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A FUNCTIONAL LOGARITHMIC FORMULA FOR THE HYPERGEOMETRIC FUNCTION ${}_3F_2$

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Dedicated to the 60th birthday of Professor Shuji Saito

Abstract. We give a sufficient condition for the hypergeometric function ${}_3F_2$ to be a linear combination of the logarithm of algebraic functions.

§1. Introduction

For $\alpha_i, \beta_j \in \mathbb{C}$ with $\beta_j \notin \mathbb{Z}_{\leqslant 0}$, the *generalized hypergeometric function* is defined by a power series expansion

$$ {}_pF_{p-1}\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_{p-1}\end{matrix};x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_{p-1})_n}\frac{x^n}{n!}, $$

where

$$ (\alpha)_0 := 1, \qquad (\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) \quad \text{for } n \geqslant 1 $$

denotes the Pochhammer symbol. When $p=2$, this is called the Gauss hypergeometric function. This has an analytic continuation to $\mathbb{C}$, and then becomes a multivalued function which is holomorphic on $\mathbb{C}\setminus\{0,1\}$. A number of formulas have been discovered since 19th century (e.g., [10, Chapters 15, 16]), and they have been applied in various areas in mathematics. At present, the theory of hypergeometric function is one of the most important tools in mathematics.

In [5], we discussed the special values of ${}_3F_2\left(\begin{smallmatrix}1,1,q\\ a,b\end{smallmatrix};x\right)$ at $x=1$, and gave a sufficient condition for it to be a $\overline{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers, namely

$$ {}_3F_2\left(\begin{matrix}1,1,q\\ a,b\end{matrix};1\right)\in\overline{\mathbb{Q}}+\overline{\mathbb{Q}}\log\overline{\mathbb{Q}}^{\times} $$

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$$:= \left\{ a + \sum_{i=1}^{n} b_i \log c_i \;\middle|\; a, b_i, c_i \in \overline{\mathbb{Q}},\ c_i \neq 0,\ n \in \mathbb{Z}_{\geqslant 0} \right\}.$$

The goal of this paper is to give its functional version. To be precise, set

$$\begin{aligned}&\overline{\mathbb{Q}(x)} + \overline{\mathbb{Q}(x)} \log \overline{\mathbb{Q}(x)}^{\times} \\ &\quad := \left\{ f + \sum_{i=1}^{n} g_i \log h_i \;\middle|\; f, g_i, h_i \in \overline{\mathbb{Q}(x)},\ h_i \neq 0,\ n \in \mathbb{Z}_{\geqslant 0} \right\}\end{aligned}$$

where $\overline{\mathbb{Q}(x)}$ denotes the algebraic closure of the field of rational functions $\mathbb{Q}(x)$. We say that the *logarithmic formula* holds for a function $F(x)$ if it belongs to the above set. The main theorem gives a sufficient condition on (a, b, q) for ${}_3F_2\left({1,1,q \atop a,b}; x\right)$ to satisfy a logarithmic formula. Recall that two proofs are presented in [5]. One of the proofs uses hypergeometric fibrations and the other uses Fermat surfaces. In this paper we follow the method of hypergeometric fibrations, while employing a new ingredient from [3]. It seems impossible to prove the functional log formula using the method of Fermat surfaces.

By developing the technique here, we can get *explicit* log formulas in some cases. For example, let

$$\begin{aligned} e_1(x) &:= \frac{1}{2} + x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} + \frac{1}{4}\sqrt{1-x}\right)^{1/3} \\ &\quad + x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} - \frac{1}{4}\sqrt{1-x}\right)^{1/3} \\ e_2(x) &:= \frac{1}{2} + e^{-2\pi i/3}x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} + \frac{1}{4}\sqrt{1-x}\right)^{1/3} \\ &\quad + e^{2\pi i/3}x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} - \frac{1}{4}\sqrt{1-x}\right)^{1/3} \\ e_3(x) &:= \frac{1}{2} + e^{2\pi i/3}x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} + \frac{1}{4}\sqrt{1-x}\right)^{1/3} \\ &\quad + e^{-2\pi i/3}x^{-1/3}\left(-\frac{1}{4} + \frac{x}{8} - \frac{1}{4}\sqrt{1-x}\right)^{1/3} \end{aligned}$$

$$p_\pm = p_\pm(x) := \left(\frac{1 \pm \sqrt{1-x}}{\sqrt{x}}\right)^{2/3}, \qquad q_j = q_j(x) := \frac{1 - \sqrt{3x} \cdot e_j(x)}{1 + \sqrt{3x} \cdot e_j(x)}.$$

Then

$$
{}_3F_2\left(\begin{matrix}1, 1, \frac{1}{2} \\ \frac{7}{6}, \frac{11}{6}\end{matrix}; x\right)
= \frac{5\sqrt{3}}{36}x^{-1/2}\left[(p_+ + p_-)\log\left(\frac{q_1}{q_2}\right) + (e^{\pi i/3}p_+ + e^{-\pi i/3}p_-)\log\left(\frac{q_2}{q_3}\right)\right].
$$

However, there remain technical difficulties arising from algebraic cycles to obtain explicit log formulas in more general cases.

§2. Main theorem

Let $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ be the completion, and $\hat{\mathbb{Z}}^\times = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times$ the group of units. The ring $\hat{\mathbb{Z}}$ acts on the additive group $\mathbb{Q}/\mathbb{Z}$ in a natural way, and then it induces $\hat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$. We denote by $\{x\} := x - \lfloor x \rfloor$ the fractional part of $x \in \mathbb{Q}$. The map $\{-\} : \mathbb{Q} \to [0, 1)$ factors through $\mathbb{Q}/\mathbb{Z}$, which we denote by the same notation.

THEOREM 2.1. (Logarithmic formula) *Let $q, a, b \in \mathbb{Q}$ satisfy the property that none of $q, a, b, q-a, q-b, q-a-b$ is an integer. Suppose*

$$
\begin{aligned}
&1 = \{sa\} + \{sb\} + 2\{-sq\} - \{s(a-q)\} - \{s(b-q)\} \\
&(\Longleftrightarrow \quad \min(\{sa\}, \{sb\}) < \{sq\} < \max(\{sa\}, \{sb\}))
\end{aligned} \tag{2.1}
$$

for $\forall s \in \hat{\mathbb{Z}}^\times$. Then

$$
{}_3F_2\left(\begin{matrix}n_1, n_2, q \\ a, \ b\end{matrix}; x\right) \in \overline{\mathbb{Q}(x)} + \overline{\mathbb{Q}(x)}\log\overline{\mathbb{Q}(x)}^\times
$$

for any integers $n_i > 0$.

As we shall see in Section 4, one can shift the indices n_i, q, a, b by arbitrary integers by applying differential operators. Thus it is enough to prove the log formula for ${}_3F_2\left(\begin{smallmatrix}1,1,q \\ a, \ b\end{smallmatrix}; x\right)$.

