RIGHT BOL QUASI-FIELDS

MICHAEL J. KALLAHER

1. Introduction. We shall consider quasi-fields which satisfy the multiplicative identity

$$(1.1)\qquad (xy \cdot z)y = x(yz \cdot y).$$

(1.1) will be called the right Bol law and a quasi-field satisfying it will be called a right Bol quasi-field. Moufang quasi-fields, i.e., those satisfying the Moufang identity

$$(1.2)\qquad (xy)(zx) = (x \cdot yz)x$$

were studied in (**5**). Quasi-fields satisfying the left Bol identity

$$(1.3)\qquad y(x \cdot yz) = (y \cdot xy)z$$

were studied by Burn (**3**) and the author (**6**). Such quasi-fields are called Bol quasi-fields.

Our investigation will parallel the investigations in (**5**; **6**). In § 2 we derive necessary and sufficient conditions for a right Bol quasi-field to be an alternative division ring and also criteria for it to be a near-field. With this information we derive in §§ 3 and 4 new characterizations of Moufang planes similar to those in (**5**; **6**).

Loops satisfying (1.1) have been studied by Robinson (**10**). He calls such loops Bol loops.

We refer the reader to either (**4**) or (**9**) for the principal definitions and concepts of quasi-fields and projective planes. In particular, we shall assume a familiarity with the notions of coordinatizing a projective plane using a ternary ring. The symbol (R, F) will be used to denote the ternary ring in which R is the set of symbols and F is the ternary function defined on R.

I wish to thank the referee for many helpful suggestions in the final preparation of this paper.

2. Right Bol quasi-fields. A *right Bol quasi-field* is a ternary ring (Q, F) which satisfies

(i) $F(a, b, c) = ab + c$ for all $a, b, c \in Q$;
(ii) Q is an abelian group under addition $(+)$;
(iii) $a(b + c) = ab + ac$ for all $a, b, c \in Q$;
(iv) $(ab \cdot c)b = a(bc \cdot b)$ for all $a, b, c \in Q$.

Received July 15, 1968. This research was partially supported by the National Research Council of Canada; the author held a summer Research Institute Fellowship from the Canadian Mathematical Congress.

Thus a right Bol quasi-field is a quasi-field whose multiplicative loop satisfies the right Bol law (iv). Robinson (**10**) proved that such loops are power-associative, and every element of such a loop has a unique inverse. Also he showed that the right inverse property holds and that the mapping of each element into its inverse is a semi-automorphism. Thus we have the following property.

LEMMA 2.1. *Let (Q, F) be a right Bol quasi-field. For $a \in Q$, $a \neq 0$, there exists an element $a^{-1} \in Q$ such that $aa^{-1} = a^{-1}a = 1$ and*

$$ba \cdot a^{-1} = b, \tag{2.1}$$

$$(ba \cdot b)^{-1} = b^{-1}a^{-1} \cdot b^{-1}, \tag{2.2}$$

for all $b \in Q$, $b \neq 0$.

Remark. The power associativity of loops satisfying the right Bol law was first proven by Bol (**2**). It also follows immediately from a result of Pickert (**9**, pp. 244–245).

Definition. If (Q, F) is a quasi-field, then the *right nucleus* of Q is the set $N_r(Q) = \{n | \, n \in Q, \; xy \cdot n = x \cdot yn \text{ for all } x, y \in Q\}$. The *middle nucleus* $N_m(Q)$ and the *left nucleus* $N_l(Q)$ are similarly defined. The *nucleus* of (Q, F) is the set $N(Q) = N_l(Q) \cap N_m(Q) \cap N_r(Q)$. The *kernel* of (Q, F) is the set $K(Q) = \{k | \, k \in N_r(Q), \; (x + y)k = xk + yk \text{ for all } x, y \in Q\}$, and the *centre* of (Q, F) is $C(Q) = \{c | \, c \in N_r(Q), \; cx = xc \text{ for all } x \in Q\}$.

Remark. In general, $N_r(Q)$ is a near-field and $K(Q)$ is an (associative) division ring, but $N_l(Q)$, $N_m(Q)$, and $C(Q)$ need not be closed under addition, even though they form groups under multiplication. Note also that $C(Q) \subset K(Q)$. Finally, any quasi-field can be regarded in a natural way as a (right) vector space over its kernel (**1**).

LEMMA 2.2. *If (Q, F) is a right Bol quasi-field, then $N_m(Q) = N_r(Q)$, and $C(Q) \subset N(Q)$.*

Proof. Let $n \in N_r(Q)$, $n \neq 0$. Then for any $x, y \in Q$ we have

$$x \cdot ny = x(n \cdot zn) = (xn \cdot z)n = xn \cdot zn = xn \cdot y,$$

where z is defined by $y = zn$. Thus $N_r(Q) \subset N_m(Q)$.

