_i}} \mathrm{Fl}_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}}.$$

It is enough to show that each projection $\widetilde{\mathrm{Fl}}'_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}} \to \widetilde{\mathrm{Fl}}_{P^{\mathrm{sc}}_i,\nu,\gamma_i}$ is an affine bundle.

We set $\overline{U}_i := U_i/U_{i+1}$. Then $\overline{U}_i \subseteq P_{i+1} = P_\psi/U_{i+1}$ is a normal subgroup, and we have $P_i \cong P_{i+1}/\overline{U}_i$. Set $I_{\overline{U}_i,\nu} := I_{P_{i+1},\nu} \cap L(\overline{U}_i)$. Then $\widetilde{\mathrm{Fl}}'_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}}$ can be identified with the locus of all $g \in L(P^{\mathrm{sc}}_{i+1})/I_{\overline{U}_i,\nu}$ such that $g^{-1}\gamma_{i+1}g \in I_{P_{i+1},\nu}$.

Step 4. Recall that the projection $p_i : P_{i+1} \to P_i$, viewed as a morphism of algebraic varieties, has a section s. Indeed, the isomorphism $P_\psi \xrightarrow{\sim} M_\psi \times U_\psi$ from Section 2.4.3(c) induces an isomorphism $P_i \xrightarrow{\sim} M_\psi \times (U_0/U_i)$. Choose an ordering of the all roots of G lying in $\mathrm{Lie}\, U_0/\mathrm{Lie}\, U_i$. Then the map $(x_\alpha)_\alpha \mapsto \prod_\alpha x_\alpha$ defines an isomorphism $\prod_\alpha U_\alpha \xrightarrow{\sim} U_0/U_i$, where U_α is the root space of α. We define s to be the composition

$$P_i \xrightarrow{\sim} M_\psi \times (U_0/U_i) \xrightarrow{\sim} M_\psi \times \prod_\alpha U_\alpha \hookrightarrow M_\psi \times (U_0/U_{i+1}) \xrightarrow{\sim} P_{i+1}.$$

By construction, we have $s(P^{\mathrm{sc}}_i) \subseteq P^{\mathrm{sc}}_{i+1}$, so using s, we identify $\widetilde{\mathrm{Fl}}'_{P^{\mathrm{sc}}_{i+1},\nu,\gamma_{i+1}}$ with the space of pairs

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(g,u), where $g \in L(P_i^{\mathrm{sc}})$ and $u \in L(\overline{U}_i)/I_{\overline{U}_i,\nu}$, satisfying

$$(s(g)u)^{-1}\gamma_{i+1}(s(g)u) \in I_{P_{i+1},\nu}. \tag{3.1}$$

Moreover, equation (3.1) implies that $g^{-1}\gamma_i g \in I_{P_i,\nu}$, and thus $g \in \widetilde{\mathrm{Fl}}_{P_i^{\mathrm{sc}},\nu,\gamma_i} \subseteq L(P_i^{\mathrm{sc}})$.
Step 5. For each $g \in \widetilde{\mathrm{Fl}}_{P_i^{\mathrm{sc}},\nu,\gamma_i}$, we set $\widetilde{g} := s(g)^{-1}\gamma_{i+1}s(g) \in L(P_{i+1}^{\mathrm{sc}})$. Then $p_i(\widetilde{g}) = g^{-1}\gamma_i g \in I_{P_i,\nu}$, so there exists a unique $u_g \in L(\overline{U}_i)$ such that $\widetilde{g} = u_g^{-1}s(g^{-1}\gamma_i g)$. Hence, we have an equality

$$(s(g)u)^{-1}\gamma_{i+1}(s(g)u) = u^{-1}\widetilde{g}u = u^{-1}(\widetilde{g}u\widetilde{g}^{-1})u_g^{-1}s(g^{-1}\gamma_i g).$$

Let $\widetilde{m} \in I_{M_\psi,\nu} \subseteq L(M_\psi)$ be the image of $g^{-1}\gamma_i g \in I_{P_i,\nu}$. Since $\overline{U}_i$ lies in the center of U_0/U_{i+1}, we have $\widetilde{g}u\widetilde{g}^{-1} = \widetilde{m}u\widetilde{m}^{-1}$. Moreover, since $g \in \widetilde{\mathrm{Fl}}_{P_i^{\mathrm{sc}},\nu,\gamma_i}$, we get that $g^{-1}\gamma_i g \in I_{P_i,\nu}$. Hence, by our construction of s, we have $s(g^{-1}\gamma_i g) \in I_{P_{i+1},\nu}$, and thus, our condition (3.1) can be rewritten as

$$u^{-1}(\widetilde{m}u\widetilde{m}^{-1}) \in u_g I_{\overline{U}_i,\nu}.$$

Step 6. Since $\overline{U}_i$ is abelian, we have a canonical isomorphism $\overline{U}_i \xrightarrow{\sim} \mathrm{Lie}\,\overline{U}_i$. Therefore, each $u_g \in L(\overline{U}_i)$ gives rise to an element $n_g \in \mathrm{Lie}\,L(\overline{U}_i)$, and $\widetilde{\mathrm{Fl}}'_{P_{i+1}^{\mathrm{sc}},\nu,\gamma_{i+1}}$ is identified with the moduli space of pairs (g,n), consisting of $g \in \widetilde{\mathrm{Fl}}_{P_i^{\mathrm{sc}},\nu,\gamma_i}$ and $n \in \mathrm{Lie}\,L(\overline{U}_i)/\mathrm{Lie}\,I_{\overline{U}_i,\nu}$ such that

$$(\mathrm{Ad}\,\widetilde{m} - 1)(n) \in n_g + \mathrm{Lie}\,I_{\overline{U}_i,\nu}. \tag{3.2}$$

Step 7. Since $\gamma \in M_\psi(K) \subseteq G(K)$ is regular semisimple, the operator $\mathrm{Ad}\,\gamma - 1$ is invertible on $\mathrm{Lie}\,\overline{U}_i(K)$, and we set $d := \mathrm{val}\det(\mathrm{Ad}\,\gamma - 1, \mathrm{Lie}\,\overline{U}_i(K))$. Since each $\widetilde{m}$ is an $M_\psi(K)$-conjugate of γ, we conclude that the valuation of determinant of $\mathrm{Ad}\,\widetilde{m} - 1$ on $\mathrm{Lie}\,\overline{U}_i(K)$ is d; thus, the linear transformation of $\mathrm{Lie}\,L(\overline{U}_i)/\mathrm{Lie}\,I_{\overline{U}_i,\nu}$, induced by $\mathrm{Ad}\,\widetilde{m} - 1$, has a kernel of dimension d. Hence, equation (3.2) implies that $\widetilde{\mathrm{Fl}}'_{P_{i+1}^{\mathrm{sc}},\nu,\gamma_{i+1}}$ is an affine subbundle of $\widetilde{\mathrm{Fl}}_{P_i^{\mathrm{sc}},\nu,\gamma_i} \times (\mathrm{Lie}\,L(\overline{U}_i)/\mathrm{Lie}\,I_{\overline{U}_i,\nu})$ of dimension d. □

Proposition 3.1.8. *Assume that we are in the situation of Section 3.1.5. Let $\overline{w} \in \widetilde{W}^{\mathcal{C}}$ be an admissible tuple, $\psi \in \Psi$, $\nu := \overline{w}_\psi \in \widetilde{W}_\psi$, and let $Z_\nu \subseteq \mathrm{Fl}$ as in Section 2.4.3(a). Then exists $m \in \mathbb{N}$ such that if $\overline{w}$ is m-regular, then*

(a) *the reduced intersections $Z_\nu \cap \mathrm{Fl}_\gamma^{\leq \overline{w}}$ and $Z_\nu \cap \left(\bigcap_{\psi \in C} \mathrm{Fl}_\gamma^{\leq_C w_C}\right)$ are equal;*

(b) *the isomorphism $Z_\nu^{T_\psi} \xrightarrow{\sim} \mathrm{Fl}_{M_\psi^{\mathrm{sc}}}$ from Sections 2.4.3(e),(f) induces an isomorphism between the reduced intersection $Z_\nu^{T_\psi} \cap \mathrm{Fl}_\gamma^{\leq \overline{w}}$ and $\mathrm{Fl}_{M_\psi^{\mathrm{sc}},\gamma}^{\leq \overline{w}_\psi}$ (see Section 1.3.9(e));*

(c) *we have an inclusion of sets $p_\nu^{-1}(p_\nu(\mathrm{Fl}^{\leq \overline{w}} \cap Z_{\nu,\gamma})) \subseteq \mathrm{Fl}^{\leq \overline{w}}$.*

Proof. (a) Let $Y \subseteq \mathrm{Fl}_\gamma$ be a closed subscheme of finite type such that $\mathrm{Fl}_\gamma = \Lambda_\gamma(Y)$ (see Section 3.1.1(c)). Then using, for example, Corollary 2.1.7(d),(e), there exists a finite stratification $Y = \bigcup_j Y_j$ such that for every j, there exists a tuple $\overline{u} \in \widetilde{W}^{\mathcal{C}}$ such that $Y_j \subseteq L(U_C)u_C$ for each $C \in \mathcal{C}$.