Recall the main theorem of [5] which asserts that if

$$
2 = \{sq\} + \{s(a-q)\} + \{s(b-q)\} + \{s(q-a-b)\} \tag{2.2}
$$

for $\forall s \in \hat{\mathbb{Z}}^\times$, then

$$
{}_3F_2\left(\begin{matrix}1, 1, q \\ a, \ b\end{matrix}; 1\right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\log\overline{\mathbb{Q}}^\times
$$

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as long as it converges ($\Leftrightarrow a+b>q+2$). It is easy to see (2.1) $\Rightarrow$ (2.2) while the converse is no longer true (e.g., $(a,b,q)=(1/6,1/4,1/2)$). Therefore Theorem 2.1 does not imply all of the main theorem of [5].

CONJECTURE 2.2. (Cf. [5, Conjecture 5.2]) *The converse of Theorem 2.1 is true.*

In the seminal paper [7], Beukers and Heckman gave a necessary and sufficient condition for ${}_pF_{p-1}$ to be an algebraic function, or equivalently for its monodromy group to be finite. Let $a_i, b_j \in \mathbb{Q}$. Then their theorem states that, under the condition that $\{a_i\} \neq \{b_j\}$ and $\{a_i\} \neq 0$,

$$ {}_pF_{p-1}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{p-1} \end{matrix};x\right) \in \overline{\mathbb{Q}(x)} $$

if and only if $(\{sa_1\},\ldots,\{sa_p\})$ and $(0,\{sb_1\},\ldots,\{sb_{p-1}\})$ interlace for all $s\in\hat{\mathbb{Z}}^\times$ [7, Theorem 4.8]. Here we say that two sets $(\alpha_1,\ldots,\alpha_p)$ and $(\beta_1,\ldots,\beta_p)$ interlace if and only if

$$ \alpha_1<\beta_1<\cdots<\alpha_p<\beta_p \qquad \text{or} \qquad \beta_1<\alpha_1<\cdots<\beta_p<\alpha_p $$

when ordering $\alpha_1<\cdots<\alpha_p$ and $\beta_1<\cdots<\beta_p$. In this terminology, (2.1) is translated into that $(0,\{sq\})$ and $(\{sa\},\{sb\})$ interlace. Our main Theorem 2.1 is not directly related to their theorem, while they are obviously comparable.

§3. Hypergeometric fibrations

We mean by a *fibration* over a ring k a projective flat morphism of quasiprojective smooth k-schemes.

3.1 Definition

Let $f:X\to\mathbb{P}^1$ be a fibration over a field k. For simplicity we assume $k=\overline{k}$ and fix an embedding $k\subset\mathbb{C}$. Let R be a finite-dimensional semisimple commutative $\mathbb{Q}$-algebra. We mean by a *multiplication* on $R^1f_*\mathbb{Q}$ by R a homomorphism $\rho:R\to\mathrm{End}_{\mathrm{VHS}}(R^1f_*\mathbb{Q}|_U)$ of rings where $U\subset\mathbb{P}^1$ is the maximal Zariski open set such that f is smooth over U. Let $e:R\to E$ be a projection onto a number field E. We say f is a *hypergeometric fibration with multiplication by* (R,e) (HG fibration) if the following conditions hold. We fix an inhomogeneous coordinate $t\in\mathbb{P}^1$.

(a) f is smooth over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,
(b) $\dim_E H^1(X_t, \mathbb{Q})(e) = 2$ where $X_t = f^{-1}(t)$ is a general fiber and we write $V(e) := E \otimes_{e,R} V$ the e-part,
(c) Let $\mathrm{Pic}^0_f \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Picard fibration whose general fiber is the Picard variety $\mathrm{Pic}^0(f^{-1}(t))$, and let $\mathrm{Pic}^0_f(e)$ be the component associated to the e-part $R^1 f_* \mathbb{Q}(e)$ (this is well defined up to isogeny). Then $\mathrm{Pic}^0_f(e) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has totally degenerate semistable reduction at $t = 1$.

The last condition (c) is equivalent to saying that the local monodromy T on $H^1(X_t, \mathbb{Q})(e)$ at $t = 1$ is unipotent and the rank of log monodromy $N := \log(T)$ is maximal, namely $\mathrm{rank}(N) = \frac{1}{2} \dim_{\mathbb{Q}} H^1(X_t, \mathbb{Q})(e)$ ($= [E : \mathbb{Q}]$ by condition (b)).

3.2 HG fibration of Gauss type

Let $f : X \to \mathbb{P}^1$ be a fibration over $\overline{\mathbb{Q}}$ whose general fiber $X_t = f^{-1}(t)$ is the nonsingular projective model of the affine curve

$$y^N = x^a (1 - x)^b (1 - tx)^{N-b}, \quad 0 < a, b < N, \ \gcd(N, a, b) = 1. \tag{3.1}$$

f is smooth outside $\{0, 1, \infty\}$ so that the condition (a) is satisfied. The group μ_N of Nth roots of unity acts on $f^{-1}(t)$ by $(x, y, t) \mapsto (x, \zeta y, t)$ for $\zeta \in \mu_N$, which gives rise to a multiplication on $R^1 f_* \mathbb{Q}$ by the group ring $R_0 := \mathbb{Q}[\mu_N]$.

LEMMA 3.1. *[4, Proposition 3.1] Let* $e_0 : R_0 := \mathbb{Q}[\mu_N] \to E_0$ *be a projection onto a number field* E_0*. Then* (R_0, e_0) *satisfies the conditions (b) and (c) if and only if* $ad \not\equiv 0$ *and* $bd \not\equiv 0$ *modulo* N *where* $d := \sharp \mathrm{Ker}[\mu_N \to R_0^\times \xrightarrow{e_0} E_0^\times]$.

DEFINITION 3.2. We say that f is a *HG fibration of Gauss type with multiplication by* $(\mathbb{Q}[\mu_N], e)$ if $ad \not\equiv 0$ and $bd \not\equiv 0$ modulo N.

Let $\chi : R_0 \to \overline{\mathbb{Q}}$ be a homomorphism of $\mathbb{Q}$-algebras factoring through e. Let n be an integer such that $\chi(\zeta) = \zeta^{-n}$ for all $\zeta \in \mu_N$. Note $\gcd(n, N) = 1$. By [1, p. 917, (13)], $H^1_{\mathrm{dR}}(X_t)(\chi) \cap H^{1,0}$ is spanned by the 1-form

$$\omega_n := \frac{x^{a_n}(1 - x)^{b_n}(1 - tx)^{c_n}}{y^n}\, dx,$$

$$a_n := \left\lfloor \frac{an}{N} \right\rfloor, \qquad b_n := \left\lfloor \frac{bn}{N} \right\rfloor, \qquad c_n := \left\lfloor \frac{Nn - bn}{N} \right\rfloor = n - b_n - 1.$$

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Let P_1 (resp. P_2) be a point of X_t above $x=0$ (resp. $x=1$). There are gcd(N,a)-points above $x=0$ (resp. gcd(N,b)-points above $x=1$). Let u be a path from P_1 to P_2 above the real interval $x\in[0,1]$. It defines a homology cycle $u\in H_1(X_t,\{P_1,P_2\};\mathbb{Z})$ with boundary. Put $d_1:=\gcd(N,a)$, $d_2:=\gcd(N,b)$. Let $\sigma\in\mu_N$ be an automorphism. Since $\sigma^{d_1}P_1=P_1$ and $\sigma^{d_2}P_2=P_2$, one has a homology cycle

$$\delta(\sigma):=(1-\sigma^{d_1})(1-\sigma^{d_2})u\in H_1(X_t,\mathbb{Z}). \tag{3.2}$$