If $n \in N_m(Q)$, $n \neq 0$, then for any $x, y \in Q$, choosing z such that $nz = y$, we have

$$x \cdot yn = x(nz \cdot n) = (xn \cdot z)n = (x \cdot nz)n = xy \cdot n.$$

Hence $N_r(Q) = N_m(Q)$.

For the second statement, let $c \in C(Q)$. Then $cx = xc$ for all x and $c \in N_r(Q) = N_m(Q)$. Thus $c \cdot xy = xy \cdot c = x \cdot yc = x \cdot cy = xc \cdot y = cx \cdot y$, for all x, y. Hence $C(Q) \subset N(Q)$.

LEMMA 2.3. *In a right Bol quasi-field* (Q, F) *we have* $-1 \in N(Q)$ *and hence*

$$(-x)y = x(-y) = -xy. \tag{2.3}$$

Proof. Assume that $1 \neq -1$. Then, since $xy + x(-y) = x[y + (-y)] = 0$, $x(-y) = -xy$, and in particular $x(-1) = -x$ for all x. Hence

$$x \cdot y(-1) = x(-y) = -xy = xy \cdot (-1),$$

and $-1 \in N_r(Q) = N_m(Q)$. Hence $x \cdot (-1)y = x(-1) \cdot y$ for all $x, y \in Q$. In particular, $(-1) \cdot (-1)y = y$, since $(-1)(-1) = 1$. Therefore

$$(-1)[x + (-1)x] = (-1)x + x,$$

which implies that $(-1)x = -x$, since $-1 \neq 1$. Then

$$\begin{aligned}(-1) \cdot xy = -xy = xy \cdot (-1) = x \cdot y(-1) = x \cdot (-1)y \\ = x(-1) \cdot y = (-1)x \cdot y\end{aligned}$$

and $-1 \in N_l(Q)$.

Our first two theorems give necessary and sufficient conditions for a right Bol quasi-field to be an alternative division ring. In (**5**; **6**) the first theorem was proven for Moufang and Bol quasi-fields, respectively.

THEOREM 2.1. *If* (Q, F) *is a right Bol quasi-field, then* (Q, F) *is an alternative division ring if and only if*

$$(a + 1)b = ab + b \tag{2.4}$$

for all $a, b \in Q$.

Proof. If Q is an alternative division ring, then (2.4) holds. Hence we need only prove the sufficiency. We first prove that

$$(ab + a)b^{-1} = a + ab^{-1}$$

for all a, b, with $b \neq 0$. For if c is defined by $a = cb^{-1}$, then

$$\begin{aligned}(ab + a)b^{-1} = a(b + 1) \cdot b^{-1} = [cb^{-1} \cdot (b + 1)]b^{-1} = c[b^{-1}(b + 1) \cdot b^{-1}] \\ = c(b^{-1} + b^{-1}b^{-1}) = cb^{-1} + cb^{-1} \cdot b^{-1} = a + ab^{-1}.\end{aligned}$$

Next we prove that for all $a, b \in Q$,

$$(ab + a)a = ab \cdot a + a^2. \tag{2.5}$$

If $b = 0$ or $a = 0$ there is nothing to prove; thus assume that $b \neq 0$ and $a \neq 0$. Then

$$\begin{aligned}(ab + a)a = (a + ab^{-1})b \cdot a = [(1 + ab^{-1} \cdot a^{-1})a \cdot b]a = (1 + ab^{-1} \cdot a^{-1})(ab \cdot a) \\ = ab \cdot a + (ab^{-1} \cdot a^{-1})(ab \cdot a) = ab \cdot a + a^2.\end{aligned}$$

For the proof of the theorem, let $a, b, c \in Q$. Then

$$[ca \cdot (b + 1)]a = (ca \cdot b + ca)a.$$

But also

$$[ca \cdot (b + 1)]a = c[a(b + 1) \cdot a] = c(ab \cdot a + a^2) = c(ab \cdot a) + ca^2,$$

using (2.5). Hence

$$(ca \cdot b + ca)a = (ca \cdot b)a + c \cdot a^2$$

for all $a, b, c \in Q$. Let $x, y, z \in Q$, $x \neq 0$ and $z \neq 0$. Choose $c, d \in Q$ such that $cz = x$, $dx = y$. Then

$$(y + x)z = (xd + x)z = (cz \cdot d + cz)z = (cz \cdot d)z + cz \cdot z = yz + xz.$$

If $x = 0$ or $z = 0$, this holds trivially; hence Q is a division ring satisfying the right Bol law. Thus Q is an alternative division ring (**11**; **12**).

