J. Inst. Math. Jussieu (2025), 1–39
doi:10.1017/S1474748024000495

MUKAI'S PROGRAM FOR NONPRIMITIVE CURVES ON K3 SURFACES

YIRAN CHENG[1], ZHIYUAN LI[2] AND HAOYU WU[3]

[1] *Shanghai Center for Mathematical Sciences, Fudan University, 2005 Songhu Road, 200438 Shanghai, China*
(yrcheng21@m.fudan.edu.cn)

[2] *Shanghai Center for Mathematical Sciences, Fudan University, 2005 Songhu Road, 200438 Shanghai, China*
(zhiyuan_li@fudan.edu.cn)

[3] *Shanghai Center for Mathematical Sciences, Fudan University, 2005 Songhu Road, 200438 Shanghai, China*
(hywu18@fudan.edu.cn)

(Received 14 December 2022; revised 11 October 2024; accepted 22 October 2024)

Abstract Mukai's program in [16] seeks to recover a K3 surface X from any curve C on it by exhibiting it as a Fourier–Mukai partner to a Brill–Noether locus of vector bundles on the curve. In the case X has Picard number one and the curve $C \in |H|$ is primitive, this was confirmed by Feyzbakhsh in [11, 13] for $g \geq 11$ and $g \neq 12$. More recently, Feyzbakhsh has shown in [12] that certain moduli spaces of stable bundles on X are isomorphic to the Brill–Noether locus of curves in $|H|$ if g is sufficiently large. In this paper, we work with irreducible curves in a nonprimitive ample linear system $|mH|$ and prove that Mukai's program is valid for any irreducible curve when $g \neq 2$, $mg \geq 11$ and $mg \neq 12$. Furthermore, we introduce the destabilising regions to improve Feyzbakhsh's analysis in [12]. We show that there are hyper-Kähler varieties as Brill–Noether loci of curves in every dimension.

1. Introduction

Let $\mathcal{F}_g$ be the moduli stack of primitively polarised K3 surface (X,H) with $H^2 = 2g-2$, and let $\mathcal{P}_{g_m}$ be the moduli stack of triples (X,H,C) such that $(X,H) \in \mathcal{F}_g$ and $C \in |mH|$ a smooth curve of genus $g_m = m^2(g-1)+1$. There are natural forgetful maps

Keywords: K3 surfaces; hyper-Kähler varieties; Mukai's program; Bridgeland stability conditions

2020 Mathematics subject classification: Primary 14J28; 14J42; 14H51; 14H60

$$\begin{array}{ccc} & \mathcal{P}_{g_m} & \\ \swarrow^{\Psi_{g_m}} & & \searrow^{\Phi_{g_m}} \\ \mathcal{M}_{g_m} & & \mathcal{F}_g, \end{array}$$

where the fibre of Φ_{g_m} over $(X,H) \in \mathcal{F}_g$ is an open subset of the linear system $|mH|$. In recent years, there is a series of works studying the rational map Ψ_{g_m} and its rational section. For instance, Mukai has proved in [16] that the rational map $\Psi_g := \Psi_{g_{m=1}}$ is dominant if $g \leq 11$ and $g \neq 10$, while Ciliberto–Lopez–Miranda [8] showed that Ψ_g is generically injective if $g \geq 11$ and $g \neq 12$. More generally, due to the results of [9] and the recent work in Ciliberto–Dedieu–Sernesi [6, 5], the map Ψ_{g_m} is generically finite when $mg \geq 11$ and $mg \neq 12$. There are other approaches for the case $m \geq 2, g \geq 8$ or $m \geq 5, g = 7$ (cf. [7, 14]).

On the other hand, Mukai has proposed a program in [18] to find the rational section of Ψ_g by relating the K3 surface with the Brill–Noether locus of vector bundles on curves. This has been confirmed by Mukai in [17] when $g = 11$ and later on Arbarello–Bruno–Sernesi [1] generalised his result to the case $g = 4k+3$ for some k. In recent years, Feyzbakhsh has verified this program in [11, 13] for all $g \geq 11$ and $g \neq 12$ by using the Bridgeland stability conditions. In this paper, we would like to investigate the rational section of the map Ψ_{g_m} for arbitrary $m \in \mathbb{Z}_{>0}$ via Mukai's program for curves in nonprimitive classes.

Main results

Let (X,H) be a primitively polarised K3 surface of genus g with Picard number one. Let $\mathrm{H}^*_{\mathrm{alg}}(X) \cong \mathbb{Z}^{\oplus 3}$ be the algebraic Mukai lattice, and let $\mathbf{M}(v)$ be the moduli space of H-Gieseker semistable coherent sheaves on X with Mukai vector $v = (r, cH, s) \in \mathrm{H}^*_{\mathrm{alg}}(X)$. For $C \in |mH|$ an irreducible curve, let $\mathbf{BN}_C(v)$ be the Brill–Noether locus of slope semistable vector bundles on C with rank r, degree $2mc(g-1)$ and $h^0 \geq r+s$. The first main result of this paper is:

Theorem 1.1. *Assume $g > 2$. Let $C \in |mH|$ be an irreducible curve. Then if $mg \geq 11$ and $mg \neq 12$, there exists a primitive Mukai vector $v = (r, cH, s)$ with $v^2 = 0$ such that the restriction map*

$$\begin{aligned} \psi\colon \mathbf{M}(v) &\to \mathbf{BN}_C(v) \\ E &\mapsto E|_C \end{aligned} \tag{1.1}$$

is an isomorphism.