By an integral expression of Gauss hypergeometric functions (e.g., [6, p. 4, 1.5] or [11, p. 20, (1.6.6)]), one has

$$\int_{\delta(\sigma)}\omega_n=(1-\zeta^{-nd_1})(1-\zeta^{-nd_2})\int_0^1\omega_n \tag{3.3}$$

$$=(1-\zeta^{-nd_1})(1-\zeta^{-nd_2})B(\alpha_n,\beta_n)_2F_1(\alpha_n,\beta_n,\alpha_n+\beta_n;t), \tag{3.4}$$

where $B(\alpha,\beta):=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function, ζ is defined by $\sigma(y)=\zeta y$ and we put

$$\alpha_n:=\left\{\frac{-an}{N}\right\},\qquad \beta_n:=\left\{\frac{-bn}{N}\right\}.$$

This shows that the monodromy on the 2-dimensional $H_1(X_t,\mathbb{C})(\chi)$ is isomorphic to the monodromy of the hypergeometric equation

$$(D_t(D_t+\alpha_n+\beta_n-1)-t(D_t+\alpha_n)(D_t+\beta_n))(y)=0,\quad D_t:=t\frac{d}{dt}$$

with the Riemann scheme

$$\left\{\begin{matrix} t=0 & t=1 & t=\infty\\ 0 & 0 & \alpha_n\\ 1-\alpha_n-\beta_n & 0 & \beta_n\end{matrix}\right\} \tag{3.5}$$

In particular, the monodromy is irreducible as $\alpha_n,\beta_n\notin\mathbb{Z}$.

3.3 Hodge numbers

Let $f:X\to\mathbb{P}^1$ be a HG fibration with multiplication by (R_0,e_0). Following [3, Section 4.1], we consider motivic sheaves $\mathscr{M}$ and $\mathscr{H}$ which are defined in the following way. Let $S:=\mathbb{A}^1_{\overline{\mathbb{Q}}}\setminus\{0,1\}$ be defined over $\overline{\mathbb{Q}}$ with coordinate λ. Let $\mathbb{P}^1_S:=\mathbb{P}^1\times S$ and denote the coordinates by (t,λ).

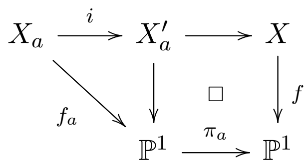
Put $\mathbb{P}^1_S \supset \mathscr{U} := (\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\} \times S) \setminus \Delta$ where Δ is the diagonal subscheme. Let $l \geqslant 1$ be an integer. Let $\pi : \mathbb{P}^1_S \to \mathbb{P}^1_S$ be a morphism over S given by $(t, \lambda) \mapsto (\lambda - t^l, \lambda)$. Then we define

$$\mathscr{M} := \pi_* \mathbb{Q} \otimes \mathrm{pr}_1^* R^1 f_* \mathbb{Q}|_{\mathscr{U}}, \qquad \mathrm{pr}_1 : \mathbb{P}^1_S = \mathbb{P}^1 \times S \to \mathbb{P}^1$$

a variation of Hodge–de Rham structures (VHdR) on $\mathscr{U}$ and

$$\mathscr{H} := R^1 \mathrm{pr}_{2*} \mathscr{M}, \qquad \mathrm{pr}_2 : \mathscr{U} \to S$$

a variation of mixed Hodge–de Rham structures (VMHdR) on S, where the terminology is as in [3, Section 2.1] or [4, Section 2.1]. For the reader's convenience, we give a description of the stalk $H_a = \mathscr{H}|_{\{a\}}$ and $\mathscr{M}_a = \mathscr{M}|_{\mathrm{pr}_2^{-1}(a)} = \mathscr{M}|_{\mathbb{A}^1 \setminus \{0,1,a\}}$ at $a \in S$ is given in the following way. Let $\pi_a : \mathbb{P}^1 \to \mathbb{P}^1$ be the map given by $t \mapsto a - t^l$. Let



be a Cartesian diagram, and i a desingularization along the singular fibers. Put $U_a := \pi_a^{-1}(\mathbb{A}^1 \setminus \{0,1,a\})$. Then

$$\begin{aligned} \mathscr{M}_a &= \pi_{a*}\mathbb{Q} \otimes R^1 f_* \mathbb{Q}|_{\mathbb{A}^1 \setminus \{0,1,a\}} = \pi_{a*}\pi_a^* R^1 f_* \mathbb{Q}|_{\mathbb{A}^1 \setminus \{0,1,a\}} \\ &\cong \pi_{a*} R^1 f_{a*} \mathbb{Q}|_{\mathbb{A}^1 \setminus \{0,1,a\}}, \\ H_a &= H^1(\mathbb{A}^1 \setminus \{0,1,a\}, \mathscr{M}_a) \\ &\cong H^1(U_a, R^1 f_{a*}\mathbb{Q}) \subset H^2(f_a^{-1}(U_a), \mathbb{Q}). \end{aligned} \tag{3.6}$$

The weights of $\mathscr{H}$ are at most 2, 3, 4, and hence there is an exact sequence

$$0 \longrightarrow W_2\mathscr{H} \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}/W_2 \longrightarrow 0 \tag{3.7}$$

of VMHdR structures on S. By (3.6), there is a canonical surjective map

$$H^2(X_a, \mathbb{Q})_0 \longrightarrow W_2 H_a \tag{3.8}$$

where we put $H^2(X_a, \mathbb{Q})_0 := \mathrm{Ker}[H^2(X_a, \mathbb{Q}) \to H^2(f_a^{-1}(t), \mathbb{Q})]$, $t \in U_a$.

Let μ_l be the group of lth roots of unity which acts on $\pi_*\mathbb{Q}$ in a natural way. Then $\mathscr{M}$ has multiplication by the group ring $R := R_0[\mu_l]$. Let $e : R \to E$ be a projection onto a number field E such that $\operatorname{Ker}(e) \supset \operatorname{Ker}(e_0)$. There is a unique embedding $E_0 \hookrightarrow E$ making the diagram

$$\begin{array}{ccc} R_0 & \xrightarrow{e_0} & E_0 \\ \downarrow & & \downarrow \\ R & \xrightarrow{e} & E \end{array}$$

commutative.

For $\chi : R \to \overline{\mathbb{Q}}$ factoring through e, we denote by $V(\chi)$ the χ-part which is defined to be the subspace on which $r \in R$ acts as multiplication by $\chi(r)$.