THEOREM 2.2. *Let (Q, F) be a right Bol quasi-field. Q is an alternative division ring if and only if for all $a, b \in Q$, $a \neq 0$,*

$$(1 - a + ba)a^{-1} = a^{-1} - 1 + b. \tag{2.6}$$

Proof. Again we need only prove that (2.6) implies Q is an alternative division ring. We will do this by showing that (Q, F) satisfies (2.4). We first prove two lemmas.

LEMMA 2.4. *For any $a, b \in Q$, $a \neq 0$, we have:*

$$(1 - a + b)a^{-1} = a^{-1} - 1 + ba^{-1}, \tag{2.7}$$

$$(a - a^2 + ba)a^{-1} = 1 - a + b, \tag{2.8}$$

$$(1 - a + b)a = a - a^2 + ba. \tag{2.9}$$

Proof. Pick c such that $b = ca$. Then applying (2.6) yields (2.7). Thus

$$\begin{aligned}(a - a^2 + ba)a^{-1} = [aa^{-1} \cdot (a - a^2 + ba)]a^{-1} &= a[a^{-1}(a - a^2 + ba) \cdot a^{-1}] \\ &= a[a^{-1} - 1 + (a^{-1} \cdot ba)a^{-1}] = 1 - a + b.\end{aligned}$$

(2.9) is obtained from (2.8) by multiplying both sides by a.

LEMMA 2.5. *For any $a \in Q$, $a \neq 0, 1$, we have $a(1 - a)^{-1} = (1 - a)^{-1}a$ and $(a^{-1} - 1)^{-1} = a(1 - a)^{-1}$.*

Proof. For the first equation we have, using (2.7) with $b = 0$,

$$\begin{aligned}a(1 - a)^{-1} = [1 - (1 - a)](1 - a)^{-1} &= (1 - a)^{-1} - 1 \\ &= (1 - a)^{-1}[1 - (1 - a)] = (1 - a)^{-1}a.\end{aligned}$$

For the second we have

$$\begin{aligned}[a(1 - a)^{-1} \cdot (a^{-1} - 1)]a &= [(1 - a)^{-1}a \cdot (a^{-1} - 1)]a \\ &= (1 - a)^{-1}[a(a^{-1} - 1) \cdot a] = (1 - a)^{-1} \cdot (1 - a)a = a.\end{aligned}$$

Hence $a(1 - a)^{-1} \cdot (a^{-1} - 1) = 1$ or $(a^{-1} - 1)^{-1} = a(1 - a)^{-1}$.

https://doi.org/10.4153/CJM-1969-155-8 Published online by Cambridge University Press

We now turn to the proof of Theorem 2.2. Let $a, b \in Q, b \neq 0, 1$, and choose c such that $(b^{-1} - 1)^{-1}c = a$. Then

$$(1 + a)b = (b^{-1} - 1)^{-1}(b^{-1} - 1 + c) \cdot b.$$

If d is given by $(b^{-1} - 1)^{-1} = db$, we have, applying (2.9),

$$\begin{aligned}(1 + a)b &= [db \cdot (b^{-1} - 1 + c)]b = d[b(b^{-1} - 1 + c) \cdot b] = d(b - b^2 + bc \cdot b)\\ &= db - db \cdot b + (db \cdot c)b = (b^{-1} - 1)^{-1} - (b^{-1} - 1)^{-1} \cdot b + ab\\ &= (b^{-1} - 1)^{-1}(1 - b) + ab = b + ab,\end{aligned}$$

(and using the second equation of Lemma 2.5 in the last step). Hence Q satisfies (2.4) and is therefore an alternative division ring.

We turn now to the question of when a right Bol quasi-field is a near-field. Our two results in this direction are in terms of the "structure" of the quasi-field. In (**5**), it was shown that a Moufang quasi-field is a near-field if it was of dimension two over its kernel and its kernel coincided with its centre. This also holds for right Bol quasi-fields as we shall now see. However, this does not hold for Bol quasi-fields. Burn (**3**) gave an infinite class of Bol quasi-fields which are neither near-fields nor division rings. In each of these quasi-fields, the kernel equals the centre and the quasi-field has dimension two over its kernel.

THEOREM 2.3. *Let (Q, F) be a right Bol quasi-field of dimension two over its kernel $K(Q)$. If $K(Q) = C(Q)$, then (Q, F) is a near-field.*

Proof. In the proof of the analogous theorem in (**5**), the only facts used were

(1) $K(Q) = C(Q) \subset N(Q)$ and

(2) Q satisfies the right alternative law.

(1) follows from Lemma 2.2 and (2) follows from the right Bol law.