As in [11], one can then reconstruct X as the moduli space of twisted sheaves on $\mathbf{BN}_C(v)$. Clearly, such reconstruction is unique for K3 surfaces in $\mathcal{F}_g$ of Picard number one. Due to the results of [10], when $m > 1$, generic curves in $|mH|$ have maximal variation, that is, the rational map

$$|mH| \overset{\Psi_{g_m}}{\dashrightarrow} \mathcal{M}_{g_m} \tag{1.2}$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

is generically finite. One can also deduce the generic quasi-finiteness of Ψ_{g_m} from Theorem 1.1 when $m > 1, g \geq 3$, $mg \geq 11$ and $mg \neq 12$. When $g_m < 11$, the map Ψ_{g_m} is not generically quasi-finite and Mukai's program will fail. We expect that Theorem 1.1 holds whenever $g_m \geq 11$. So far, the missing values of (g,m) are

$$(2,m) \text{ with } m \geq 4, \quad (3,3), \quad (3,4), \quad (4,2), \quad (4,3), \quad (5,2), \quad (6,2).$$

A mysterious case is when $g = 2$, where our method fails for any m.

More generally, one may consider the restriction map (1.1) for $v^2 = 2n > 0$. Most recently, Feyzbakhsh [12] has generalised her construction in [11, 13] and showed that for each Mukai vector $v = (r, cH, s)$ satisfying

$$c < r, \quad \gcd(r,c) = \gcd(c,s) = 1 \quad \text{and} \quad -2 \leq v^2 \leq -2 + r, \tag{1.3}$$

the restriction gives an isomorphism $\mathbf{M}(v) \cong \mathbf{BN}_C(v)$ when g is sufficiently large and the class of C is primitive. As mentioned in [12], the analysis in [12] also works for the nonprimitive case and one can actually show that Feyzbakhsh's construction still gives an isomorphism for $C \in |mH|$ if g is sufficiently large (depending on r and m). This gives many examples of Brill–Noether loci on curves as hyper-Kähler varieties of dimension $2g - 2r\lfloor \frac{g}{r} \rfloor$. In this paper, we also improve her result (see Theorem 7.1) and obtain an explicit condition of v for ψ being an isomorphism (see Theorem 7.3). As an application, we show that one can construct hyper-Kähler varieties as the Brill–Noether loci of curves in every dimension.

Theorem 1.2. *For any $n > 0 \in \mathbb{Z}$, there exists an integer $N = N(n)$ satisfying that if $g > N$, there is a primitive Mukai vector $v \in \mathrm{H}^*_{alg}(X)$ with $v^2 = 2n$ such that the restriction map $\psi : \mathbf{M}(v) \to \mathbf{BN}_C(v)$ is an isomorphism for all irreducible curves C on X.*

In other words, the bound N does not depend on the class of C. This makes use of the boundedness result of prime character nonresidues (See Lemma 8.2). The strategy of our proof is similar to [11, 13, 12]. Roughly speaking, we prove that ψ will be a well-defined and injective morphism if the Gieseker chambers for objects with Mukai vector v and $v(-m) := e^{-mH}v$ are large enough, and ψ is bijective if further the Harder–Narasimhan polygon of i_*F for $F \in \mathbf{BN}_C(v)$ achieves its maximum. The main ingredient is the use of a wall-crossing argument to analyse the existence of walls. There are two crucial improvement in our approach. One is that we find the strongest criterion (Proposition 3.4) to characterise the stability conditions which are not lying on the wall of objects with a given Mukai vector. This leads to a more explicit condition for ψ being an isomorphism. The other one is that we develop a method in analysing the relative position of HN polygons towards the surjectivity of ψ. This allows us to get a sharper bound of (g,m) without using the computer program.

Organization of this paper

In Section 2, we review the basic knowledge of the Bridgeland stability condition on K3 surfaces and the wall-chamber structure on a section. In Section 3, we introduce the (strictly) destabilising regions $\Omega_v^{(+)}(-)$ of a Mukai vector v. They characterise the stability

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

conditions which are not lying on the wall of objects in $\mathrm{D}^b(X)$ with Mukai vector v. This will play a key role in the proof of our main theorems.

In Section 4, we show that the map $\psi : \mathbf{M}(v) \to \mathbf{BN}_C(v)$ is a well-defined morphism and $h^0(X,E) = r+s$ for any $E \in \mathbf{M}(v)$ if the positive integers r,c and s satisfy

$$\gcd(r,c)=1, r > \frac{v^2+2}{2} \text{ and } g-1 \geq \begin{cases} r, & \text{if } v^2 = 0; \\ \max\left\{\frac{r^2}{c} > r, \frac{r^2}{mr-c}\right\}, & \text{if } v^2 > 0. \end{cases} \tag{1.4}$$

The first two assumptions provide that any stable sheaf in $\mathbf{M}(v)$ is locally free while the third assumption essentially ensures that there is no wall between the large volume limit and the Brill–Noether wall. As a by-product, we obtain a numerical criterion for the injectivity of ψ.