Thus, it is enough to show that $Z_\nu \cap (\Lambda_\gamma(Y_j))^{\leq \overline{w}}$ equals $Z_\nu \cap \left(\bigcap_{C \ni \psi} \Lambda_\gamma(Y_j)^{\leq_C w_C}\right)$ for each j. In other words, we have to show that for every $\mu \in \Lambda_\gamma \subseteq \Lambda$ and $y \in Y_j$ such that $\mu(y) \in \mathrm{Fl}^{\leq_C w_C}$ for every $C \ni \psi$ and $\mu(y) \in Z_\nu$, we have $\mu(y) \in \mathrm{Fl}^{\leq_C w_C}$ for every $C \in \mathcal{C}$.

Let $\overline{u}$ be a tuple of elements of $\widetilde{W}$ such that $y \in L(U_C)u_C$ for every C. Then $y \in \mathrm{Fl}^{\leq \overline{u}}$, and it follows from Corollary 2.1.7(d) that $\overline{u}$ is admissible. By the assumption, for every $C \ni \psi$, we have $\mu u_C \leq_C w_C$ and also $(\mu\overline{u})_\psi = \nu = \overline{w}_\psi$.

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By Lemma 1.2.7(a), this implies that $\mu u_C \leq_{C^\psi} w_C$ for every $C \ni \psi$. Hence, it follows from Lemma 1.3.12 that if $\overline{w}$ is sufficiently regular, then $\mu u_C \leq_C w_C$ for every $C \in \mathcal{C}$; thus, $\mu(y) \in \bigcap_{C\in\mathcal{C}} \mathrm{Fl}^{\leq_C w_C}$, as claimed.

(b) By part (a), the reduced intersections $Z_\nu^{T_\psi} \cap \mathrm{Fl}_\gamma^{\leq \overline{w}}$ and $Z_\nu^{T_\psi} \cap \left(\bigcap_{C\ni\psi} \mathrm{Fl}_\gamma^{\leq_C w_C}\right)$ are equal. Therefore, it suffices to show that the isomorphism $Z_\nu^{T_\psi} \xrightarrow{\sim} \mathrm{Fl}_{M_\psi^{\mathrm{sc}}}$ induces an isomorphism between the reduced intersection $Z_\nu^{T_\psi} \cap \mathrm{Fl}^{\leq_C w_C}$ and $\mathrm{Fl}_{M_\psi^{\mathrm{sc}}}^{\leq_{C^\psi} w_{C^\psi}}$ for all $C \ni \psi$. Since $\mathrm{Fl}^{\leq_C w_C}$ is a closed $L(U_C)$-invariant ind-subscheme of Fl, we conclude that a closed ind-subscheme $Z_\nu^{T_\psi} \cap \mathrm{Fl}^{\leq_C w_C}$ of $Z_\nu^{T_\psi}$ corresponds to a closed $L(U_C) \cap L(M_\psi^{\mathrm{sc}}) = L(U_{C^\psi})$-invariant ind-subscheme of $\mathrm{Fl}_{M_\psi^{\mathrm{sc}}}$. Using Proposition 2.1.6, the question is equivalent to the assertion that if $w' \in \widetilde{W}^\psi$, then $w' w_\psi \leq_C w^\psi w_\psi$ if and only if $w' \leq_{C^\psi} w^\psi$. But this was shown in Lemma 1.2.7(b).

(c) By part (a), it is enough to show that $p_\nu^{-1}(p_\nu(\mathrm{Fl}^{\leq_C w_C} \cap Z_{\nu,\gamma})) \subseteq \mathrm{Fl}^{\leq_C w_C}$ for each $C \ni \psi$. Since every fiber of p_ν lies in a single $L(U_\psi)$-orbit, the assertion follows from the inclusion $U_\psi \subseteq U_C$. □

3.2. *Finiteness of homology*

3.2.1. Homology. We fix a prime number ℓ different from the characteristic of k.

(a) For a scheme Y of finite type over k and $\mathcal{F} \in D_c^b(Y, \overline{\mathbb{Q}}_\ell)$, one can form the homology groups $H_i(Y, \mathcal{F}) := (H^i(Y, \mathcal{F}))^*$. We also set $H_i(Y) := H_i(Y, \overline{\mathbb{Q}}_\ell)$.

(b) A closed embedding $\iota : X \hookrightarrow Y$ induces a morphism

$$\iota^* : H^i(Y, \mathcal{F}) \to H^i(Y, \iota_* \iota^* \mathcal{F}) = H^i(X, \iota^* \mathcal{F}) = H^i(X, \mathcal{F}|_X),$$

and hence a morphism $\iota_* : H_i(X, \mathcal{F}|_X) \to H_i(Y, \mathcal{F})$.

(c) By part (b), a closed embedding $\iota : X \hookrightarrow Y$ induces a morphism $\iota_* : H_i(X) \to H_i(Y)$. Therefore, for every ind-scheme $Y = \mathrm{colim}_i Y_i$ over k, one can form a homology $H_i(Y) := \mathrm{colim}_i H_i(Y_i)$.

The main goal of this section is to show the following finiteness property of homology of affine Springer fibers:

Proposition 3.2.2. *In the situation of Section 3.1.1(e), there exists an integer r such that for every tuple $\overline{x} \in \mathbb{Z}^\Psi$ and every $\psi \in \Psi$, we have an equality of kernels*

$$\mathrm{Ker}\Big(H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x}}) \to H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x} + r\overline{e}_\psi})\Big) = \mathrm{Ker}\Big(H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x}}) \to H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x} + (r+1)\overline{e}_\psi})\Big). \tag{3.3}$$

In order to prove this, we need to introduce certain notation, generalizing [BV, Sections A.4.2 and 3.1.2].

3.2.3. Filtrations. Let Γ be an ordered monoid – that is, a monoid and a partially ordered set such that $\sigma\tau \leq \sigma'\tau'$ for each $\sigma \leq \sigma'$ and $\tau \leq \tau'$.

(a) By a Γ-*filtered set* (or a set with a Γ-*filtration*), we mean a set X together with collections of subsets $\{X_\sigma\}_{\sigma\in\Gamma}$ such that $X_\sigma \subseteq X_\tau$ for all $\sigma \leq \tau$, and $X = \bigcup_\sigma X_\sigma$.

(b) By a Γ-filtered group we mean a group A with a Γ-filtration such that $1 \in A_1$ and $A_\sigma \cdot A_\tau \subseteq A_{\sigma\tau}$.

(c) Let A be a Γ-filtered group, and X is a set equipped with an A-action and a Γ-filtration. We say that Γ-filtration on X is *compatible* with a filtration of A if for every $\sigma, \tau \in \Gamma$ we have $A_\sigma(X_\tau) \subseteq X_{\sigma\tau}$.

(d) In the situation of (c), we will say that the Γ-filtration on X is *finitely generated over A* if there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\{X_\sigma\}_{\sigma\in\Gamma}$ is *generated by* $\{X_\sigma\}_{\sigma\in\Gamma_0}$; that is, for every $\sigma \in \Gamma$, we have $X_\sigma = \bigcup_{\{(\tau,\sigma')\in\Gamma\times\Gamma_0 \mid \tau\sigma'=\sigma\}} A_\tau(X_{\sigma'})$.