André (**1**) defined a class of quasi-fields as follows: Let F be a field and σ an automorphism of F having finite order n. Let K be the fixed field of σ and define the multiplicative homomorphism ν: $F \to K$ by $\nu(x) = x\, x^{\sigma} \ldots x^{\sigma^{n-1}}$. Finally, let μ be any mapping of K into the set $\{0, 1, \ldots, n - 1\}$ subject only to the condition that $\mu(0) = \mu(1) = 0$. The elements of the quasi-field are the elements of F with addition "$\oplus$" and multiplication "$\circ$" defined as follows:

$$x \oplus y = x + y, \qquad x \circ y = x \cdot y^{\sigma(x)},$$

where "$+$" and "$\cdot$" are the addition and multiplication, respectively, of F and $\sigma(x) \equiv \sigma^{\mu\nu(x)}$.

Burn (**3**) proved that an André quasi-field is a near-field if (1) it is Moufang and (2) it satisfies the following reflexive law: $(x \circ y) \circ x = x \circ (y \circ x)$, or if (3) it is finite and satisfies the left Bol law. He did, however, construct infinite André quasi-fields satisfying the left Bol quasi-field. We shall show that every André quasi-field satisfying the right Bol law is a near-field. For this we need the following facts about André quasi-fields:

(i) $\nu(x \circ y) = \nu(x)\nu(y) = \nu(y \circ x)$,
(ii) $\sigma(x \circ y) = \sigma(y \circ x)$.

THEOREM 2.4. *Let (Q, F) be an André quasi-field which satisfies the right alternative law. Then (Q, F) is a near-field.*

Proof. From $[(x \circ y) \circ z] \circ y = x \circ [(y \circ z) \circ y]$, for all x, y, z we obtain:

$$xy^{\sigma(x)}z^{\sigma(x \circ y)}y^{\sigma((x \circ y) \circ z)} = xy^{\sigma(x)}z^{\sigma(y)\sigma(x)}y^{\sigma(y \circ z)\sigma(x)}.$$

Then $z = 1$ implies, since $x \circ 1 = x$ for all $x \in Q$, that

$$y^{\sigma(x \circ y)} = y^{\sigma(y)\sigma(x)}$$

for all $x, y \in Q$. Now $\sigma(x \circ y) = \sigma(y \circ x)$ implies that

$$(x \circ y) \circ x = xy^{\sigma(x)} \cdot x^{\sigma(x \circ y)} = xy^{\sigma(x)}x^{\sigma(y)\sigma(x)}$$

and

$$x \circ (y \circ x) = xy^{\sigma(x)}x^{\sigma(y)\sigma(x)}.$$

Hence, the reflexive law holds and (Q, F) is a near-field.

3. Right Bol planes. Let $\mathscr{P}$ be a projective plane and let A, B, and C be three distinct non-collinear points. The Bol configuration with basis points A, B, and C is as follows: From

$$\begin{aligned} &E_1C = E_2C, \quad E_2B = Q_2B, && E_iA = P_iA \ (i = 1, 2),\\ &P_1C = P_2C, \quad P_1B = R_1B, && Q_iA = R_iA \ (i = 1, 2),\\ &Q_1C = Q_2C, \quad E_1B = Q_1B = P_2B = R_2B, &&\\ &\qquad E_i, P_i, Q_i, R_i \notin AB, AC, BC && (i = 1, 2),\end{aligned}$$

follows

$$R_1C = R_2C$$

(Figure 1).

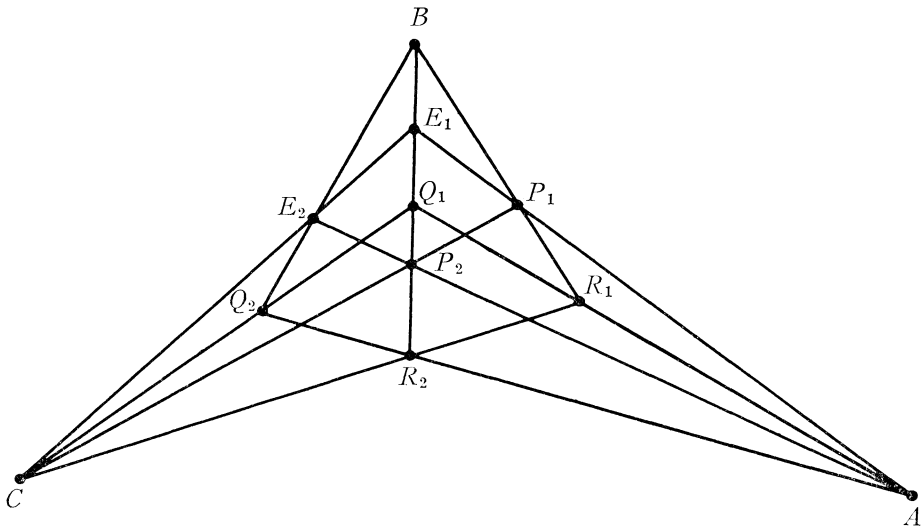

FIGURE 1

The Reidemeister configuration is obtained if we replace the condition $E_1B = \cdots = R_2B$ by the separate conditions $E_1B = Q_1B$ and $P_2B = R_2B$.