THEOREM 3.3. *Let T_p denote the local monodromy on $R^1 f_*\overline{\mathbb{Q}}(\chi)$ at $t = p$. Let α_j^χ (resp. β_j^χ) for $j = 1, 2$ be rational numbers such that $e^{2\pi i\alpha_j^\chi}$ (resp. $e^{2\pi i\beta_j^\chi}$) are eigenvalues of T_0 (resp. T_∞). Let k be an integer such that $\chi(\zeta_l) = \zeta_l^k$ for $\zeta_l \in \mu_l$. Suppose that $k/l, -k/l + \beta_j^\chi \notin \mathbb{Z}$ and $\alpha_1^\chi \in \mathbb{Z}$. Write $h_\chi^{p,2-p} := \dim_{\overline{\mathbb{Q}}} \operatorname{Gr}_F^p W_2\mathscr{H}(\chi)$. Put*

$$d_\chi := 2\{-k/l\} + \sum_{i=1}^{2} \{\beta_i^\chi\} - \{\beta_i^\chi - k/l\}.$$

Then

$$(h_\chi^{2,0}, h_\chi^{1,1}, h_\chi^{0,2}) = \begin{cases} (1, 1, 0) & \textit{if } d_\chi = 2, \\ (0, 2, 0) & \textit{if } d_\chi = 1, \\ (0, 1, 1) & \textit{if } d_\chi = 0. \end{cases}$$

Note that d_χ takes values only in $0, 1$ or 2. Indeed

$$d_\chi = \overbrace{\{\beta_1^\chi\} + \{-k/l\} - \{\beta_1^\chi - k/l\}}^{\delta_1} + \overbrace{\{\beta_2^\chi\} + \{-k/l\} - \{\beta_2^\chi - k/l\}}^{\delta_2}$$

and each δ_i is either 0 or 1.

Proof. We first note that $\dim_E W_2\mathscr{H}(e) = 2$ [3, Section 4.3]. We employ two results from [2] and [9], respectively. First of all, it follows from [2, Theorem 4.2] that one has the Hodge numbers of the determinant

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$D := \det_E W_2\mathscr{H}(e) = \bigwedge_E^2 W_2\mathscr{H}(e)$. The result is

$$(D_\chi^{4,0}, D_\chi^{3,1}, D_\chi^{2,2}, D_\chi^{1,3}, D_\chi^{0,4}) = \begin{cases} (0,1,0,0,0) & \text{if } d_\chi = 2, \\ (0,0,1,0,0) & \text{if } d_\chi = 1, \\ (0,0,0,1,0) & \text{if } d_\chi = 0 \end{cases}$$

where we put $D_\chi^{p,4-p} := \dim \mathrm{Gr}_F^p D(\chi)$. Since $D_\chi^{p,4-p} = 1 \Leftrightarrow 2h_\chi^{2,0} + h_\chi^{1,1} = p$, this implies

$$(h_\chi^{2,0}, h_\chi^{1,1}, h_\chi^{0,2}) = \begin{cases} (1,1,0) & \text{if } d_\chi = 2, \\ (0,2,0) \text{ or } (1,0,1) & \text{if } d_\chi = 1, \\ (0,1,1) & \text{if } d_\chi = 0 \end{cases}$$

which completes the proof in the case $d_\chi \neq 1$. Suppose $d_\chi = 1$. We want to show that $(h_\chi^{2,0}, h_\chi^{1,1}, h_\chi^{0,2}) = (1,0,1)$ cannot happen. By [3, Theorem 5.8], the underlying connection of $W_2\mathscr{H}(\chi)$ is defined by the hypergeometric differential operator as in *loc. cit.* One can apply the main theorem in [9] and then the possible triplets of the Hodge numbers are at most $(2,0,0), (0,2,0), (0,0,2)$. In particular, the case $(h_\chi^{2,0}, h_\chi^{1,1}, h_\chi^{0,2}) = (1,0,1)$ is excluded. This completes the proof in case $d_\chi = 1$. □

REMARK 3.4. For the latter half of the proof of Theorem 3.3, there is an alternative discussion without using the main theorem of [9]. Let $\pi_0 : \mathbb{P}^1 \to \mathbb{P}^1$ be the map given by $t \mapsto -t^l$. Let $\mathscr{M}_0 := \pi_{0*}\mathbb{Q} \otimes R^1 f_*\mathbb{Q}$ be a VHdR on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Put $H_0 := H^1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \mathscr{M}_0)$. Let $\psi_{\lambda=0}$ denote the nearby cycle functor. Then one can construct an injection

$$E \cong W_2H_0(e) \hookrightarrow \psi_{\lambda=0} W_2\mathscr{H}(e)$$

of mixed Hodge–de Rham structures. The cohomology group $W_2H_0(e)$ is studied in detail in [4]. In particular, if $d_\chi = 1$, then the Hodge type of $W_2H_0(\chi)$ is $(1,1)$. Hence $h_\chi^{1,1} > 0$ by the above injection, which excludes the case $(h_\chi^{2,0}, h_\chi^{1,1}, h_\chi^{0,2}) = (1,0,1)$.

COROLLARY 3.5. *$W_2\mathscr{H}(e)$ is a Tate VHdR of type $(1,1)$ if and only if $d_\chi = 1$ for all $\chi : R \to \overline{\mathbb{Q}}$, equivalently*

$$2\{-sk_0/l\} + \textstyle\sum_{i=1}^2 \{s\beta_i^{\chi_0}\} - \{s(\beta_i^{\chi_0} - k_0/l)\} = 1$$
$$\Longleftrightarrow \quad \{s\beta_1^{\chi_0}\} < \{sk_0/l\} < \{s\beta_2^{\chi_0}\} \quad \text{or} \quad \{s\beta_2^{\chi_0}\} < \{sk_0/l\} < \{s\beta_1^{\chi_0}\}$$

for $\forall s \in \hat{\mathbb{Z}}^\times$ where χ_0 is a fixed one and $\beta_j^{\chi_0}, k_0$ are the rational numbers arising from χ_0.

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3.4 Beilinson regulator

Let $\psi_{t=1}$ be the nearby cycle functor along the function $t-1$ on $\mathscr{U}$, and put

$$C := \mathrm{Gr}_2^W \psi_{t=1}\mathscr{M} \cong \pi_*\mathbb{Q}|_{\{1\}\times S} \otimes (\mathrm{Gr}_2^W \psi_{t=1} R^1 f_*\mathbb{Q})$$

a VHdR on S. The condition (c) in Section 3.1 implies that the e-part $C(e)$ is of Hodge type $(1,1)$. Recall from [3, Proposition 4.2] that there is a natural embedding

$$C(e) \otimes \mathbb{Q}(-1) \longrightarrow \mathscr{H}(e)/W_2.$$

This gives a 1-extension

$$0 \longrightarrow W_2\mathscr{H}(e) \longrightarrow \mathscr{H}'(e) \longrightarrow C(e) \otimes \mathbb{Q}(-1) \longrightarrow 0 \tag{3.9}$$

of VMHdR with multiplication by E which is induced from (3.7). Note $C(e)$ is one-dimensional over E and endowed with Hodge type $(1,1)$ by (c) in Section 3.1.