3.2.4. Rees algebras and modules. Let L be a field, and assume that we are in the situation of Section 3.2.3.

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(a) For a Γ-filtered group A, the group algebra $L[A]$ is also equipped with a Γ-filtration $L[A]_\sigma := \mathrm{Span}_L(A_\sigma)$, and we denote by $R(L[A]) := \bigoplus_{\sigma\in\Gamma} L[A]_\sigma$ the corresponding Rees algebra. Note that $R(L[A])$ is the monoid algebra of the monoid $R(A) := \{(a,\sigma) \in A\times\Gamma \mid a \in A_\sigma\}$.

(b) Let X be a scheme locally of finite type over k equipped with an action of A. Assume that A is a Γ-filtered group, and that X is equipped with a Γ-filtration compatible with Γ-filtration on A and such that $X_\sigma \subseteq X$ is a closed subscheme of finite type over k for each $\sigma \in \Gamma$.

(c) For every A-equivariant element $\mathcal{F} \in D^b_c(X,\overline{\mathbb{Q}}_\ell)$, we can form a Γ-graded $R(\overline{\mathbb{Q}}_\ell[A])$-module $R(H_i(X,\mathcal{F})) := \bigoplus_\sigma H_i(X_\sigma,\mathcal{F}|_{X_\sigma})$ for every $i \in \mathbb{Z}$. Explicitly, the action of $a \in A_\tau$ on X defines a closed embedding $a : X_\sigma \hookrightarrow X_{\tau\sigma}$, and hence a homomorphism $H_i(X_\sigma,\mathcal{F}|_{X_\sigma}) \to H_i(X_{\tau\sigma},\mathcal{F}|_{X_{\tau\sigma}})$ (see Section 3.2.1(b)).

In particular, we form a Γ-graded $R(\overline{\mathbb{Q}}_\ell[A])$-module $R(H_i(X)) := R(H_i(X,\overline{\mathbb{Q}}_\ell))$.

Lemma 3.2.5. *In the situation of Section 3.2.4(b), assume that*

(i) *the group A acts on the set of irreducible components of X with finite stabilizers;*

(ii) *the filtration $\{X_\sigma\}_\sigma$ is finitely generated over A;*

(iii) *the Rees algebra $R(\overline{\mathbb{Q}}_\ell[A])$ is Noetherian.*

Then for every A-equivariant object $\mathcal{F} \in D^b_c(X,\overline{\mathbb{Q}}_\ell)$ and $i \in \mathbb{Z}$, the $R(\overline{\mathbb{Q}}_\ell[A])$-module $R(H_i(X,\mathcal{F}))$ is finitely generated.

Proof. The argument is identical to that of [BV, Lemma 3.1.3], where the case of $\Gamma = \mathbb{Z}_{\geq 0}$ is treated. □

Example 3.2.6. (a) Let Γ be an ordered monoid $\mathbb{Z}^{\Psi}_{\geq 0}$, which we identify with a corresponding submonoid of the group of quasi-admissible tuples in Λ via the correspondence of Section 1.3.4(a),(b).

(b) Let $\Lambda' \subseteq \Lambda$ be a subgroup. Consider a Γ-filtration on Λ', where for every $\overline{x} \in \Gamma$, we set $\Lambda'_{\overline{x}} := \Lambda' \cap V^{\leq\overline{x}}$, where $V^{\leq\overline{x}}$ is defined in Section 1.3.9(d). Then $\{\Lambda'_{\overline{x}}\}_{\overline{x}}$ is a Γ-filtered semigroup.

(c) Note that $R(\Lambda') = \{(\mu,\overline{x}) \in \Lambda'\times\mathbb{Z}^{\Psi}_{\geq 0} \mid \langle\psi,\mu\rangle \leq \overline{x}(\psi) \text{ for every } \psi \in \Psi\}$. Therefore, by Gordan's lemma (see, for example, [Ew, Lemma 3.4, page 154]), $R(\Lambda')$ is a finitely generated commutative monoid. Therefore, the Rees algebra $R(\overline{\mathbb{Q}}_\ell[\Lambda']) = \overline{\mathbb{Q}}_\ell[R(\Lambda')]$ is a finitely generated commutative algebra over $\overline{\mathbb{Q}}_\ell$; hence, it is Noetherian.

(d) We apply the construction of part (b) to $\Lambda' := \Lambda_\gamma$, and equip the ind-scheme $X = \mathrm{Fl}_\gamma$ (resp. $X = \mathrm{Gr}_\gamma$) with a Γ-filtration $\mathrm{Fl}_\gamma^{\leq'\overline{x}}$ (resp. $\mathrm{Gr}_\gamma^{\leq\overline{x}}$). Then it follows from definitions that this filtration is compatible with a Γ-filtration on Λ_γ.

Lemma 3.2.7. *The Γ-filtrations $\{\mathrm{Gr}_\gamma^{\leq\overline{x}}\}_{\overline{x}}$ on Gr_γ and $\{\mathrm{Fl}_\gamma^{\leq'\overline{x}}\}_{\overline{x}}$ on Fl_γ are finitely generated over Λ_γ.*

Proof. Since the filtration $\{\mathrm{Fl}_\gamma^{\leq'\overline{x}}\}_{\overline{x}}$ on Fl_γ is defined to be the preimage of the filtration $\{\mathrm{Gr}_\gamma^{\leq\overline{x}}\}_{\overline{x}}$ on Gr_γ, it will suffice to show the assertion for $\{\mathrm{Gr}_\gamma^{\leq\overline{x}}\}_{\overline{x}}$.

Notice that for every Λ_γ-invariant subset of $X \subseteq \mathrm{Gr}_\gamma$, the Γ-filtration on Gr_γ induces a Γ-filtration on X. Moreover, if Gr_γ is a finite union $\bigcup_j X_j$ of Λ_γ-invariant subsets, then the filtration on Gr_γ is finitely generated if and only if the corresponding filtration on each X_j is finitely generated.

Recall that there exists a closed subscheme of finite type $Y \subseteq \mathrm{Gr}_\gamma$ such that $\mathrm{Gr}_\gamma = \Lambda_\gamma(Y)$. Moreover, using Corollary 2.1.7(d), there exists a finite decomposition $Y = \bigcup_j Y_j$ such that for each j, there exists a tuple $\overline{y} = \overline{y}_j$ such that $Y_j \subseteq L(U_C)y_C$ for all $C \in \mathcal{C}$. Then $\mathrm{Gr}_\gamma = \bigcup_j \Lambda_\gamma(Y_j)$, and it suffices to show that the filtration $\{\Lambda_\gamma(Y_j)^{\leq\overline{x}}\}_{\overline{x}}$ on each $\Lambda_\gamma(Y_j)$ is finitely generated over Λ_γ.

Note that for every $\overline{x} \in \Gamma$, we have an equality $\Lambda_\gamma(Y_j)^{\leq\overline{x}} = \Lambda_\gamma^{\leq\overline{x}-\overline{y}_j}(Y_j)$. Indeed, it follows from [MV, Proposition 3.1] (or can be deduced from Proposition 2.1.6) that for every $\mu \in \Lambda_\gamma$ and $z \in Y_j$, we have $\mu z \in \Lambda_\gamma(Y_j)^{\leq\overline{x}}$ if and only if $\mu y_C \leq_C x_C$ for all $C \in \mathcal{C}$. Hence, $\mu z \in \Lambda_\gamma(Y_j)^{\leq\overline{x}}$ if and only if $\mu \in \Lambda_\gamma^{\leq\overline{x}-\overline{y}_j}$, as claimed.

Therefore, it is enough to show that the Γ-filtration $\{(\Lambda_\gamma)_{\overline{x}-\overline{y}_j}\}_{\overline{x}}$ is finitely generated over Λ_γ. Since $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$ is a finitely generated $\overline{\mathbb{Q}}_\ell$-algebra (by Section 3.2.6(c)), the assertion follows. □

Corollary 3.2.8. *The Rees module $R(H_i(\mathrm{Fl}_\gamma))$ is a finitely generated $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$-module.*

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Proof. Since Rees algebra $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$ is Noetherian (see Section 3.2.6(c)), the assertion follows from Lemmas 3.2.5 and 3.2.7. □

Now we are ready to prove Proposition 3.2.2.