We shall use Hall's method of coordinatization (**4**) with "slope" written on the left. If a projective plane $\mathscr{P}$ is coordinatized by a ternary ring (R, F), then with (R, F) can be associated four points U, V, O, I of $\mathscr{P}$. U is the point (0) or the point at infinity of the x-axis, V is the point (∞) or the point at infinity of the y-axis, O is the point $(0, 0)$, and I the point $(1, 1)$. We will call these four points the basis points of (R, F) and, in particular, U and V will be called the points at infinity of (R, F). When (R, F) is a quasi-field, we will sometimes say that $\mathscr{P}$ is coordinatized by (R, F) with respect to U and V. Note, however, that order is important in this terminology: the first point is always the point (0), the second the point (∞), etc.

The Bol configuration is intimately tied up with the right and left Bol laws for loops and with the Moufang law. We summarize here what is contained in (**9**, pp. 50–57). Let $\mathscr{P}$ be a projective plane coordinatized by a ternary ring (R, F) with basis points U, V, O, I. (R, F) satisfies the left Bol law if and only if the Bol condition holds for $A = V$, $B = O$, $C = U$.

The right Bol law for (R, F) is equivalent to the Bol condition for $A = O$, $B = V$, $C = U$. The Moufang law for (R, F) is equivalent to the Bol condition holding for the above two cases as well as the case $A = O$, $B = U$, $C = V$. Furthermore, the Bol condition holding for any two of these three cases implies that it holds for the third case also and thus (R, F) satisfies the Moufang law. For example, if (R, F) satisfies the right Bol law and $\mathscr{P}$ has a collineation fixing O and interchanging U and V, then (R, F) satisfies the Moufang identity.

The Reidemeister condition for $A = V$, $B = O$, $C = U$ is equivalent to (R, F) having associative multiplication. We also should mention that the Bol condition is symmetric in A and C while the Reidemeister condition is symmetric in A, B, C.

Finally we mention that if one ternary ring satisfies the right Bol law (left Bol law, Moufang law) then every ternary ring having the same first three basis points satisfies the right Bol law (left Bol law, Moufang law).

In (**8**) Klingenberg proved that the Reidemeister condition holding generally in a projective plane implies the Theorem of Desargues and in (**7**) he showed that the Bol condition holding generally implies the small Theorem of Desargues.

Definition. A projective plane $\mathscr{P}$ is a *right Bol plane* (with respect to the points U and V) if one of the ternary rings of $\mathscr{P}$ (with U and V as its points at infinity) is a right Bol quasi-field.

THEOREM 3.1. *A projective plane $\mathscr{P}$ is a Moufang plane if there exist in $\mathscr{P}$ distinct points U, V, U' with $UV = U'V$ such that $\mathscr{P}$ is a right Bol plane with respect to U and V and also with respect to U' and V.*

Proof. Choose a Bol quasi-field (Q, F) of $\mathscr{P}$ such that U and V are its points at infinity and $U' = (1)$. This we may do since $\mathscr{P}$ is (UV, UV)-transitive and the fact that the right Bol law is an isotopic invariant. Pick $a, b \in Q$ with $a \neq 0, 1, b \neq 0, 1$. Consider the points

$$E_1 = (1, 1 - a), \quad E_2 = (a, 0), \quad P_1 = (x, x - 1), \quad P_2 = (1, 0),$$
$$Q_1 = (1, 1 - a + ba), \quad Q_2 = (a, ba),$$
$$R_1 = (x, (1 - a + ba)x), \quad R_2 = (1, b),$$

where x is given by the equation

$$(1 - a)x = x - 1 \tag{3.1}$$

(Figure 2). These eight points satisfy the hypothesis of the Bol configuration for the points $A = (0, 0)$, $B = V$, $C = U'$. Thus $R_1U' = R_2U'$ or