In [3, Section 5] we discussed the extension data of (3.9). More precisely, let $\mathscr{O}^{\mathrm{zar}}$ be the Zariski sheaf of polynomial functions (with coefficients in $\overline{\mathbb{Q}}$) on $S = \mathbb{A}^1_{\overline{\mathbb{Q}}} \setminus \{0,1\}$ with coordinate λ. Let $\mathscr{O}^{an}$ be the sheaf of analytic functions on $S^{an} = \mathbb{C}^{an} \setminus \{0,1\}$. Let $a : S^{an} \to S^{\mathrm{zar}}$ be the canonical morphism from the analytic site to the Zariski site. Set

$$\mathscr{J} := \mathrm{Coker}[a^{-1}F^2 W_2\mathscr{H}_{\mathrm{dR}} \oplus \iota(W_2\mathscr{H}_B) \to \mathscr{O}^{an} \otimes_{a^{-1}\mathscr{O}^{\mathrm{zar}}} a^{-1}W_2\mathscr{H}_{\mathrm{dR}}]$$

a sheaf on the analytic site $\mathbb{C}^{an} \setminus \{0,1\}$ where $\iota : \mathscr{H}_B \to a^{-1}\mathscr{H}_{\mathrm{dR}}$ is the comparison map. Let $h : \widetilde{S} \to S$ be a generically finite and dominant map such that $\sqrt[l]{\lambda - 1} \in \overline{\mathbb{Q}}(\widetilde{S})$. Then $h^*C(e)$ is a direct sum of copies of the constant VHdR $\mathbb{Q}(-1)$. The connecting homomorphism arising from (3.9) gives a map

$$h^*C(e) \otimes \mathbb{Q}(1) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHdR}}(\mathbb{Q}, W_2\mathscr{H}(e) \otimes \mathbb{Q}(2))$$

to the Yoneda extension group of VMHdR's on S where we simply write $h^*C(e) \otimes \mathbb{Q}(1) = \Gamma(\widetilde{S}, h^*C(e) \otimes \mathbb{Q}(1))$. Combining this with the Carlson isomorphism (cf. [3, Proposition 2.1]), we have

$$\rho : h^*C(e) \otimes \mathbb{Q}(1) \longrightarrow \Gamma(\widetilde{S}^{an}, h^*\mathscr{J}(e)). \tag{3.10}$$

A down-to-earth description of ρ is the following. Let $x \in h^*C(e) \otimes \mathbb{Q}(1)$. Let $e_{\mathrm{dR},x} \in \mathscr{H}'(e)_{\mathrm{dR}} \otimes \mathbb{Q}(2)$ and $e_{B,x} \in \mathscr{H}'(e)_B \otimes \mathbb{Q}(2)$ be liftings of x. Then $\rho(x) = \pm(\iota(e_{B,x}) - e_{\mathrm{dR},x})$ (see also [3, Section 5.2]).

The map ρ agrees with the *Beilinson regulator map* on the motivic cohomology supported on singular fibers up to sign in the following sense. Let $\widetilde{\pi}:\mathbb{P}^1_{\widetilde{S}}:=\mathbb{P}^1\times_{\overline{\mathbb{Q}}}\widetilde{S}\to\mathbb{P}^1$ be given by $(s,\lambda')\mapsto h(\lambda')-s^l$. Consider the diagram

$$\begin{array}{ccccc}
X_{\widetilde{S}} & \xrightarrow{\ i\ } & \mathbb{P}^1_{\widetilde{S}}\times_{\mathbb{P}^1}X & \longrightarrow & X \\
{\scriptstyle g}\downarrow & \searrow^{f_{\widetilde{S}}} & \downarrow & & \downarrow{\scriptstyle f} \\
\widetilde{S} & \xleftarrow{\ p\ } & \mathbb{P}^1_{\widetilde{S}} & \xrightarrow{\ \widetilde{\pi}\ } & \mathbb{P}^1
\end{array}$$

with i desingularization and p the 2nd projection. Let

$$\mathrm{reg}:H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))\longrightarrow H^3_{\mathscr{D}}(X_{\widetilde{S}},\mathbb{Q}(2))=\mathrm{Ext}^3_{\mathrm{MHM}(X_{\widetilde{S}})}(\mathbb{Q},\mathbb{Q}(2))$$

be the Beilinson regulator map where $\mathrm{MHM}(\widetilde{S})$ denotes the category of mixed Hodge modules on $\widetilde{S}$. There is a canonical surjective map

$$\mathrm{Ext}^3_{\mathrm{MHM}(X_{\widetilde{S}})}(\mathbb{Q},\mathbb{Q}(2))\longrightarrow \mathrm{Ext}^1_{\mathrm{VMHdR}(\widetilde{S})}(\mathbb{Q},R^2g_*\mathbb{Q}(2)).$$

Let $U_{\widetilde{S}}\subset\mathbb{P}^1_{\widetilde{S}}$ be a Zariski open set on which $f_{\widetilde{S}}$ is smooth and projective. Put

$$H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))_0:=\mathrm{Ker}[H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))\longrightarrow H^3_{\mathscr{M}}(f^{-1}_{\widetilde{S}}(U_{\widetilde{S}}),\mathbb{Q}(2))]$$

and $(R^2g_*\mathbb{Q}(2))_0:=\mathrm{Ker}[R^2g_*\mathbb{Q}(2)\to p_*(R^2(f_{\widetilde{S}})_*\mathbb{Q}(2)|_{U_{\widetilde{S}}})]$. Then the regulator map induces a map

$$H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))_0\longrightarrow \mathrm{Ext}^1_{\mathrm{VMHdR}(\widetilde{S})}(\mathbb{Q},(R^2g_*\mathbb{Q}(2))_0).$$

Recall from (3.8) that there is a canonical surjective map $(R^2g_*\mathbb{Q}(2))_0\to h^*W_2\mathscr{H}(2)$. We thus have a composition

$$\mathrm{reg}_0:H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))_0\longrightarrow \mathrm{Ext}^1_{\mathrm{VMHdR}(\widetilde{S})}(\mathbb{Q},h^*W_2\mathscr{H}(2))\longrightarrow \Gamma(\widetilde{S}^{an},h^*\mathscr{J})$$

of the maps. The compatibility with (3.10) is given by the commutate diagram

$$(3.11)\qquad \begin{array}{ccc}
H^3_{\mathscr{M},D_{\widetilde{S}}}(X_{\widetilde{S}},\mathbb{Q}(2)) & \longrightarrow & h^*C(e)\otimes\mathbb{Q}(1) \\
\downarrow & & \downarrow{\scriptstyle\rho} \\
H^3_{\mathscr{M}}(X_{\widetilde{S}},\mathbb{Q}(2))_0 & \xrightarrow{\ \mathrm{reg}_0\ } & \Gamma(\widetilde{S}^{an},h^*\mathscr{J})
\end{array}$$

where $D_{\widetilde{S}}:=X_{\widetilde{S}}\setminus U_{\widetilde{S}}$.

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3.5 Regulator formula for HG fibrations of Gauss type

One of the main results in [3] (which we call *regulator formula*) is an explicit description of the map ρ in (3.10). Here we apply [3, Theorem 5.9] (=a precise version of regulator formula) to the case that f is a HG fibration of Gauss type (see Definition 3.2).