Section 5 and Section 6 are devoted to studying the surjectivity of the restriction map ψ. They contain the most technical part of this paper. In Section 5, we show that ψ is surjective if the Harder–Narasimhan polygon of i_*F for arbitrary $F \in \mathbf{BN}_C(v)$ is maximal when g is relatively large. It involves a dedicated analysis of the slope of destabilising factors of i_*F via a geometric vision of the destabilising region. In Section 6, we analyse the sharpness of HN-polygons for special Mukai vectors with zero square. The concept of sharpness is used to detect how far the HN-polygon stays away from the convex polygon given by the critical position of the first wall. This makes the construction valid for small genera.

In Section 7, we analyse the surjectivity of the tangent map $d\psi$ of ψ and show that it is always surjective if $g-1 \geq 4r^2$. In Section 8, we prove Theorem 1.1 and Theorem 1.2 by showing the existence of Mukai vectors satisfying all conditions. Here, we make use of the bound of prime character nonresidues.

Notation and conventions

Throughout this paper, we always assume (X,H) is a primitively polarised K3 surface of genus g of Picard number one.

For any two points $p,q \in \mathbb{R}^n$, let $\mathrm{L}_{p,q}$ be the line passing through them and let $\mathrm{L}^+_{p,q}$ be the ray starting from p. We use $\mathrm{L}_{[p,q]}$, $\mathrm{L}_{(p,q)}$, $\mathrm{L}_{(p,q]}$ and $\mathrm{L}_{[p,q)}$ to denote the closed, open and half open line segment, respectively. For any line segment I, we set

$$\Delta_p(I) = \bigcup_{q \in I} \mathrm{L}_{(p,q]}$$

to be a (half open) triangular region. We denote by $\mathbf{P}_{p_1 \dots p_n}$ the polygon with vertices $p_1, \dots, p_n$.

2. Stability condition on K3 surfaces

Let $\mathrm{D}^{\mathrm{b}}(X)$ be the bounded derived category of coherent sheaves on X. We let $K_{\mathrm{num}}(X)$ be the Grothendieck group of X modulo numerical equivalence. There is an onto map to the (algebraic) Mukai lattice $\mathrm{H}^*_{\mathrm{alg}}(X) := \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus \mathrm{H}^4(X,\mathbb{Z})$ by sending

$$v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \in \mathrm{H}^*_{\mathrm{alg}}(X).$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

As X has Picard number one, we may identify $\mathrm{H}^*_{\mathrm{alg}}(X)$ as $\mathbb{Z}^{\oplus 3}$. So in the sequel of this paper, we shall abuse the notation and simply write $v(E) = (r,c,s)$ with $r = \mathrm{rk}(E)$, $c_1(E) = cH$ and $s = \chi(E) - r$. Here, $\chi(E) = \chi(\mathcal{O}_X, E)$ is the Euler characteristic. The Mukai pairing $\langle , \rangle$ on $\mathrm{H}^*_{\mathrm{alg}}(X)$ defined by $\langle v(E), v(F)\rangle = -\chi(E,F)$ can be viewed as an integral quadratic form on $\mathbb{Z}^{\oplus 3}$ given by

$$\langle (x,y,z),(x',y',z')\rangle = 2yy'(g-1) - xz' - zx'.$$

We may write $v^2 = \langle v, v\rangle$ for $v \in \mathrm{H}^*_{\mathrm{alg}}(X)$. Consider the projection map

$$\mathrm{pr}: \mathrm{H}^*_{\mathrm{alg}}(X) \otimes \mathbb{R} \setminus \{s = 0\} \to \mathbb{R}^2$$

sending a vector $v = (r,c,s)$ to $(\frac{r}{s}, \frac{c}{s})$. We write $\pi_v = \mathrm{pr}(v)$ and $\pi_E = \mathrm{pr}(v(E))$ for $E \in \mathrm{D}^b(X)$ for simplicity. We let $O = (0,0,0)$ be the origin of $\mathrm{H}^*_{\mathrm{alg}}(X) \otimes \mathbb{R}$ and denote by $o = (0,0)$ the origin of $\mathbb{R}^2$.

A **numerical (Bridgeland) stability condition** on X is a pair $\sigma = (\mathcal{A}_\sigma, Z_\sigma)$ consisting a heart $\mathcal{A}_\sigma \subset \mathrm{D}^{\mathrm{b}}(X)$ of a bounded t-structure and an $\mathbb{R}$-linear map

$$Z_\sigma : K_{\mathrm{num}}(X) \otimes \mathbb{R} \to \mathbb{C}$$

satisfying the conditions

(i) For any $0 \neq E \in \mathcal{A}$,

$$Z_\sigma(E) \in \mathbb{R}_{>0} \exp(i\pi\phi_\sigma(E)) \text{ with } 0 < \phi_\sigma(E) \leq 1,$$

where $\phi_\sigma(E)$ is the **phase** of $Z_\sigma(E)$ in the complex plane.

(i) The Harder–Narasimhan (HN) property, cf. [3, Definition 2.3].

The σ-slope is defined by

$$\mu_\sigma(E) = -\frac{\mathrm{Re}\, Z(E)}{\mathrm{Im}\, Z(E)},$$

and we set the σ-phase to be

$$\phi_\sigma(E) = \frac{1}{\pi}[\pi - \cot^{-1}(\mu_\sigma(E))] \in (0,1].$$

An object $E \in \mathcal{A}$ is called **σ-(semi)stable** if $\mu_\sigma(F) < (\leq)\, \mu_\sigma(E)$ or equivalently $\phi_\sigma(F) < (\leq)\, \phi_\sigma(E)$ whenever $F \subset E$ is a subobject of E in $\mathcal{A}$. We say an object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is **σ-(semi)stable** if $E[k] \in \mathcal{A}$ for some k, and $E[k]$ is σ-(semi)stable.