3.2.9. *Proof of Proposition 3.2.2.* Since Ψ is finite, it will suffice to show the existence of r for a fixed ψ. For every $r \in \mathbb{N}$, the embeddings $\mathrm{Fl}^{\leq' \overline{x}} \hookrightarrow \mathrm{Fl}^{\leq' \overline{x}+r\overline{e}_\psi}$ for all $\overline{x}$ induce a homomorphism of $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$-modules $\iota_{r\overline{e}_\psi} : R(H_i(\mathrm{Fl}_\gamma)) \to R(H_i(\mathrm{Fl}_\gamma))$, and Proposition 3.2.2 asserts that $\operatorname{Ker} \iota_{r\overline{e}_\psi} = \operatorname{Ker} \iota_{(r+1)\overline{e}_\psi}$ for some r.

Since $\{\operatorname{Ker} \iota_{r\overline{e}_\psi}\}_r$ is an increasing sequence of $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$-submodules of $R(H_i(\mathrm{Fl}_\gamma))$, the Rees algebra $R(\overline{\mathbb{Q}}_\ell[\Lambda_\gamma])$ is Noetherian (by Section 3.2.6(c)), while $R(H_i(\mathrm{Fl}_\gamma))$ is finitely generated (by Corollary 3.2.8), this sequence stabilizes. □

The following lemma will be used in the proof of Theorem 0.3.

Lemma 3.2.10. *There exists $m \in \mathbb{N}$ such that for every m-regular tuple $\overline{x} \in \mathbb{Z}^\Psi$ and every $\psi \in \Psi$ such that $\check{\psi} \notin (\Lambda_\gamma)_\mathbb{Q}$, we have $\mathrm{Fl}_\gamma^{\leq' \overline{x}} = \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi}$.*

Proof. Our argument is similar to that of Lemma 3.2.7. It is enough to show that $\mathrm{Gr}_\gamma^{\leq \overline{x}} = \mathrm{Gr}_\gamma^{\leq \overline{x}+\overline{e}_\psi}$. Let Y, Y_j and $\overline{y}_j$ be as in the proof of Lemma 3.2.7, and choose $m \in \mathbb{N}$ such that for every m-regular $\overline{x}$, the tuples $\overline{x} - \overline{y}_j + \overline{e}_\psi$ is regular. We claim that this m satisfies the required property.

It suffices to show that $\Lambda_\gamma(Y_j)^{\leq \overline{x}} = \Lambda_\gamma(Y_j)^{\leq \overline{x}+\overline{e}_\psi}$ for each j. For this, it suffices to show that $\Lambda_\gamma^{\leq \overline{x}-\overline{y}_j} = \Lambda_\gamma^{\leq \overline{x}-\overline{y}_j+\overline{e}_\psi}$. In other words, we have to show that every $\mu \in \Lambda^{\leq \overline{x}-\overline{y}_j+\overline{e}_\psi} \setminus \Lambda^{\leq \overline{x}-\overline{y}_j}$ does not belong to Λ_γ.

We are going to deduce the assertion from Lemma 1.3.11(b). Since $\check{\psi} \notin (\Lambda_\gamma)_\mathbb{Q}$, it follows from Section 3.1.4(d) that there exists a root $\alpha \in \Phi$ such that $\alpha \in (\Lambda_\gamma)^\perp$ and $\langle \alpha, \check{\psi} \rangle > 0$. Since $\langle \psi, \mu \rangle = (\overline{x} - \overline{y}_j + \overline{e}_\psi)_\psi$, and the tuple $\overline{x} - \overline{y}_j + \overline{e}_\psi$ is regular by assumption, we conclude from Lemma 1.3.11(b) that $\langle \alpha, \mu \rangle > 0$. Therefore, $\mu \notin \Lambda_\gamma$ because $\alpha \in (\Lambda_\gamma)^\perp$. □

4. Proof of Theorem 0.3

4.1. Localization theorem for equivariant cohomology

In this section we will review basic facts about equivariant cohomology (with compact support), including a version of a localization theorem.

4.1.1. Total cohomology of Artin stacks. For an Artin stack $\mathcal{X}$ of finite type over k and $\mathcal{F} \in D_c^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$, we denote by $H^\bullet(\mathcal{X}, \mathcal{F}) := \bigoplus_i H^i(\mathcal{X}, \mathcal{F})$ its total cohomology, and set $H^\bullet(\mathcal{X}) := H^\bullet(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$.

(a) Note that $H^\bullet(\mathcal{X}) = \operatorname{Ext}^\bullet_\mathcal{X}(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ is a graded $\overline{\mathbb{Q}}_\ell$-algebra, and identification $\mathcal{F}[\bullet] = \overline{\mathbb{Q}}_\ell[\bullet] \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}$ give to $H^\bullet(\mathcal{X}, \mathcal{F})$ a natural structure of a graded $H^\bullet(\mathcal{X})$-module.

(b) Every morphism $\mathcal{F}_1 \to \mathcal{F}_2$ in $D_c^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$ induces a homomorphism $H^\bullet(\mathcal{X}, \mathcal{F}_1) \to H^\bullet(\mathcal{X}, \mathcal{F}_2)$ of graded $H^\bullet(\mathcal{X})$-modules.

(c) For every homomorphism $f : \mathcal{X} \to \mathcal{Y}$ of Artin stacks of finite type over k, the pullback $f^* : H^\bullet(\mathcal{Y}) \to H^\bullet(\mathcal{X})$ is a homomorphism of graded $\overline{\mathbb{Q}}_\ell$-algebras. Moreover, for every $\mathcal{F} \in D_c^b(\mathcal{Y}, \overline{\mathbb{Q}}_\ell)$ the pullback f^* gives rise to a homomorphism

$$H^\bullet(\mathcal{X}) \otimes_{H^\bullet(\mathcal{Y})} H^\bullet(\mathcal{Y}, \mathcal{F}) \to H^\bullet(\mathcal{X}, f^*\mathcal{F})$$

of graded $H^\bullet(\mathcal{X})$-modules.

4.1.2. Equivariant cohomology (compare [BL, GKM, Ac, AF]). Let G be an algebraic group over k, let X be a separated scheme of finite type over k equipped with a G-action, set $\mathrm{pt} := \operatorname{Spec} k$, let $[\mathrm{pt}/G]$ be the classifying stack of G, and let $\mathrm{pr}_X : [X/G] \to [\mathrm{pt}/G]$ be the natural projection.

(a) For every $\mathcal{F} \in D_c^b([X/G], \overline{\mathbb{Q}}_\ell)$, we define its equivariant cohomology

$$H_G^\bullet(X, \mathcal{F}) := H^\bullet([X/G], \mathcal{F})) = H^\bullet([\mathrm{pt}/G], R(\mathrm{pr}_X)_*(\mathcal{F})),$$

equivariant cohomology with compact support

$$H_{c,G}^\bullet(X, \mathcal{F}) := H^\bullet([\mathrm{pt}/G], R(\mathrm{pr}_X)_!(\mathcal{F})),$$

and set $H_G^\bullet(X) := H_G^\bullet(X, \overline{\mathbb{Q}}_\ell)$ and $H_{c,G}^\bullet(X) := H_{c,G}^\bullet(X, \overline{\mathbb{Q}}_\ell)$.

(b) By Section 4.1.1(a), $H_G^\bullet(\mathrm{pt})$ is a graded $\overline{\mathbb{Q}}_\ell$-algebra, while both $H_G^\bullet(X, \mathcal{F})$ and $H_{c,G}^\bullet(X, \mathcal{F})$ have natural structures of graded $H_G^\bullet(\mathrm{pt})$-modules.

(c) Note that $H_G^\bullet(X) = \mathrm{Ext}_{[X/G]}^\bullet(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$ is a graded $\overline{\mathbb{Q}}_\ell$-algebra; hence, both $H_G^\bullet(X, \mathcal{F})$ and $H_{c,G}^\bullet(X, \mathcal{F})$ have natural structures of graded $H_G^\bullet(X)$-modules.