Let $f : X \to \mathbb{P}^1$ be a HG fibration of Gauss type with multiplication by $(R_0 := \mathbb{Q}[\mu_N], e_0)$ as in Definition 3.2. Let $\chi : E_0 \to \overline{\mathbb{Q}}$ be a homomorphism such that $\sigma(\zeta) = \zeta^{-n}$. Recall from Section 3.2 that $F^1 H^1_{\mathrm{dR}}(X_t)(\chi)$ is one-dimensional and spanned by a 1-form

$$\omega_n := \frac{x^{a_n}(1-x)^{b_n}(1-tx)^{c_n}}{y^n}\, dx,$$

$$a_n := \left\lfloor \frac{an}{N} \right\rfloor, \qquad b_n := \left\lfloor \frac{bn}{N} \right\rfloor, \qquad c_n := \left\lfloor \frac{Nn - bn}{N} \right\rfloor = n - b_n - 1,$$

where $n \in \{1, 2, \ldots, N-1\}$ such that $\chi(\zeta) = \zeta^{-n}$ for $\forall \zeta \in \mu_N$.

LEMMA 3.6. *Let D_0, D_1 be the reduced singular fibers over $t = 0, 1$. We assume that $D_0 + D_1$ is a normal crossing divisor (abbreviated NCD). Then $t\omega_n \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, f_*\Omega^1_{X/\mathbb{P}^1}(\log D_0 + D_1))$.*

Proof. Put $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $U = f^{-1}(S)$. Let $\mathscr{V} := H^1_{\mathrm{dR}}(U/S)$ be the bundle and $\nabla : \mathscr{V} \to \Omega^1_S \otimes \mathscr{V}$ the Gauss–Manin connection. Let D_∞ be the reduced singular fibers over $t = \infty$ and assume that it is a NCD. Put $T := \{0, 1, \infty\}$. Recall that the sheaf $\Omega^1_{X/\mathbb{P}^1}(\log D)$ $(D := D_0 + D_1 + D_\infty)$ is defined by the exact sequence

$$0 \longrightarrow f^*\Omega^1_{\mathbb{P}^1}(\log T) \longrightarrow \Omega^1_X(\log D) \longrightarrow \Omega^1_{X/\mathbb{P}^1}(\log D) \longrightarrow 0.$$

Let $\mathscr{V}_e$ be Deligne's canonical extension over $\mathbb{P}^1$. This is characterized as the subbundle $\mathscr{V}_e \subset j_*\mathscr{V}$ $(j : S \hookrightarrow \mathbb{P}^1)$ which satisfies

- ∇ has at most log poles, $\nabla : \mathscr{V}_e \to \Omega^1_{\mathbb{P}^1}(\log(0 + 1 + \infty)) \otimes \mathscr{V}_e$,
- The eigenvalues of residue $\mathrm{Res}(\nabla)$ at $t = 0, 1, \infty$ belong to $[0, 1)$.

Then there is an isomorphism

$$\mathscr{V}_e \cong R^1 f_*\Omega^\bullet_{X/\mathbb{P}^1}(\log D)$$

[12, 2.20] and $F^1\mathscr{V}_e := \mathscr{V}_e \cap j_*F^1\mathscr{V} \cong f_*\Omega^1_{X/\mathbb{P}^1}(\log D)$ (*loc. cit.* 4.20 (ii)). Hence the desired assertion is equivalent to

$$t\omega_n \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, \mathscr{V}_e). \tag{3.12}$$

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To show this, we give a local frame of $\mathscr{V}_e$ at $t=0,1$ explicitly. Let

$$\eta_n := \frac{x^{a_n}(1-x)^{b_n+1}(1-tx)^{c_n}}{y^n}\,dx,$$

and put

$$\beta_1^\chi := \left\{\frac{-an}{N}\right\}, \qquad \beta_2^\chi := \left\{\frac{-bn}{N}\right\}.$$

Recall from (3.2) a homology cycle $\delta := (1-\sigma^{d_1})(1-\sigma^{d_2})u \in H_1(X_t,\mathbb{Z})$. Then

$$\int_\delta \omega_n = (1-\zeta^{-nd_1})(1-\zeta^{-nd_2})B(\beta_1^\chi,\beta_2^\chi)\,{}_2F_1(\beta_1^\chi,\beta_2^\chi,\beta_1^\chi+\beta_2^\chi;t), \tag{3.13}$$

$$\int_\delta \eta_n = (1-\zeta^{-nd_1})(1-\zeta^{-nd_2})B(\beta_1^\chi,\beta_2^\chi+1)\,{}_2F_1(\beta_1^\chi,\beta_2^\chi,1+\beta_1^\chi+\beta_2^\chi;t). \tag{3.14}$$

This shows that ω_n and η_n are basis of the χ-part $\mathscr{V}(\chi)$ of the bundle (over a Zariski open set of $\mathbb{P}^1\setminus\{0,1,\infty\}$). Denote by $\mathscr{V}(\chi)^*$ the dual connection, and by $\{\omega_n^*,\eta_n^*\}$ the dual basis. Then

$$\left(\int_\delta \omega_n\right)\omega_n^* + \left(\int_\delta \eta_n\right)\eta_n^*$$

is annihilated by the dual connection, and hence

$$d\left(\int_\delta \omega_n\right)\omega_n^* + d\left(\int_\delta \eta_n\right)\eta_n^* + \left(\int_\delta \omega_n\right)\nabla(\omega_n^*) + \left(\int_\delta \eta_n\right)\nabla(\eta_n^*) = 0. \tag{3.15}$$

Now (3.13)–(3.15) together with the formulas

$$(1-t)\frac{d}{dt}{}_2F_1(a,b,a+b;t) = \frac{ab}{a+b}{}_2F_1(a,b,a+b+1;t),$$

$$t\frac{d}{dt}{}_2F_1(a,b,a+b+1;t)$$
$$= (a+b)\left({}_2F_1(a,b,a+b;t) - {}_2F_1(a,b,a+b+1;t)\right)$$

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imply

$$(\nabla(\omega_n^*), \nabla(\eta_n^*)) = dt \otimes (\omega_n^*, \eta_n^*) \begin{pmatrix} 0 & -\beta_1^\chi/(1-t) \\ -\beta_2^\chi/t & (\beta_1^\chi+\beta_2^\chi)/t \end{pmatrix}$$
$$\iff \quad (\nabla(\omega_n), \nabla(\eta_n)) = dt \otimes (\omega_n, \eta_n) \begin{pmatrix} 0 & \beta_2^\chi/t \\ \beta_1^\chi/(1-t) & -(\beta_1^\chi+\beta_2^\chi)/t \end{pmatrix}.$$

Then it is an elementary linear algebra to compute local frames of $\mathscr{V}_e$:

$$\mathscr{V}_e(\chi)|_{t=0} = \begin{cases} \langle \omega_n, t(\beta_2^\chi\omega_n + (\beta_1^\chi+\beta_1^\chi)\eta_n)\rangle & \beta_1^\chi+\beta_2^\chi \leqslant 1, \\ \langle t\omega_n, (\beta_1^\chi+\beta_2^\chi-1)\omega_n + t\beta_1^\chi\eta_n\rangle & \beta_1^\chi+\beta_2^\chi > 1, \end{cases}$$
$$\mathscr{V}_e(\chi)|_{t=1} = \langle \omega_n, \eta_n\rangle.$$