If E is a sheaf with $v(E) = (r,c,s)$, we write $\mu_H(E) = \frac{c}{r}$ for the H-slope of E and $\mu_H^{\pm}(E)$ for the H-slope of the first/last HN factor of E. In [4], Bridgeland has constructed a continuous family of stability conditions on X as follows: For $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, for any $\beta \in \mathbb{R}$, the β-tilt of $\mathrm{Coh}(X)$ is defined by

$$\mathrm{Coh}^\beta(X) := \left\{ E \in \mathrm{D}^{\mathrm{b}}(X) \,\middle|\, \mu_H^+(\mathrm{H}^{-1}(E)) \leq \beta, \mu_H^-(\mathrm{H}^0(E)) > \beta,\, \mathrm{H}^i(E) = 0 \text{ for } i \neq 0, -1 \right\}$$

which is the heart of a t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ with

$$Z_{\alpha,\beta}(E) = \left\langle (1, \beta, \frac{H^2}{2}(\beta^2 - \alpha^2)), v(E) \right\rangle + \sqrt{-1} \left\langle (0, \frac{1}{H^2}, \beta), v(E) \right\rangle;$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

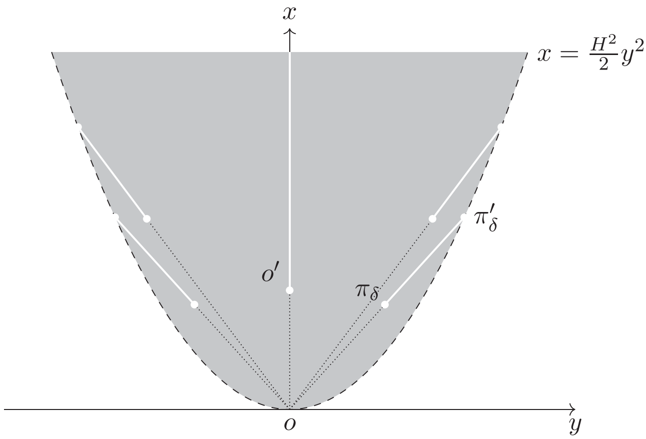


Figure 1. Visualization of $V(X)$.

here, $\mathrm{H}^i(E)$ is the i-th cohomology of E with respect to the standard heart $\mathrm{Coh}(X)$ of $\mathrm{D^b}(X)$. Let $\mathrm{R}(X)$ be the collection of roots in $\mathrm{H}^*_{\mathrm{alg}}(X)$, that is,

$$\mathrm{R}(X) := \left\{ v \in \mathrm{H}^*_{\mathrm{alg}}(X) \,\middle|\, \langle v,v\rangle = -2 \right\}.$$

Theorem 2.1 [4]. *The pair $\sigma_{\alpha,\beta} := (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ is a Bridgeland stability condition on $D^b(X)$ if $\mathrm{Re}\, Z_{\alpha,\beta}(\delta) > 0$ for all roots $\delta \in \mathrm{R}(X)$ with $\mathrm{rk}(\delta) > 0$ and $\mathrm{Im}\, Z_{\alpha,\beta}(\delta) = 0$.*

The stability condition $\sigma_{\alpha,\beta}$ is uniquely characterised by its kernel

$$\ker Z_{\alpha,\beta} = \left\{ (r,c,s) \in \mathrm{H}^*_{\mathrm{alg}}(X) \,\middle|\, c = r\beta, s = \frac{rH^2}{2}(\alpha^2+\beta^2) \right\}.$$

According to [11, Lemma 2.4], if we set $k(\alpha,\beta) = \mathrm{pr}(\ker Z_{\alpha,\beta}) \in \mathbb{R}^2$, then $k(\alpha,\beta)$ are parameterised by the space

$$V(X) = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x > \frac{H^2y^2}{2} \right\} \setminus \bigcup_{\delta \in \mathrm{R}(X)} \mathrm{L}_{(\pi'_\delta, \pi_\delta]}, \tag{2.1}$$

where π'_δ is the intersection point of the parabola $\left\{ x = \frac{H^2y^2}{2} \right\}$ and the line $\mathrm{L}_{o,\pi_\delta}$. See Figure 1 for a picture of $V(X)$. Therefore, we may view the stability condition $\sigma_{\alpha,\beta}$ as the point $k(\alpha,\beta)$ in $V(X)$.

The following are some simple observations that will be frequently used in this paper:

(A) If $\sigma \in V(X)$, then the line segment $\mathrm{L}_{(o,\sigma]}$ is contained in $V(X)$.

(B) If $\gcd(r,c) = 1$ and $r > 0$, the line $ry = cx$ contains a (unique) projection of root if and only if $r \mid c^2(g-1)+1$ (cf. [20]). In particular, the unique projection of root on the x-axis is $(1,0)$, which we denote by o'.

A simple observation is, for elements in the same heart, we can read their phases from the plane.