(d) Note that the structures of $H_G^\bullet(X, \mathcal{F})$ and $H_{c,G}^\bullet(X, \mathcal{F})$ of graded $H_G^\bullet(\mathrm{pt})$-modules from part (b) are obtained from structures of graded $H_G^\bullet(X)$-modules from part (c) by the homomorphism

$$(p_X)^* : H_G^\bullet(\mathrm{pt}) = H^\bullet([\mathrm{pt}/G]) \to H^\bullet([X/G]) = H_G^\bullet(X)$$

of graded $\overline{\mathbb{Q}}_\ell$-algebras from Section 4.1.1(c).

4.1.3. Simple properties. Let G, X and $\mathcal{F}$ be as in Section 4.1.2.

(a) Using Section 4.1.1(b), for each closed G-invariant subscheme $Z \subseteq X$, the long exact sequence for cohomology with compact support naturally upgrades to an exact sequence

$$H_{c,G}^\bullet(Z)[-1] \xrightarrow{\delta} H_{c,G}^\bullet(X \setminus Z) \to H_{c,G}^\bullet(X) \to H_{c,G}^\bullet(Z) \xrightarrow{\delta} H_{c,G}^\bullet(X \setminus Z)[1]$$

of graded $H_G^\bullet(\mathrm{pt})$-modules, functorial in (X, Z).

(b) If G acts trivially on X, then we have canonical isomorphism

$$H_{c,G}^\bullet(X) \simeq H_G^\bullet(\mathrm{pt}) \otimes_{\overline{\mathbb{Q}}_\ell} H_c^\bullet(X)$$

of graded $H_G^\bullet(\mathrm{pt})$-modules, functorial in X. Indeed, since $[X/G] \simeq X \times [\mathrm{pt}/G]$, the assertion follows from Künneth formula. Alternatively, choose a compactification $j : X \hookrightarrow \overline{X}$ of X, and apply [Ac, Proposition 6.7.5] for $H_{\{1\}\times G}^\bullet(\overline{X} \times \mathrm{pt}, (j_!\overline{\mathbb{Q}}_\ell) \boxtimes \overline{\mathbb{Q}}_\ell)$.

(c) Using observation of Section 4.1.1(c) applied to the projection $\pi : \mathrm{pt} \to [\mathrm{pt}/G]$ and an object $R(p_X)_!(\mathcal{F}) \in D_c^b([\mathrm{pt}/G], \overline{\mathbb{Q}}_\ell)$, we have a homomorphism $\pi^* : H_G^\bullet(\mathrm{pt}) \to H^\bullet(\mathrm{pt}) = \overline{\mathbb{Q}}_\ell$ of graded $\overline{\mathbb{Q}}_\ell$-algebras and a homomorphism

$$\overline{\mathbb{Q}}_\ell \otimes_{H_G^\bullet(\mathrm{pt})} H_{c,G}^\bullet(X, \mathcal{F}) \to H_c^\bullet(X, \mathcal{F}) \tag{4.1}$$

of graded vector spaces (compare [Ac, equation (6.7.2)]).

Moreover, if $H_{c,S}^\bullet(X, \mathcal{F})$ is a free graded $H_S^\bullet(\mathrm{pt})$-module, then morphism (4.1) is an isomorphism. Indeed, as in the proof of [Ac, Lemma 6.7.4], one first reduces to the case $X = \mathrm{pt}$ in which case the assertion follows from [Ac, Lemma 6.7.3].

4.1.4. Localization theorem (compare [GKM, AF]). Let S be an algebraic torus acting on a separated scheme X of finite type over k.

(a) Recall that graded $\overline{\mathbb{Q}}_\ell$-algebra $H_S^\bullet(\mathrm{pt})$ is canonically isomorphic with the symmetric algebra $\mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}^\bullet(X^*(S) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell(-1)[-2])$, where $X^*(S)$ denote the group of characters of S, while $[-2]$ indicates that the vector space $X^*(S) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell(-1)$ is placed in degree 2 (see, for example, [Ac, Theorem 6.7.7]).

We fix an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces $\overline{\mathbb{Q}}_\ell(-1) \simeq \overline{\mathbb{Q}}_\ell$; thus, we can view $X^*(S)$ as a subset of $\mathrm{Sym}^\bullet_{\overline{\mathbb{Q}}_\ell}(X^*(S) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Sym}^\bullet_{\overline{\mathbb{Q}}_\ell}(X^*(S) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell(-1)) \simeq H^\bullet_S(\mathrm{pt})$.

(b) By Section 4.1.2(b), both $H^\bullet_{c,S}(X,\mathcal{F})$ and $H^\bullet_S(X,\mathcal{F})$ are graded $H^\bullet_S(\mathrm{pt})$-modules for every $\mathcal{F} \in D^b_c([X/S],\overline{\mathbb{Q}}_\ell)$. We claim that if $X^S = \emptyset$, then there exists $\lambda \in X^*(S) \subseteq H^\bullet_S(\mathrm{pt})$, which acts on each $H^\bullet_{c,S}(X,\mathcal{F})$ and $H^\bullet_S(X,\mathcal{F})$ as zero.

Indeed, by a particular case of the localization theorem (see, for example, [AF, Chapter 7, Theorem 1.1]), there exists $\lambda \in X^*(S)$ such that the image of λ under the pullback $(p_X)^* : H^\bullet_S(\mathrm{pt}) \to H^\bullet_S(X)$ is zero, so the assertion follows by the observation of Section 4.1.2(d).

(c) The pullback $H^\bullet_{c,S}(X,\mathcal{F}) \to H^\bullet_{c,S}(X^S,\mathcal{F}|_{X^S})$ induces an isomorphism of localizations

$$(X^*(S))^{-1}H^\bullet_{c,S}(X,\mathcal{F}) \xrightarrow{\sim} (X^*(S))^{-1}H^\bullet_{c,S}(X^S,\mathcal{F}|_{X^S}).$$

Indeed, by part (b), we have $(X^*(S))^{-1}H^\bullet_{c,S}(X \setminus X^S,\mathcal{F}) = 0$, so the assertion follows from the exact sequence of Section 4.1.3(a).

(d) If $H^\bullet_{c,S}(X,\mathcal{F})$ is a free (or, more generally, torsion free) $H^\bullet_S(\mathrm{pt})$-module, then the restriction map $H^\bullet_{c,S}(X,\mathcal{F}) \to H^\bullet_{c,S}(X^S,\mathcal{F}|_{X^S})$ is injective. Indeed, our assumption implies that the canonical map $H^\bullet_{c,S}(X,\mathcal{F}) \to (X^*(S))^{-1}H^\bullet_{c,S}(X,\mathcal{F})$ is injective, so the assertion follows from part (c) and Section 4.1.3(b).

4.2. *Criterion of injectivity*

4.2.1. Borel–Moore homology. To every scheme X of finite type over k, one associates the Borel-Moore homology groups $H_{i,BM}(X) := H^i_c(X,\overline{\mathbb{Q}}_\ell)^*$. In particular, we have $H_{i,BM}(X) = H_i(X)$ if X is proper over k. Also for every closed subscheme $Z \subseteq X$, we have a long exact sequence

$$\to H_{i,BM}(Z) \to H_{i,BM}(X) \to H_{i,BM}(X \setminus Z) \to H_{i-1,BM}(Z) \to .$$

Lemma 4.2.2. *Let X be a closed subscheme of Y, and let $\iota : H_{i,BM}(X) \to H_{i,BM}(Y)$ be the natural map.*

(a) *The map ι is injective if there exists a closed subscheme $Z \subseteq X$ such that*

$$\mathrm{Ker}\big(H_{i,BM}(Z) \to H_{i,BM}(X)\big) = \mathrm{Ker}\big(H_{i,BM}(Z) \to H_{i,BM}(X) \to H_{i,BM}(Y)\big),$$

and the map $H_{i,BM}(X \setminus Z) \to H_{i,BM}(Y \setminus Z)$ is injective.