Now (3.12) is immediate. ∎

Let $e_0 : \mu_N \to E_0^\times$ be an injective homomorphism. Then the condition in Lemma 3.1 is satisfied. Let $e : R := \mathbb{Q}[\mu_l, \mu_N] \to E$ be a projection such that $\mathrm{Ker}(e) \supset \mathrm{Ker}(e_0)$. Let $\chi : R \to \overline{\mathbb{Q}}$ be a homomorphism factoring through e. Fix integers k, n such that

$$\chi(\zeta_1, \zeta_2) = \zeta_1^k\zeta_2^n, \quad \forall(\zeta_1, \zeta_2) \in \mu_l \times \mu_N.$$

Note $\gcd(n, N) = 1$ as $e_0 : \mu_N \to E_0^\times$ is injective. Let

(3.16)
$$\beta_1^\chi := \left\{\frac{-na}{N}\right\}, \qquad \beta_2^\chi := \left\{\frac{-nb}{N}\right\}, \qquad \alpha_1^\chi := 0, \qquad \alpha_2^\chi := 1 - \beta_1^\chi - \beta_2^\chi$$

which do not depend on the choice of n. Then $e^{2\pi i\alpha_j^\chi}$ (resp. $e^{2\pi i\beta_j^\chi}$) are eigenvalues of the local monodromy T_0 at $t = 0$ (resp. T_∞ at $t = \infty$) on $R^1 f_*\mathbb{C}(\chi) \cong \mathbb{C}^2$ (see (3.5)). The relative 1-form $\omega := t\omega_n$ satisfies the conditions (P1), (P2) in [3, Section 4.5]:

(P1) $\int_{\gamma_t} \omega(\gamma_t \in H_1(X_t))$ is spanned by $t\,{}_2F_1(\beta_1^\chi, \beta_2^\chi, 1; 1-t)$ and $t\,{}_2F_1(\beta_1^\chi, \beta_2^\chi, \beta_1^\chi+\beta_2^\chi; t)$. (This follows from (3.4).)
(P2) $\omega \in \varGamma(\mathbb{P}^1 \setminus \{\infty\}, f_*\Omega^1_{X/\mathbb{P}^1}(\log D))$. (This is Lemma 3.6.)

We thus can apply the *regulator formula* [3, Theorem 5.9]. In our particular case, it is stated as follows (the notation is slightly changed for the use in below).

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THEOREM 3.7. *Let* e_0, e, χ *be as above, and let* α_i^χ, β_j^χ *be as in (3.16). Assume that* $k/l, k/l-\beta_1^\chi, k/l-\beta_2^\chi, k/l-\beta_1^\chi-\beta_2^\chi \notin \mathbb{Z}$. *Put*

$$\mathscr{F}_1(\lambda) := (1-\lambda)^{k/l-1}\ {}_3F_2\left(\begin{matrix}1,1,1-k/l\\ 2-\beta_1^\chi, 2-\beta_2^\chi\end{matrix}; (1-\lambda)^{-1}\right),$$
$$\mathscr{F}_2(\lambda) := (1-\lambda)^{k/l-1}\ {}_3F_2\left(\begin{matrix}1,1,2-k/l\\ 2-\beta_1^\chi, 2-\beta_2^\chi\end{matrix}; (1-\lambda)^{-1}\right).$$

Let $\rho({}^t\chi)$ *be the* ${}^t\chi$*-part of the map* ρ *in (3.10). Let*

$$\rho({}^t\chi) = (\phi_1(\lambda), \phi_2(\lambda)) \in (\mathscr{O}^{an})^{\oplus 2} \cong \mathscr{O}^{an} \otimes W_2\mathscr{H}_{\mathrm{dR}}({}^t\chi)$$

be a local lifting where the above isomorphism is with respect to $\overline{\mathbb{Q}}$*-frame of* $W_2\mathscr{H}_{\mathrm{dR}}({}^t\chi)$. *Define rational functions* $E_i^{(r)} = E_i^{(r)}(\lambda) \in \mathbb{Q}(\lambda)$ *for* $r \in \mathbb{Z}_{\geqslant -1}$ *in the following way. Write* $a := 2-\beta_1^\chi$, $b := 2-\beta_2^\chi$. *Put*

$$A(s) := \frac{s(a+b+2s-3-s(1-\lambda)^{-1})}{(a+s-1)(b+s-1)},$$
$$B(s) := \frac{s(1-s)(1-(1-\lambda)^{-1})}{(a+s-1)(b+s-1)}.$$

Define $C_i(s)$ *and* $D_i(s)$ *by*

$$\begin{pmatrix}C_{i+1}(s)\\ D_{i+1}(s)\end{pmatrix} = \begin{pmatrix}A(s) & 1\\ B(s) & 0\end{pmatrix}\begin{pmatrix}C_i(s+1)\\ D_i(s+1)\end{pmatrix}, \qquad \begin{pmatrix}C_{-1}(s)\\ D_{-1}(s)\end{pmatrix} := \begin{pmatrix}0\\ 1\end{pmatrix},$$

and define $E_i^{(r)}$ *by*

$$E_1^{(r)} = \lambda C_r(k/l) + (1-\lambda)C_{r+1}(k/l),$$
$$E_2^{(r)} = \lambda D_r(k/l) + (1-\lambda)D_{r+1}(k/l). \tag{3.17}$$

Then for infinitely many integers $r > 0$, *we have*

$$\phi_1(\lambda) \equiv C_1(1-\lambda)^r[E_1^{(r)}(\lambda)\mathscr{F}_1(\lambda) + E_2^{(r)}(\lambda)\mathscr{F}_2(\lambda)],$$
$$\phi_2(\lambda) \equiv C_2(1-\lambda)^{r-1}[E_1^{(r-1)}(\lambda)\mathscr{F}_1(\lambda) + E_2^{(r-1)}(\lambda)\mathscr{F}_2(\lambda)]$$

modulo $\overline{\mathbb{Q}(\lambda)}$ *with some* $C_1, C_2 \in \overline{\mathbb{Q}}^\times$.

We note that (N, l, k, n, a, b) in Theorem 3.7 can run over the set of all 6-tuples of integers satisfying

- $0 < a, b < N$, $\gcd(N, a, b) = 1$ and $\gcd(n, N) = 1$,
- $k/l, k/l-\beta_1^\chi, k/l-\beta_2^\chi, k/l-\beta_1^\chi-\beta_2^\chi \notin \mathbb{Z}$ (see (3.16) for definition of β_j^χ).

§4. Proof of main theorem

We are now in a position to prove Theorem 2.1 (log formula).