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

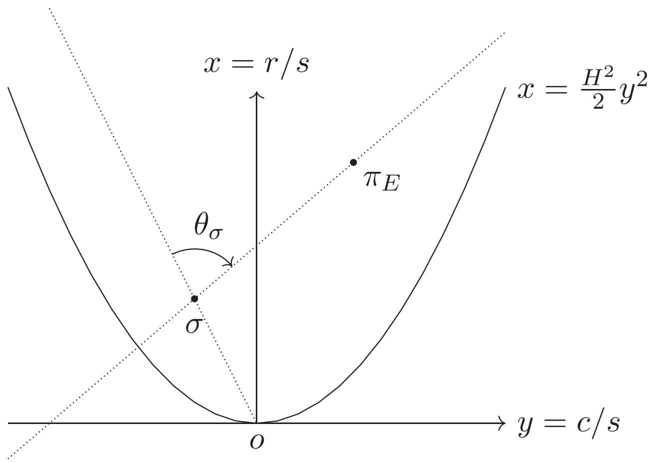


Figure 2. An example of $\theta_\sigma(E)$.

Proposition 2.2 (Phase reading, see Figure 2). *Fix $\sigma_{\alpha,\beta} \in V(X)$. For $E \in \mathrm{Coh}^\beta(X)$, let $0 < \theta_\sigma \leq \pi$ be the directed angle from $\overrightarrow{\sigma\pi_E}$ to $\overrightarrow{o\sigma}$ modulo π. Then, ϕ_σ is a strictly increasing function of θ_σ.*

Proof. Note that $\phi_\sigma(E_1) = \phi_\sigma(E_2)$ if and only if

$$v(E_1) + \lambda v(E_2) \in \ker Z_{\alpha,\beta}$$

for some $\lambda \in \mathbb{R}^*$, which is equivalent to $\sigma, \pi_{v(E_1)}, \pi_{v(E_2)}$ being colinear, as $\sigma \in V(X)$ is precisely the projection of the kernel of Z_σ. This already proves ϕ_σ is a strictly monotonic function of θ_σ due to continuity. It is increasing since $\phi_\sigma(0+) < \phi_\sigma(\pi)$. The interchange phase $\phi_\sigma = 1$ corresponds to the line $\mathrm{L}_{o,\sigma}$. □

Wall and chamber structure

For any object $E \in \mathrm{D^b}(X)$, there is a wall and chamber structure of $V(X)$ described as follows.

Proposition 2.3 (cf. [11, Proposition 2.6]). *Given an object $E \in D^b(X)$, there exists a locally finite set of walls (line segments) in $V(X)$ with the following properties:*

(a) *The $\sigma_{\alpha,\beta}$-(semi)stability of E is independent of the choice of the stability condition $\sigma_{\alpha,\beta}$ in any chamber.*

(b) *If $\sigma_{\alpha_0,\beta_0}$ is on a wall $\mathcal{W}_E$, that is, the point $k(\alpha_0,\beta_0) \in \mathcal{W}_E$, E is strictly $\sigma_{\alpha_0,\beta_0}$-semistable.*

(c) *If E is semistable in one of the adjacent chambers to a wall, then it is unstable in other adjacent chambers.*

(d) *Any wall $\mathcal{W}_E$ is a connected component of $\mathrm{L} \cap V(X)$, where L is a line passing through the point π_E if $\chi(E) \neq \mathrm{rk}(E)$ or with slope $\mathrm{rk}(E)/c_H(E)$ if $\chi(E) = \mathrm{rk}(E)$.*

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

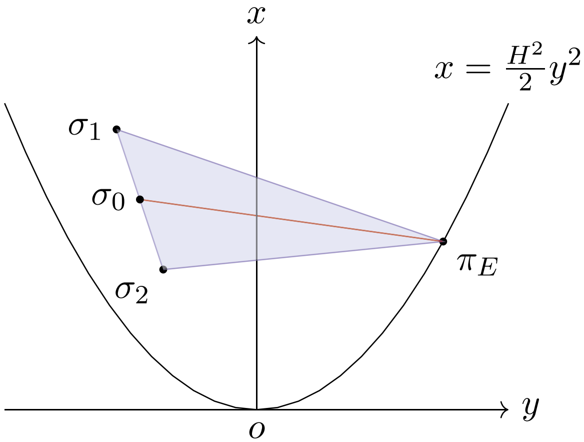


Figure 3. An example of triangle rule: if any point in the colored region is a stability condition, then there is no wall between σ_1 and σ_2.

By definition, if $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable, then $\pi_E \neq \sigma_{\alpha,\beta}$ since $Z_{\alpha,\beta}(E) \neq 0$ (recall that in Section 2 we identify a stability $\sigma_{\alpha,\beta}$ with the projection of kernel $k(\alpha,\beta) := \mathrm{pr}(\ker Z_{\alpha,\beta})$ by abuse of notation). Combined with Proposition 2.3, one can see that for any line segment $\mathrm{L}_{[\sigma_1,\sigma_2]} \subseteq V(X)$ containing $\sigma_{\alpha,\beta}$ with σ_1,σ_2, and π_E colinear, we have

$$\pi_E \notin \mathrm{L}_{[\sigma_1,\sigma_2]}, \tag{2.2}$$

that is, $v(E)$ cannot lie in the kernel of any stability condition in $V(X)$. (In the case where E is stable, this follows directly from the $v(E)^2 \geq -2$ and hence $\pi_E \notin V(X)$.) This will be used in later sections.