(b) *The map ι is injective if there exists a closed subscheme $Z \subseteq Y$ containing $Y \setminus X$ such that the natural map $H_{i,BM}(Z \cap X) \to H_{i,BM}(Z)$ is injective.*

Proof. Both assertions follow from a straightforward diagram chase. Namely, assertion (a) follows from the commutative diagram

$$\begin{array}{ccccc} H_{i,BM}(Z) & \longrightarrow & H_{i,BM}(X) & \longrightarrow & H_{i,BM}(X \setminus Z) \\ & & \downarrow & & \downarrow \\ & & H_{i,BM}(Y) & \longrightarrow & H_{i,BM}(Y \setminus Z) \end{array}$$

with an exact first row, while assertion (b) follows from the commutative diagram with exact rows

$$\begin{array}{ccccc} H_{i+1,BM}(Z \setminus (Z \cap X)) & \longrightarrow & H_{i,BM}(Z \cap X) & \longrightarrow & H_{i,BM}(Z) \\ \| & & \downarrow & & \downarrow \\ H_{i+1,BM}(Y \setminus X) & \longrightarrow & H_{i,BM}(X) & \longrightarrow & H_{i,BM}(Y). \end{array}$$

□

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4.2.3. Acyclic morphisms. (a) We say that a scheme X of finite type over k is *acyclic* if the canonical morphism $\overline{\mathbb{Q}}_\ell \to R\Gamma(X, \overline{\mathbb{Q}}_\ell)$ is an isomorphism.

(b) We say that a morphism $f : X \to Y$ between schemes of finite type over k is *acyclic* if it is smooth and all geometric fibers of f are acyclic.[1]

(c) Note that if $f : X \to Y$ is acyclic, then for every connected component Y' of Y, the restriction $f|_{Y'} : f^{-1}(Y') \to Y'$ is smooth of some relative dimension N, and we have $Rf_!(\overline{\mathbb{Q}}_\ell)|_{Y'} \simeq \overline{\mathbb{Q}}_\ell[2N](N)$ (use, for example, [BKV, Lemma 1.1.3]).

The following result uses notation of Section 4.1.

Lemma 4.2.4. (a) *Let S be a torus, let Y be an S-equivariant scheme of finite type over k, and let $X \subseteq Y$ be a closed S-invariant subscheme. Assume that*

(i) *the restriction map $H^\bullet_c(Y^S) \to H^\bullet_c(X^S)$ is surjective and*

(ii) *both $H^\bullet_{c,S}(X)$ and $H^\bullet_{c,S}(Y \setminus X)$ are free graded $H^\bullet_S(\mathrm{pt})$-modules.*

Then $H^\bullet_{c,S}(Y)$ is a free graded $H^\bullet_S(\mathrm{pt})$-module, and the restriction map $H^\bullet_c(Y) \to H^\bullet_c(X)$ is surjective.

(b) *Assume that Y has a finite S-invariant filtration $\emptyset = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n = Y$ by closed reduced subschemes such that for each $j = 1, \ldots, n-1$,*

(i) *the restriction map $H^\bullet_c(Y^S_j) \to H^\bullet_c(Y^S_{j-1})$ is surjective and*

(ii) *there exists an S-equivariant acyclic morphism $\pi_j : Y_j \setminus Y_{j-1} \to (Y_j \setminus Y_{j-1})^S$.*

Then $H^\bullet_{c,S}(Y)$ is a free graded $H^\bullet_S(\mathrm{pt})$-module.

Proof. (a) By Section 4.1.3(a), we have a commutative diagram

$$\begin{array}{ccccc} & & H^\bullet_{c,S}(X) & \xrightarrow{\delta_1} & H^\bullet_{c,S}(Y \setminus X)[1] \\ & & \downarrow & & \downarrow \\ H^\bullet_{c,S}(Y^S) & \longrightarrow & H^\bullet_{c,S}(X^S) & \xrightarrow{\delta_2} & H^\bullet_{c,S}(Y^S \setminus X^S)[1] \end{array}$$

of graded $H^\bullet_S(\mathrm{pt})$-modules with exact bottom row, where vertical arrows are induced by the inclusion $Y^S \hookrightarrow Y$. By Section 4.1.3(b), we have canonical isomorphisms

$$H^\bullet_{c,S}(Y^S) \cong H^\bullet_S(\mathrm{pt}) \otimes_{\overline{\mathbb{Q}}_\ell} H^\bullet_c(Y^S) \text{ and } H^\bullet_{c,S}(X^S) \cong H^\bullet_S(\mathrm{pt}) \otimes_{\overline{\mathbb{Q}}_\ell} H^\bullet_c(X^S)$$

of $H^\bullet_S(\mathrm{pt})$-modules. Hence, by assumption (i), the map $H^\bullet_{c,S}(Y^S) \to H^\bullet_{c,S}(X^S)$ is surjective; therefore, the connecting homomorphism δ_2 is zero.

By assumption (ii) and the localization theorem (see Section 4.1.4(d)), the right vertical map is injective; hence, the connecting homomorphism δ_1 is zero as well. Thus, by Section 4.1.3(a), we get a short exact sequence

$$0 \to H^\bullet_{c,S}(Y \setminus X) \to H^\bullet_{c,S}(Y) \to H^\bullet_{c,S}(X) \to 0$$

of graded $H^\bullet_S(\mathrm{pt})$-modules; hence, $H^\bullet_{c,S}(Y)$ is a free graded $H^\bullet_S(\mathrm{pt})$-module by assumption (ii). In this case, we have canonical isomorphisms

$$H^\bullet_c(Y) \cong \overline{\mathbb{Q}}_\ell \otimes_{H^\bullet_S(\mathrm{pt})} H^\bullet_{c,S}(Y) \text{ and } H^\bullet_c(X) \cong \overline{\mathbb{Q}}_\ell \otimes_{H^\bullet_S(\mathrm{pt})} H^\bullet_{c,S}(X)$$

of graded vector spaces (see Section 4.1.3(c)); therefore, surjectivity of the map $H^\bullet_c(Y) \to H^\bullet_c(X)$ follows from the surjectivity of $H^\bullet_{c,S}(Y) \to H^\bullet_{c,S}(X)$.

(b) We are going to show the assertion by induction on n. Assume first that $n = 1$. Then $\pi := \pi_1 : Y \to Y^S$ is an S-equivariant acyclic morphism, so we conclude from Sections 4.1.3(b) and 4.2.3(c) that $H^\bullet_{c,S}(Y, \overline{\mathbb{Q}}_\ell) \cong H^\bullet_{c,S}(Y^S, R\pi_!\overline{\mathbb{Q}}_\ell)$ is a free graded $H^\bullet_S(\mathrm{pt})$-module.

[1]In [BKV, Section 1.1.2], such morphisms are called unipotent.

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Now assume that $n > 1$, and set $X := Y_{n-1}$. By the induction hypothesis, both $H^\bullet_{c,S}(X)$ and $H^\bullet_{c,S}(Y \setminus X)$ are free graded $H^\bullet_S(\mathrm{pt})$-modules. Therefore, by assumption (i), all assumptions of part (a) are satisfied; thus, $H^\bullet_{c,S}(Y)$ is a free graded $H^\bullet_S(\mathrm{pt})$-module. □

Lemma 4.2.5. *Let Y be an ind-scheme of ind-finite type over k equipped with an action of a torus S, let $p : Y \to Y^S$ be an S-equivariant acyclic morphism such that its restriction $p|_{Y^S}$ is the identity, and let $X \subseteq Y$ be a reduced locally closed ind-subscheme such that we have an inclusion of sets $p^{-1}(p(X)) \subseteq X$.*

Then X is equal to the schematic preimage $p^{-1}(X^S) \subseteq Y$. In particular, X is S-invariant, and p induces an S-equivariant acyclic morphism $p_X : X \to X^S$ such that $p_X|_{X^S}$ is the identity.

Proof. Notice that since the inclusion $p^{-1}(p(X)) \supseteq X$ always holds, we have an equality of sets $p^{-1}(p(X)) = X$; thus, the ind-subscheme $X \subseteq Y$ is S-invariant.