$$x + b - 1 = (1 - a + ba)x. \tag{3.2}$$

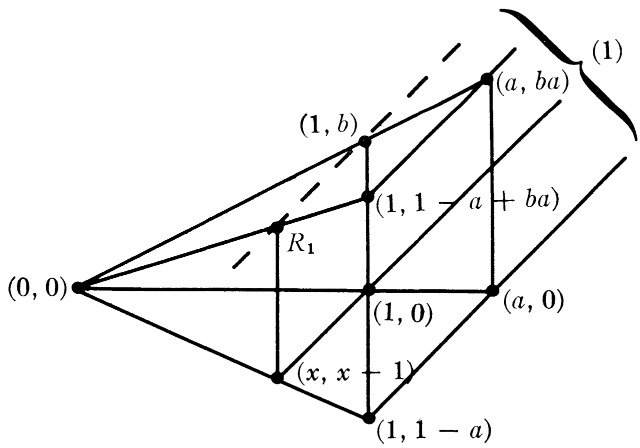

FIGURE 2

We have shown that given $a \neq 0, 1$, if x satisfies (3.1), then for any $b \neq 0, 1$, (3.2) holds. Thus x depends only on a. Assume, in addition, that $a \neq -1$ and let $b = -a^{-1}$ in (3.2). This yields

$$x - a^{-1} - 1 = (-a)x.$$

$x = a^{-1}$ is a solution of this equation. If $a = -1$, then it follows directly from (3.1) that $x = -1$. Therefore $x = a^{-1}$ in (3.1) and (3.2).

Hence, given $a \neq 0, 1, b \neq 0, 1$, we have proven:

$$(1 - a)a^{-1} = a^{-1} - 1, \tag{3.3}$$

$$(1 - a + ba)a^{-1} = a^{-1} + b - 1. \tag{3.4}$$

If $a = 1$ or $b = 1$, (3.4) is trivial. If $b = 0$, (3.4) reduces to (3.3). Hence Theorem 2.2 implies that Q is an alternative division ring and therefore $\mathscr{P}$ is Moufang.

COROLLARY 3.1.1. *Let $\mathscr{P}$ be a right Bol plane with respect to the points U and V. $\mathscr{P}$ is a Moufang plane if and only if there exists a collineation of $\mathscr{P}$ fixing V and moving U.*

Proof. If $\mathscr{P}$ is Moufang, such collineations exist. On the other hand, if σ is such a collineation, then $U^\sigma V \neq UV$ implies that $\mathscr{P}$ is a translation plane with respect to two lines and hence $\mathscr{P}$ is a Moufang plane (**4**, p. 372). If $U^\sigma V = UV$, then Theorem 3.1 applies and $\mathscr{P}$ is Moufang.

Remark. More general results than Theorem 3.1 and its corollary have been obtained by André (**1**) in the case of near-field planes and by the author for planes coordinatized by Moufang quasi-fields (**5**) or Bol quasi-fields (**6**).

4. Integration. In this section we consider what happens if a projective plane $\mathscr{P}$ is coordinatized by a Bol quasi-field with respect to the points U and V and also by a right Bol quasi-field with respect to the points U' and V'. If $U' = U$, $V' = V$ (or if $U' = V$, $V' = U$), then $\mathscr{P}$ is coordinatized by a Moufang quasi-field with respect to U and V. This is the best we can hope for; for example, $\mathscr{P}$ could be coordinatized by a near-field.

If U' and V' are both distinct from U and V, then it is still possible for $\mathscr{P}$ not to be Moufang. André's exceptional plane (the plane over the Hall near-field of order 9) is an example of this (**1**). Thus we have to add an additional restriction. The restriction needed is the same one that was needed in (**5**; **6**). With this restriction, we shall show that $\mathscr{P}$ is Moufang.

THEOREM 4.1. *Assume that $\mathscr{P}$ is a projective plane coordinatized by a Bol quasi-field with respect to the points U and V. If there exists a point U' with $U' \neq U$, V, $U'V = UV$ such that $\mathscr{P}$ is a right Bol plane with respect to U' and V, then $\mathscr{P}$ is Moufang.*

Proof. Let (Q, F) be a Bol quasi-field coordinatizing $\mathscr{P}$ with respect to U and V. We denote the points of $\mathscr{P}$ by their coordinates (x, y), (m) with respect to (Q, F). Without loss of generality, we may assume that $U' = (1)$, since $\mathscr{P}$ is (UV, UV)-transitive and the fact that the right Bol law is an isotopic invariant. Since every quasi-field of order less than 9 is a field, we may assume that Q has at least nine elements. The following two mappings are collineations of $\mathscr{P}$:

$$\sigma\colon (x, y) \to (y, x), \qquad (m) \to (m^{-1}),\ U \to V,\ V \to U, \tag{4.1}$$

$$\tau\colon (x, y) \to (a^{-1}y, ax), \qquad (m) \to (a \cdot m^{-1}a),\ U \to V,\ V \to U, \tag{4.2}$$

where $a \in Q$, $a \neq 0, \pm 1$. Their product $\sigma\tau$ maps (1) into (a^2) and fixes U and V. $a^2 \neq 1$ since this would imply that $a = \pm 1$; see (**3**). Thus $\sigma\tau$ fixes V and moves U'. By Corollary 3.1.1, $\mathscr{P}$ is Moufang.