3. The destabilising regions

In this section, we characterize the stability conditions which are not lying on the walls of an object $E \in \mathrm{D}^b(X)$. As a warm-up, we first assume $\pi_E \in \partial V(X)$ and hence $v(E)^2 = 0$ or -2. Then we have

Proposition 3.1 (Triangle rule, see Figure 3). *Let $E \in D^b(X)$, and let $I \subseteq V(X)$ be a line segment. Assume*

$$\Delta_{\pi_E}(I) \subseteq V(X). \tag{3.1}$$

Then any point in I is not on a wall. In particular, if $I = \mathrm{L}_{[\sigma_1,\sigma_2]}$, then E is σ_1-stable if and only if it is σ_2-stable.

Proof. Assume on the contrary, that is, there is a wall $\mathcal{W}_E \subseteq \mathrm{L} \cap V(X)$ where L passes through π_E and intersects with I. Let $\sigma_0 = I \cap \mathcal{W}_E$. By our assumption, one has

$$\mathrm{L}_{(\pi_E,\,\sigma_0]} \subseteq \mathcal{W}_E \subseteq V(X).$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

By Proposition 2.3 (b), E is strictly σ-semistable for any $\sigma \in \mathrm{L}_{(\pi_E,\sigma_0]}$. Up to a shift, one may assume that $E \in \mathrm{Coh}^{\beta(\sigma_0)}(X)$. Since σ_0 is on a wall, there exists some semistable factor $F \subset E$ in $\mathrm{Coh}^{\beta(\sigma_0)}(X)$ such that $\phi_{\sigma_0}(F) = \phi_{\sigma_0}(E)$ and $\phi_\sigma(F) > \phi_\sigma(E)$ for σ in an adjacent chamber. In particular, $\pi_F \neq \pi_E$. Applying Proposition 2.3(b) to E, F, and $\mathrm{cok}(F \to E)$, respectively, we know that they remain in the heart for any $\sigma \in \mathrm{L}_{(\pi_E,\sigma_0]}$. Hence, $F \subset E$ is a proper subobject in the corresponding $\mathrm{Coh}^\beta(X)$. As a consequence, we get

$$0 < |Z_\sigma(F)| < |Z_\sigma(E)|.$$

Now, if we tend σ to π_E, then $|Z_\sigma(E)| \to 0$ while $|Z_\sigma(F)| \to \epsilon > 0$ since $\pi_F \neq \pi_E$. This is a contradiction. □

Destabilising regions

The proposition above only works for $\pi_E \in \partial V(X)$ due to Equation (2.2). For the case $v(E)^2 \geq 0$, we need to make use of the three-dimensional region defined as below: For any $\sigma \in V(X)$ and $v \in \mathrm{H}^*_{\mathrm{alg}}(X)$, let $\mathrm{L}_{(\sigma',\sigma'')} \subseteq \mathrm{L}_{\sigma,\pi_v} \cap V(X)$ be the connected component containing σ. Let $[\sigma] \subseteq \mathbb{R}^3$ be the preimage of σ via the projection $\mathrm{pr} : \mathbb{R}^3 \dashrightarrow \mathbb{R}^2$. Consider the plane Π spanned by $[\sigma]$ and $v(E)$. Then $[\sigma_0] \subset \Pi$ for any $\sigma_0 \in \mathrm{L}_{(\sigma',\sigma'')}$. We define the **destabilising region of v with respect to σ** as

$$\Omega_v(\sigma) = (\mathbf{P}_{Ov^+_\sigma v v^-_\sigma} \setminus \{o,v\}) \cap \{u \in \mathbb{R}^3 \mid u^2 \geq -2, (u-v)^2 \geq -2\},$$

where $v^+_\sigma = [\sigma'] \cap ([\sigma''] + v)$ and $v^-_\sigma = [\sigma''] \cap ([\sigma'] + v)$. Note that for any $\sigma \in V(X)$, we have $x \cdot z \geq 0$ for any $(x,y,z) \in [\sigma]$. Consider the (open) shadow area in Figure 4 which is bounded by the two lines and consists of nonzero $(x,y,z) \in [\sigma]$ for some σ. Since $v^2 \geq 0$, $\pi_E \notin V(X)$ and hence v is not in this shadow area. Therefore, the x-coordinates of v^+_σ and v^-_σ have opposite sign. We may simply put

$$v^+_\sigma \in \{(x,y,z) \mid x \geq 0, z \geq 0\} \quad \text{and} \quad v^-_\sigma \in \{(x,y,z) \mid x \leq 0, z \leq 0\}$$

up to switching σ' and σ''.

There is a natural decomposition

$$\Omega_v(\sigma) = \Omega^+_v(\sigma) \sqcup \mathrm{L}_{(O,v)} \sqcup \Omega^-_v(\sigma),$$

where $\Omega^\pm_v(\sigma) = \Omega_v(\sigma) \cap \mathbf{P}_{Ov^\pm_\sigma v} \setminus \mathrm{L}_{(O,v)}$. We call $\Omega^+_v(\sigma)$ the **strictly destabilising region of v with respect to σ**. A key result is:

Lemma 3.2. *For $E \in D^b(X)$ with $v(E)^2 \geq 0$ and $\sigma \in V(X)$, if σ is lying on a wall of E, then there exists an integer point in $\Omega^+_{v(E)}(\sigma)$.*