Next, we claim that we have an equality of sets $p(X) = X^S$. Indeed, $p|_{Y^S}$ is the identity, we get $p(X^S) = X^S$ and $p(X) \subseteq p^{-1}(p(X))$. Therefore, we have inclusions

$$X^S = p(X^S) \subseteq p(X) \subseteq p^{-1}(p(X)) \cap Y^S \subseteq X \cap Y^S = X^S.$$

By the proven above, we have an equality of sets $p^{-1}(X^S) = p^{-1}(p(X)) = X$, and from this, the assertion follows: Indeed, since X is reduced and S is a torus, we conclude that X^S is reduced. Since p is smooth, the schematic preimage $p^{-1}(X^S)$ is reduced, so the equality of reduced ind-subschemes $p^{-1}(X^S) = X$ follows from the corresponding equality of the underlying sets. □

Corollary 4.2.6. *Let Z be an S-equivariant ind-scheme of ind-finite type over k, $\{Z_\nu\}_{\nu\in\Xi}$ an S-invariant stratification of Z, $Y \subseteq Z$ an S-invariant locally closed subscheme of finite type over k, and $X \subseteq Y$ an S-invariant closed subscheme.*

Assume that for each $\nu \in \Xi$,

(a) *the stratum Z^S_ν is an open and closed ind-subscheme of Z^S;*

(b) *the map $H_{i,BM}(X \cap Z^S_\nu) \to H_{i,BM}(Y \cap Z^S_\nu)$ is injective for all i;*

(c) *there exists an S-equivariant acyclic morphism $p_\nu : Y \cap Z_\nu \to Y \cap Z^S_\nu$ between reduced intersections such that $p_\nu|_{Y\cap Z^S_\nu}$ is the identity, and we have an inclusion of sets $p_\nu^{-1}(p_\nu(X \cap Z_\nu)) \subseteq X$.*

Then the map $H_{i,BM}(X) \to H_{i,BM}(Y)$ is injective for all i.

Proof. We are going to apply the criterion of Lemma 4.2.4(a).

It follows from assumption (a) that Y^S (resp. X^S) is a disjoint union of the $Y \cap Z^S_\nu$'s (resp. $X \cap Z^S_\nu$'s). This observation together with assumption (b) implies that the map $H_{i,BM}(X^S) \to H_{i,BM}(Y^S)$ is injective for all i, which by duality implies that the map $H^\bullet_c(Y^S) \to H^\bullet_c(X^S)$ is surjective.

It thus suffices to show that both $H^\bullet_{c,S}(X)$ and $H^\bullet_{c,S}(Y \setminus X)$ are free graded $H^\bullet(S)$-modules. Indeed, Lemma 4.2.4(a) then would imply that the restriction map $H^\bullet_c(Y) \to H^\bullet_c(X)$ is surjective, from which our assertion would follow by duality.

We are going to apply the criterion of Lemma 4.2.4(b):

By assumption (a), the disjoint union $Y^S = \coprod_\nu (Y \cap Z^S_\nu)$ is of finite type; hence, the set $\Xi_0 := \{\nu \in \Xi \,|\, Y \cap Z^S_\nu \neq \emptyset\}$ is finite. However, by assumption (c), we have $\Xi_0 = \{\nu \in \Xi \,|\, Y \cap Z_\nu \neq \emptyset\}$. Define a standard partial order on Ξ_0 requiring that $\alpha \le \beta$ if and only if $Z_\alpha \subseteq \overline{Z_\beta}$. Denote the cardinality of Ξ_0 by n, and write Ξ_0 in the form $\Xi_0 = \{\nu_1, \ldots, \nu_n\}$ such that ν_j is a minimal element of the set $\{\nu_j, \ldots, \nu_n\}$ for all $j = 1, \ldots, n$.

For each $j = 1, \ldots, n$, we denote by Y_j the reduced intersection $Y \cap (\bigcup_{t=1}^j Z_{\nu_t})$. Then by construction, each $Y_j \subseteq Y$ is closed, $Y_n = Y$, and $Y_j \setminus Y_{j-1} = Y \cap Z_{\nu_j}$. It suffices to show that the induced filtrations $X_j := X \cap Y_j$ of X and $(Y \setminus X)_j := Y_j \cap (Y \setminus X)$ of $Y \setminus X$ satisfy the assumptions of Lemma 4.2.4(b).

Since Y^S is a disjoint union of the $(Y_j \setminus Y_{j-1})^S$'s (by assumption (a)), assumption (i) of Lemma 4.2.4(b) follows. Next, since $Y_j \setminus Y_{j-1} = Y \cap Z_{\nu_j}$, we get $X_j \setminus X_{j-1} = X \cap Z_{\nu_j}$ and $(Y \setminus X)_j \setminus (Y \setminus X)_{j-1} = (Y \setminus X) \cap Z_{\nu_j}$. Hence, it remains to construct S-equivariant acyclic morphisms $X \cap Z_{\nu_j} \to X \cap Z^S_{\nu_j}$ and $(Y \setminus X) \cap Z_{\nu_j} \to$

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$(Y \setminus X) \cap Z^S_{\nu_j}$. But both morphisms are induced from acyclic morphism $p_{\nu_j} : Y \cap Z_{\nu_j} \to Y \cap Z^S_{\nu_j}$ from assumption (c) using Lemma 4.2.5. □

4.3. The proof

Now we are ready to prove our main result (Theorem 0.3).

Theorem 4.3.1. *There exists $m \in \mathbb{N}$ (depending on γ) such that for all m-regular admissible tuples $\overline{w}_1, \ldots, \overline{w}_n \in \widetilde{W}^{\mathcal{C}}$, the natural map $H_i(\bigcup_{j=1}^n \mathrm{Fl}_\gamma^{\leq \overline{w}_j}) \to H_i(\mathrm{Fl}_\gamma)$ is injective for all $i \in \mathbb{Z}$.*

Proof. Set $Z' := \bigcup_{j=1}^n \mathrm{Fl}^{\leq \overline{w}_j} \subseteq \mathrm{Fl}$. We want to show that if each $\overline{w}_j$ is sufficiently regular, then the natural map $H_i(Z'_\gamma) \to H_i(\mathrm{Fl}_\gamma)$ is injective for all $i \in \mathbb{Z}$. To make the proof more structural, we will divide it into steps.

Step 1. Let $\overline{x}_0 \in \Lambda^{\mathcal{C}}$ be an admissible tuple constructed in Lemma 2.3.9 and such that $\mathrm{Fl}^{\leq' \bar{x}_0} \subseteq \mathrm{Fl}^{\leq \overline{w}_1}$, and let $\{\overline{x}_l\}_{l\geq 0}$ be a sequence of admissible tuples from Lemma 2.3.10. Moreover, it follows from Lemmas 2.3.9 and 2.3.10 that each $\overline{x}_l$ is sufficiently regular if $\overline{w}_1$ is sufficiently regular.

Notice that $\{\mathrm{Fl}_\gamma^{\leq' \overline{x}_l}\}_{l\geq 0}$ form an exhausting increasing union of closed subsets of Fl_γ; hence, it is enough to show that for every $l > 0$, the map

$$H_i(Z'_\gamma \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}_{l-1}}) \to H_i(Z'_\gamma \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}_l})$$

is injective for all l. Using inclusion

$$(Z'_\gamma \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}_l}) \setminus (Z'_\gamma \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}_{l-1}}) \subseteq \mathrm{Fl}_\gamma^{\leq' \overline{x}_l},$$

we conclude from Lemma 4.2.2(b) that it suffices to show that the map

$$H_i((Z'_\gamma \cap \mathrm{Fl}_\gamma^{\leq' \overline{x}_l}) \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}_{l-1}}) \to H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x}_l})$$

is injective. We set $\overline{x} := \overline{x}_{l-1}$. Then $\overline{x}_l = \overline{x} + \overline{e}_\psi$ for some $\psi \in \Psi$, and we want to show that the map

$$H_i((Z'_\gamma \cap \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi}) \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}}) \to H_i(\mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi})$$

is injective.

Step 2. If $\check{\psi} \notin (\Lambda_\gamma)_{\mathbb{Q}}$, then we have an equality $\mathrm{Fl}_\gamma^{\leq' \overline{x}} = \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi}$ (by Lemma 3.2.10), so the assertion is tautological.

From now on, assume that $\check{\psi} \in (\Lambda_\gamma)_{\mathbb{Q}}$. Let $r \in \mathbb{N}$ be as in Proposition 3.2.2. Then, by Lemma 4.2.2(a), it is enough to show that the map

$$H_{i,BM}([(Z'_\gamma \cap \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi}) \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}}] \setminus \mathrm{Fl}_\gamma^{\leq' \overline{x}-r\overline{e}_\psi}) \to H_{i,BM}(\mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi} \setminus \mathrm{Fl}_\gamma^{\leq' \overline{x}-r\overline{e}_\psi})$$

is injective.