THEOREM 4.2. *Assume that $\mathscr{P}$ is a projective plane coordinatized by a Bol quasi-field with respect to the points U and V. If there exists a point V' with $V' \neq U, V, UV' = UV$ such that $\mathscr{P}$ is a right Bol plane with respect to U and V', then $\mathscr{P}$ is Moufang.*

Proof. Choose a coordinate system (Q, F) for $\mathscr{P}$ with U and V' as its points at infinity. Then (Q, F) is a right Bol quasi-field. If we represent the points of P by their coordinates (x, y), (m) with respect to (Q, F), then, as before, we may assume that V has coordinate (1). Since $\mathscr{P}$ is coordinatized by a Bol quasi-field with respect to U and V, the Bol configuration holds for $A = V$, $B = (0, 0)$, $C = U$.

Pick $b, c \in Q$ with $b \neq 0, 1$, $c \neq 0, 1$. Consider the points

$$E_1 = (0, c), \quad E_2 = (1, c), \quad P_1 = (-1, -1 + c), \quad P_2 = (0, -1 + c),$$
$$Q_1 = (0, cb), \quad Q_2 = (b, cb),$$
$$R_1 = (f, f + cb), \quad R_2 = (0, -b + cb),$$

where f is defined by the equation $(1 - c)f = f + cb$ (Figure 3).

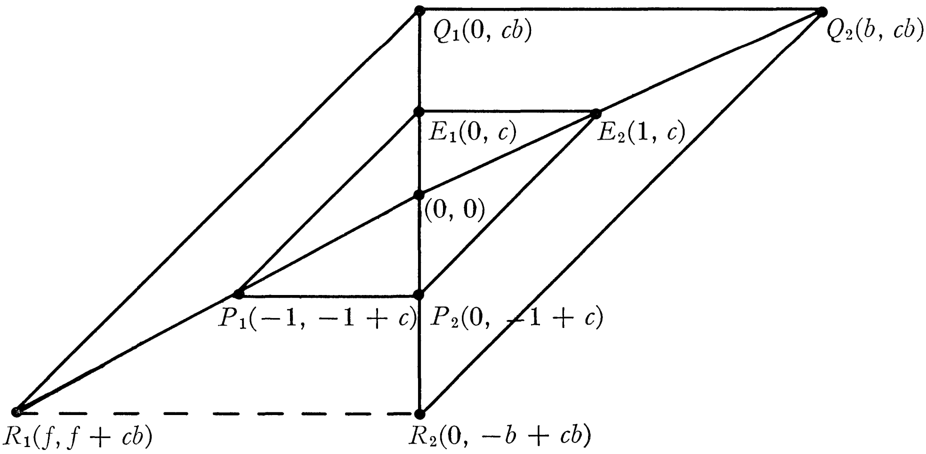

FIGURE 3

These eight points satisfy the hypothesis of the Bol configuration for $A = V$, $B = (0, 0)$, $C = U$. Therefore $R_1U = R_2U$, or $f = -b$. Hence

$$(1 - c)(-b) = -b + cb \tag{4.3}$$

for all $b, c \in Q$, $b \neq 0, 1$, $c \neq 0, 1$. Since $-1 \in C(Q) \subset N(Q)$ we have, replacing $-c$ by d and $-b$ by e,

$$(1 + d)e = e + de \tag{4.4}$$

for all $d, e \in Q$ with $d \neq 0, 1$, $e \neq 0, -1$. (4.4) still holds if $d = 0$ or -1, or if

$e = 0$ or -1. Thus by Theorem 2.1, Q is an alternative division ring and $\mathscr{P}$ is Moufang.

Definition. A right Bol quasi-field (Q, F) is *non-exceptional* if $N_r(Q)$ is neither the Hall near-field of nine elements, nor GF(3), nor GF(5). A projective plane $\mathscr{P}$ is a *non-exceptional right Bol plane* (with respect to the points U and V) if one of the ternary rings of $\mathscr{P}$ (with U and V as its points at infinity) is a non-exceptional right Bol quasi-field.

LEMMA 4.1. *If (Q, F) is a non-exceptional right Bol quasi-field with characteristic not 2, there exists $b \in N_r(Q)$ with $b \neq 0, 1, -1$ and $b^2 \neq -1$.*

Proof. If $N_r(Q)$ is a proper near-field, the assertion follows from André's result that the only near-field with no such element b is the Hall near-field of order 9 (**1**, pp. 141–142). Clearly, the only fields not fulfilling the assertion are GF(3) and GF(5). In the case of a non-commutative division ring, the lemma follows from the fact that to every subfield there exists a pure extension field.