Proof. Set $v = v(E)$ for temporary notation. Firstly, for any $G \subseteq F \subseteq E$ in $\mathrm{Coh}^{\beta(\sigma)}(X)$ satisfying that E, F, G have the same σ-phase, we always have that $v(F/G)$ is lying in the parallelogram $\mathbf{P}_{Ov^+_\sigma v v^-_\sigma}$. This is because there are inclusions

$$\mathrm{L}_{[0,Z_\tau(F/G)]} \subseteq \mathrm{L}_{[0,Z_\tau(F)]} \subseteq \mathrm{L}_{[0,Z_\tau(E)]}$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

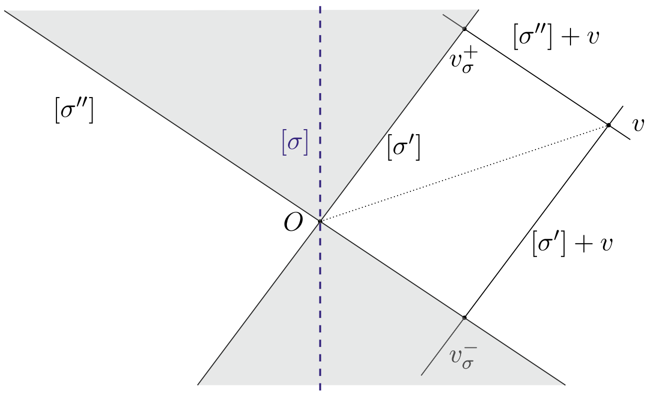


Figure 4. The (open) shawdow area is covered by kernel of stability conditions.

for any $\tau\in \mathrm{L}_{(\sigma',\sigma'')}$, which yields that

$$v(F/G)\in \bigcap_{\tau\in \mathrm{L}_{(\sigma',\sigma'')}} Z_\tau^{-1}(\mathrm{L}_{[0,Z_\tau(v)]})=\mathbf{P}_{Ov_\sigma^+vv_\sigma^-}.$$

In particular, if $0=\widetilde{E}_0\subset\cdots\subset\widetilde{E}_k=E$ is a σ-Jordan–Holder filtration of E with $E_i=\widetilde{E}_i/\widetilde{E}_{i-1}$ its JH-factors, then any $v(E_i)$ and also $v-v(E_i)$ is lying $\mathbf{P}_{Ov_\sigma^+vv_\sigma^-}$.

If necessary, reordering these factors E_i such that the angles between $\mathrm{L}^+_{O,v(E_i)}$ and $\mathrm{L}^+_{O,v_\sigma^+}$ increase with respect to i. And we get a polygon with vertexes $\sum_{i=1}^j v(E_i)$ $(0\leq j\leq k)$. Since $v(E_1)$ and $v-v(E_k)$ lie in $\mathbf{P}^+_{Ov_\sigma^+v}$, one sees that $\sum\limits_{i=1}^{j} v(E_i)$ is an integer point in $\mathbf{P}_{Ov_\sigma^+v}\setminus \mathrm{L}_{(O,v)}$ for any j. We claim that either $v(E_1)$ or $\sum\limits_{i=1}^{k-1} v(E_i)$ is lying in $\Omega_v(\sigma)$. This can be proved by using purely Euclidean geometry. Suppose this fails, then we have

$$\begin{aligned} v(E_1)^2\geq -2 \quad &\text{and} \quad (v(E_1)-v)^2<-2,\\ (\sum_{i=1}^{k-1} v(E_i)-v)^2\geq -2 \quad &\text{and} \quad (\sum_{i=1}^{k-1} v(E_i))^2<-2, \end{aligned} \tag{3.2}$$

as E_i is stable. If we restrict the quadratic equation $u^2=-2$ to the plane of $\mathbf{P}_{ovv_\sigma^+}$, we can obtain a hyperbola whose center is O. The edge $\mathrm{L}_{[O,\,v_\sigma^+]}$ can meet the connected component of this hyperbola at most one point. Similarly, $\mathrm{L}_{[v,\,v_\sigma^+]}$ can intersect with the connected component of the hyperbola defined by $(u-v)^2=-2$ at most one point. Note that the edge $\mathrm{L}_{[O,v]}$ is lying outside the area

$$\Big\{u^2<-2 \text{ and } (u-v)^2<-2\Big\}. \tag{3.3}$$

See the shadow part in Figure 5. By Equation (3.2), v_σ^+ has to lie in the region (3.3). Moreover, as one can see from the picture, there is a point $w\in \mathbf{P}^\circ_{Ov_\sigma^+v}$ lying on the

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

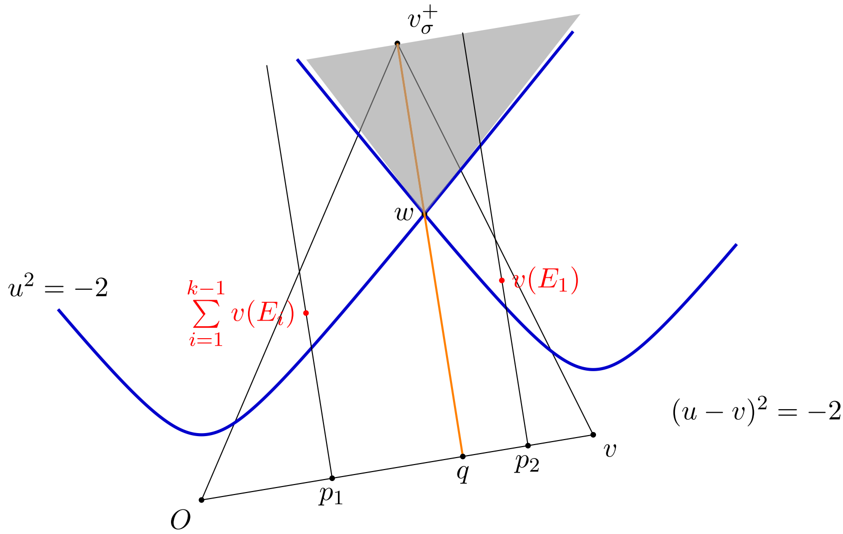


Figure 5. If this plane corresponds to a wall, there must be some integer point inside this triangle and below these two hyperbolas.