Step 3. We are going to apply the criterion of Corollary 4.2.6 in the case $Z = \mathrm{Fl}$, $S = T_\psi$,

$$X = [(Z'_\gamma \cap \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi}) \cup \mathrm{Fl}_\gamma^{\leq' \overline{x}}] \setminus \mathrm{Fl}_\gamma^{\leq' \overline{x}-r\overline{e}_\psi},$$

$Y = \mathrm{Fl}_\gamma^{\leq' \overline{x}+\overline{e}_\psi} \setminus \mathrm{Fl}_\gamma^{\leq' \overline{x}-r\overline{e}_\psi}$, and $\{Z_\nu\}_{\nu \in \widetilde{W}_\psi}$ is the stratification of Fl by $L(P_\psi^{\mathrm{sc}})$-orbits, considered in Section 2.4.3.

Since X and Y are locally closed subschemes of Z of finite type over k, it remains to show that X and Y are S-invariant and properties (a)–(c) of Corollary 4.2.6 are satisfied. Property (a) was mentioned in Section 2.4.3(g).

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Step 4. We claim that the reduced intersections $Y \cap Z_\nu$ and $X \cap Z_\nu$ are either empty or are of the form $\bigcup_{t=1}^m (\mathrm{Fl}_\gamma^{\leq \overline{u}_t} \cap Z_\nu)$, where each $\overline{u}_t$ is sufficiently regular, and $(\overline{u}_t)_\psi = \nu$.

First, we claim that it follows from Corollary 2.3.3 that for every stratum Z_ν such that $Y \cap Z_\nu \neq \emptyset$, we have

$$\overline{x}(\psi) - r < \langle \psi, \pi(\nu) \rangle \leq \overline{x}(\psi) + 1. \tag{4.2}$$

Indeed, our assumption implies that $\mathrm{pr}(Z_\nu) \cap (\mathrm{Gr}^{\leq \overline{x} + \overline{e}_\psi} \setminus \mathrm{Gr}^{\leq \overline{x} - r\overline{e}_\psi}) \neq \emptyset$. Then, using equality $\mathrm{pr}(Z_\nu) = L(P_\psi^{\mathrm{sc}})(\pi(\nu)) \subseteq \mathrm{Gr}$, we conclude from Corollary 2.3.3(c) that $\pi(\nu)$ belongs to $\mathrm{Gr}^{\leq_\psi \overline{x}(\psi)+1} \setminus \mathrm{Gr}^{\leq_\psi \overline{x}(\psi) - r}$, from which inequalities (4.2) follow from Corollary 2.3.3(a),(b).

Similarly, we claim that we have equalities

$$Y \cap Z_\nu = (\mathrm{Fl}^{\leq' \overline{x} + \overline{e}_\psi} \cap \mathrm{Fl}^{\leq_\psi \nu}) \cap Z_{\nu,\gamma}, \text{ if } \overline{x}(\psi) - r < \langle \psi, \pi(\nu) \rangle \leq \overline{x}(\psi) + 1; \tag{4.3}$$

$$X \cap Z_\nu = (\mathrm{Fl}^{\leq' \overline{x}} \cap \mathrm{Fl}^{\leq_\psi \nu}) \cap Z_{\nu,\gamma}, \text{ if } \overline{x}(\psi) - r < \langle \psi, \pi(\nu) \rangle \leq \overline{x}(\psi); \tag{4.4}$$

$$X \cap Z_\nu = (Z' \cap \mathrm{Fl}^{\leq' \overline{x} + \overline{e}_\psi} \cap \mathrm{Fl}^{\leq_\psi \nu}) \cap Z_{\nu,\gamma}, \text{ if } \langle \psi, \pi(\nu) \rangle = \overline{x}(\psi) + 1. \tag{4.5}$$

For this, we have to show that our assumption on ν in (4.3) (resp. (4.4), resp. (4.5)) implies that $Z_\nu \cap \mathrm{Fl}^{\leq' \overline{x} - r\overline{e}_\psi} = \emptyset$ (resp. $Z_\nu \cap \mathrm{Fl}^{\leq' \overline{x} + \overline{e}_\psi} = Z_\nu \cap \mathrm{Fl}^{\leq' \overline{x}}$ and $Z_\nu \cap \mathrm{Fl}^{\leq' \overline{x} - r\overline{e}_\psi} = \emptyset$, resp. $Z_\nu \cap \mathrm{Fl}^{\leq' \overline{x}} = \emptyset$). But this follows from Corollary (a),(c).

Next, using inequalities (4.2), we deduce from a combination of Lemma 2.3.5(a),(b) and Corollary 2.1.7(f), that the reduced intersections $\mathrm{Fl}^{\leq' \overline{x} + \overline{e}_\psi} \cap \mathrm{Fl}^{\leq_\psi \nu}$, $\mathrm{Fl}^{\leq' \overline{x}} \cap \mathrm{Fl}^{\leq_\psi \nu}$ and $Z' \cap \mathrm{Fl}^{\leq' \overline{x} + \overline{e}_\psi} \cap \mathrm{Fl}^{\leq_\psi \nu}$ decompose as finite unions $\bigcup_t \mathrm{Fl}^{\leq \overline{u}_t}$, where each $\overline{u}_t$ is sufficiently regular. Therefore, using formulas (4.3)-(4.5) and Lemma 2.3.2(a), we see that the reduced intersections $X \cap Z_\nu$ and $Y \cap Z_\nu$ are of the form $\bigcup_t (\mathrm{Fl}_\gamma^{\leq \overline{u}_t} \cap Z_\nu)$, where each $\overline{u}_t$ is sufficiently regular, and $(\overline{u}_t)_\psi = \nu$.

Step 5. Now we are going to show property (b) of Corollary 4.2.6. It is enough to show that the composition

$$H_{i,BM}(X \cap Z_\nu^S) \to H_{i,BM}(Y \cap Z_\nu^S) \to H_{i,BM}(Z_{\nu,\gamma}^S) \tag{4.6}$$

is injective.

By a combination of Step 4 and Proposition 3.1.8(b), the reduced intersection $X \cap Z_\nu^S$ is of the form $\bigcup_t \mathrm{Fl}_{M_\psi^{\mathrm{sc}},\gamma}^{\leq \overline{u}_t^\psi}$, and each $\overline{u}_t^\psi \in \widetilde{W}^\psi$ is sufficiently regular (by Lemma 1.3.10(d)).

By induction on the semisimple rank of G, we can assume that Theorem 4.3.1 holds for the Levi subgroup M_ψ. Therefore, the map

$$H_{i,BM}(\bigcup_t \mathrm{Fl}_{M_\psi^{\mathrm{sc}},\gamma}^{\leq \overline{u}_t^\psi}) \to H_{i,BM}(\mathrm{Fl}_{M_\psi^{\mathrm{sc}},\gamma})$$

is injective, from which the injectivity of (4.6) and hence property (b) of Corollary 4.2.6 follows.

Step 6. It remains to show X and Y are S-invariant and satisfy property (c) of Corollary 4.2.6. Recall that in Proposition 3.1.7, we constructed an S-equivariant retraction $p_{\nu,\gamma} : Z_{\nu,\gamma} \to Z_{\nu,\gamma}^S$, which is a composition of affine bundles; hence, it is acyclic.

By Lemma 4.2.5, it is enough to show that $p_{\nu,\gamma}$ satisfies inclusions of sets

$$p_{\nu,\gamma}^{-1}(p_{\nu,\gamma}(Y \cap Z_\nu)) \subseteq Y \cap Z_\nu \text{ and } p_{\nu,\gamma}^{-1}(p_{\nu,\gamma}(X \cap Z_\nu)) \subseteq X \cap Z_\nu. \tag{4.7}$$

By Step 4, it suffices to show that for every sufficiently regular admissible tuple $\overline{u}_t$ such that $(\overline{u}_t)_\psi = \nu$, we have an inclusion of sets $p_{\nu,\gamma}^{-1}(p_{\nu,\gamma}(\mathrm{Fl}^{\leq \overline{u}_t} \cap Z_{\nu,\gamma})) \subseteq \mathrm{Fl}^{\leq \overline{u}_t}$. But this follows from Proposition 3.1.8(c). □

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