If the non-exceptional right Bol quasi-field (Q, F) is actually Moufang, then our definition agrees with that of a special Moufang quasi-field given in (**5**), since in this case $N_r(Q) = N(Q)$. By the lemma, (Q, F) will then also be a non-exceptional Bol quasi-field if it has characteristic not 2 (**6**).

For our next theorem we need the following result proved in (**6**).

THEOREM A. *Let $\mathscr{P}$ be a projective plane coordinatized by a non-exceptional Bol quasi-field with respect to the distinct points U and V. If there exist distinct points U' and V' with $U' \neq U$, V such that $\mathscr{P}$ is also coordinatized by a Bol quasi-field with respect to U' and V', then $\mathscr{P}$ is a Moufang plane.*

THEOREM 4.3. *Let $\mathscr{P}$ be a non-exceptional right Bol plane with respect to the points U and V. If there exist distinct points U', V' with*

$$UV' = UV, \quad U' \neq U, \quad V, V' \neq U, V$$

such that $\mathscr{P}$ is coordinatized by a Bol quasi-field with respect to U' and V', then $\mathscr{P}$ is a Moufang plane.

Proof. Choose a coordinate system (Q, F) for $\mathscr{P}$ with U' and V' as its points at infinity. If we denote the points of $\mathscr{P}$ by their coordinates with respect to (Q, F), then we may assume that $V = (1)$. Assume $U = (a)$, $a \neq -1$. Then the mapping σ defined in (4.1) is a collineation of $\mathscr{P}$ since (Q, F) is a Bol quasi-field. σ fixes V and moves U since $a^{-1} = a$ implies $a = 1$ or -1 (**3**). Corollary 3.1.1 implies that $\mathscr{P}$ is Moufang.

Assume now that $U = (-1)$. The characteristic of (Q, F) is not 2. Since $-1 \in C(Q) \cap N(Q)$ (see **3**), the mapping $(x, y) \to (x, -y)$, $(m) \to (-m)$, $V' \to V'$ is a collineation of $\mathscr{P}$ which interchanges U and V. Thus, if (Q', F') is a non-exceptional right Bol quasi-field coordinatizing $\mathscr{P}$ and having basic points U and V, then it is a Moufang quasi-field. Furthermore, (Q, F) having characteristic not 2 implies that (Q', F') has characteristic not 2. Thus (Q', F') is a non-exceptional Bol quasi-field. By Theorem A, $\mathscr{P}$ is Moufang.

THEOREM 4.4. *Let $\mathscr{P}$ be a non-exceptional right Bol plane with respect to the points U and V. If there exist points U', V' in $\mathscr{P}$ with $U' \neq U, V$ such that $\mathscr{P}$ is coordinatized by a Bol quasi-field with respect to U' and V', then $\mathscr{P}$ is a Moufang plane.*

Proof. If $U'V' \neq UV$, then $\mathscr{P}$ is a translation plane with respect to two lines and thus is Moufang. If $U'V' = UV$, then the theorem follows from the previous theorems.

REFERENCES

1. Johannes André, *Projektiven Ebenen über Fastkörpern*, Math. Z. *62* (1955), 137–160.
2. G. Bol, *Topologische Fragen der Differentialgeometrie*, 65, *Gewebe und Gruppen*, Math. Ann. *114* (1937), 414–431.
3. Robert Burn, *Bol quasi-fields and Pappus's theorem*, Math. Z. *105* (1968), 351–364.
4. Marshall Hall, Jr., *Theory of groups* (Macmillan, New York, 1959).
5. Michael J. Kallaher, *Moufang Veblen-Wedderburn systems*, Math. Z. *105* (1968), 1.
6. ——— *Projective planes over Bol quasi-fields*, Math. Z. *109* (1969), 53–65.
7. Wilhelm Klingenberg, *Beziehungen zwischen einigen affinen Schliessungssätzen*, Abh. Math. Sem. Univ. Hamburg *18* (1952), 120–143.
8. ——— *Beweis des Desarguesschen Satzes aus der Reidemeisterfigur und verwandte Sätze*, Abh. Math. Sem. Univ. Hamburg *19* (1955), 158–175.
9. Günter Pickert, *Projektive Ebenen* (Springer-Verlag, Berlin, 1955).
10. D. A. Robinson, *Bol loops*, Trans. Amer. Math. Soc. *123* (1966), 341–354.
11. R. L. San Soucie, *Right alternative division rings of characteristic two*, Proc. Amer. Math. Soc. *6* (1955), 291–296.
12. L. A. Skornyakov, *Right-alternative fields*, Izv. Akad. Nauk SSSR Ser. Mat. *15* (1951), 177–184. (Russian)

The University of Manitoba,
Winnipeg, Manitoba;
Washington State University,
Pullman, Washington