There are the following formulas

$$(b_1-1){}_3F_2\left({a_1,a_2,a_3 \atop b_1-1,b_2};x\right)=\left(b_1-1+x\frac{d}{dx}\right){}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};x\right),$$
$$a_1\cdot {}_3F_2\left({a_1+1,a_2,a_3 \atop b_1,b_2};x\right)=\left(a_1+x\frac{d}{dx}\right){}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};x\right),$$

$$(a_2-b_1)(a_1-b_1)(a_3-b_1){}_3F_2\left({a_1,a_2,a_3 \atop b_1+1,b_2};x\right)=\theta_1\left({}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};x\right)\right),$$
$$(a_1-b_1)(a_1-b_2){}_3F_2\left({a_1-1,a_2,a_3 \atop b_1,b_2};x\right)=\theta_2\left({}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};x\right)\right),$$

where

$$\begin{aligned}\theta_1 &:= -a_1a_2a_3+(a_2-b_1)(a_1-b_1)(a_3-b_1)\\ &\qquad + b_1(b_2+(b_1-a_1-a_2-a_3-1)x)\frac{d}{dx}+b_1(x-x^2)\frac{d^2}{dx^2}\\ \theta_2 &:= (a_1-b_1)(a_1-b_2)-a_2a_3x\\ &\qquad + ((b_1+b_2-a_1)-(a_2+a_3+1)x)x\frac{d}{dx}+(1-x)x^2\frac{d^2}{dx^2}.\end{aligned}$$

Therefore if one can prove the log formula for ${}_3F_2\left({1,1,q \atop a,\ b};x\right)$ then one immediately has the log formula for ${}_3F_2\left({n_1,n_2,q+n_3 \atop a+n_4,\ b+n_5};x\right)$ for arbitrary integers $n_1,n_2>0$ and $n_3,n_4,n_5\in\mathbb{Z}$.

We keep the setting and the notation in Section 3.5. Suppose that

$$1=2\{-sk/l\}+\sum_{i=1}^{2}\{s\beta_2^\chi\}-\{s(\beta_i^\chi-k/l)\},\quad \forall s\in\hat{\mathbb{Z}}^\times. \tag{4.1}$$

Then it follows from Corollary 3.5 that $W_2\mathscr{H}(e)$ is a Tate HdR structure of type $(1,1)$. Let us look at the map $\rho({}^t\chi)$ in Theorem 3.7. This turns out to be the Beilinson regulator by the diagram (3.11). Since $W_2\mathscr{H}(e)$ is Tate, it is generated by the divisor classes of the geometric generic fiber $X_{\overline{\eta}}$ of $f_{\widetilde{S}}$. This implies that the image of reg_0 in (3.11) is generated by the images of $H^1_{\mathscr{M}}(\widetilde{D}_i,\mathbb{Q}(1))$ where D_i runs over the generators of the Neron–Severi group $\mathrm{NS}(X_{\overline{\eta}})\otimes\mathbb{Q}$ and $\widetilde{D}_i\to D_i$ is the desingularization. As is well known,

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$H^1_{\mathscr{M}}(\widetilde{D}_i, \mathbb{Q}(1)) \cong \overline{\eta}^{\times} \otimes \mathbb{Q}$ as $\widetilde{D}_i$ is smooth projective, and the Beilinson regulator on it is given by the logarithmic function. Therefore we have

$$\phi_1(\lambda),\ \phi_2(\lambda) \in \overline{\mathbb{Q}(\lambda)} + \overline{\mathbb{Q}(\lambda)} \log \overline{\mathbb{Q}(\lambda)}^{\times}. \tag{4.2}$$

We now apply Theorem 3.7. If one can show that

$$\begin{vmatrix} E_1^{(r)} & E_2^{(r)} \\ E_1^{(r-1)} & E_2^{(r-1)} \end{vmatrix} \neq 0$$

for almost all $r > 0$, then we have $\mathscr{F}_i(\lambda) \in \overline{\mathbb{Q}(\lambda)} + \overline{\mathbb{Q}(\lambda)} \log \overline{\mathbb{Q}(\lambda)}^{\times}$, which would finish the proof of Theorem 2.1. To do this, recall (3.17). Letting

$$E_1^{(r)}(s) := \lambda C_r(s) + (1-\lambda) C_{r+1}(s),$$
$$E_2^{(r)}(s) := \lambda D_r(s) + (1-\lambda) D_{r+1}(s),$$

we want to show

$$\begin{vmatrix} E_1^{(r)}(k/l) & E_2^{(r)}(k/l) \\ E_1^{(r-1)}(k/l) & E_2^{(r-1)}(k/l) \end{vmatrix} \neq 0 \tag{4.3}$$

for almost all $r > 0$. Since

$$\begin{pmatrix} E_1^{(r+1)}(s) & E_1^{(r)}(s) \\ E_2^{(r+1)}(s) & E_2^{(r)}(s) \end{pmatrix} = \begin{pmatrix} A(s) & 1 \\ B(s) & 0 \end{pmatrix} \begin{pmatrix} E_1^{(r)}(s+1) & E_1^{(r-1)}(s+1) \\ E_2^{(r)}(s+1) & E_2^{(r-1)}(s+1) \end{pmatrix}$$

(4.3) is reduced to showing that

$$\begin{vmatrix} E_1^{(0)}(k/l+r) & E_2^{(0)}(k/l+r) \\ E_1^{(-1)}(k/l+r) & E_2^{(-1)}(k/l+r) \end{vmatrix} \neq 0$$

for all integers r. However, this follows from

$$\begin{aligned} \begin{vmatrix} E_1^{(0)}(s) & E_2^{(0)}(s) \\ E_1^{(-1)}(s) & E_2^{(-1)}(s) \end{vmatrix} &= \begin{vmatrix} \lambda + (1-\lambda)A(s) & (1-\lambda)B(s) \\ 1-\lambda & \lambda \end{vmatrix} \\ &= \lambda \frac{(a-1)(b-1)\lambda + s(a+b-2)}{(s+a-1)(s+b-1)}, \\ &\quad (a := 2-\beta_1^{\chi}, b := 2-\beta_2^{\chi}) \end{aligned}$$

and the fact $\beta_i^{\chi} \notin \mathbb{Z}$ (see (3.16)) and $k/l - \beta_i^{\chi} \notin \mathbb{Z}$ as is assumed. This completes the proof of Theorem 2.1.

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References

[1] N. Archinard, *Hypergeometric abelian varieties*, Canad. J. Math. **55** (2003), 897–932.
[2] M. Asakura and J. Fresán, *On the Gross–Deligne conjecture for variations of Hodge–de Rham structures*, preprint.
[3] M. Asakura and N. Otsubo, *Regulators on K_1 of hypergeometric fibrations*, to appear in the Proceedings of Conference "Arithmetic L-functions and Differential Geometric Methods (Regulators IV)", arXiv:1709.04144.
[4] M. Asakura and N. Otsubo, *CM periods, CM regulators and hypergeometric functions, II*, Math. Z. **289**(3–4) (2018), 1325–1355.
[5] M. Asakura, N. Otsubo and T. Terasoma, *An algebro-geometric study of special values of hypergeometric functions* ${}_3F_2$, Nagoya Math. J. **236** (2019), 47–62.
[6] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics **32**, Stechert-Hafner, Inc., New York, 1964.
[7] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95**(2) (1989), 325–354.
[8] A. Erdélyi (eds), *Higher transcendental functions,* Vol. 1, McGrow-Hill, New York, 1953.
[9] R. Fedorov, *Variations of Hodge structures for hypergeometric differential operators and parabolic Higgs bundles*, arXiv:1505.01704.
[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds), *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, 2010.
[11] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.
[12] J. Steenbrink, *Limits of Hodge structures*, Invent. Math. **31**(3) (1976), 229–257.

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