intersection of two hyperboloids

$$\left\{u \in \mathbb{R}^3 \mid u^2 = -2\right\} \cap \left\{u \in \mathbb{R}^3 \mid (u-v)^2 = -2\right\}, \tag{3.4}$$

and the line $\mathrm{L}_{w,v_\sigma^+}$ will intersect the edge $\mathrm{L}_{[O,v]}$ at a point, denoted by q. Thus, we get that the point $v(E_1)$ is contained in the triangle $\mathbf{P}_{qvv_\sigma^+}$, while the point $\sum_{i=1}^{k-1} v(E_i)$ is contained in the triangle $\mathbf{P}_{Oqv_\sigma^+}$. If we define L_u to be the line passing through $u \in \mathbb{R}^3$ and parallel to the line $\mathrm{L}_{w,v_\sigma^+}$, the discussion above exactly means

$$\mathrm{L}_{[O,\,p_1]} \subset \mathrm{L}_{[O,\,q)} \subset \mathrm{L}_{[O,\,p_2]}, \tag{3.5}$$

where $p_1 = \mathrm{L}_{\sum_{i=1}^{k-1} v(E_i)} \cap \mathrm{L}_{[O,\,v]}$ and $p_2 = \mathrm{L}_{v(E_1)} \cap \mathrm{L}_{[O,\,v]}$; see Figure 5.

Next, one can regard $\mathrm{L}_{w,v_\sigma^+}$ as a stability condition in a natural way. Consider the line ℓ passing through the origin O on the plane Π, which is parallel to the line passing through w and v_σ^+. By construction, the line ℓ lies between $[\sigma']$ and $[\sigma'']$. Thus, there is a stability condition $\tau \in (\sigma', \sigma'')$ such that $[\tau]$ corresponds to the line ℓ. Since $Z_\tau(\ell) = 0$, we obtain $Z_\tau(E_1) = Z_\tau(p_2)$, and the same holds for p_1.

Using the inclusions (3.5), we obtain the inequality

$$\frac{\left|Z_\tau(v(E_1))\right|}{\left|Z_\tau(\sum_{i=1}^{k-1} v(E_i))\right|} = \frac{\left|Z_\tau(p_2)\right|}{\left|Z_\tau(p_1)\right|} = \frac{\|\mathrm{L}_{[O,\,p_2]}\|}{\|\mathrm{L}_{[O,\,p_1]}\|} > 1. \tag{3.6}$$

However, this contradicts to the relation $\sum_{i=1}^{k-1} \left|Z_\tau(v(E_i))\right| = \left|Z_\tau\left(\sum_{i=1}^{k-1} v(E_i)\right)\right|$ which finishes the proof. □

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Remark 3.3. If $v(\widetilde{E}_i)$ already lies in $\mathbf{P}_{Ovv_\sigma^+} \setminus \mathrm{L}_{(O,v)}$ for all i, then our argument actually implies that we can always take a destabilising sequence

$$F \hookrightarrow E \twoheadrightarrow Q$$

such that $v(F) \in \Omega^+_{v(E)}(\sigma)$ and $v(Q) \in \Omega^-_{v(E)}(\sigma)$. This will happen, for instance, if $E = i_*G$ for some slope stable vector bundle G on C. Indeed, as any subobject A of the sheaf E (in the heart $\mathrm{Coh}^{\beta(\sigma)}(X)$) is also a sheaf (cf., for example, [2, Proposition 2.4]), we have $v(A) \in \mathbf{P}_{Ov_\sigma^+ v} \setminus \mathrm{L}_{(O,v)}$ (since $r(A) \geq 0$ and the case $r(A) = 0$ cannot happen as it is on a wall). In particular, this holds for $v(\widetilde{E}_i)$.

Then we can obtain a generalisation of Proposition 3.1.

Proposition 3.4. *Given $v^2 \geq 0$ and a region $\mathcal{I} \subseteq V(X)$, we define*

$$\Omega_v(\mathcal{I}) = \bigcup_{\sigma \in \mathcal{I}} \Omega_v(\sigma) \quad \text{and} \quad \Omega_v^+(\mathcal{I}) = \bigcup_{\sigma \in \mathcal{I}} \Omega_v^+(\sigma).$$

Then any $\sigma \in \mathcal{I}$ is not lying on a wall of any E with $v(E) = v$ if and only if

$$\Omega_v^+(\mathcal{I}) \cap \mathrm{H}^*_{alg}(X) = \emptyset. \tag{3.7}$$

Similarly, any E with $v(E) = v$ cannot be strictly σ-semistable for any $\sigma \in \mathcal{I}$ if and only if

$$\Omega_v(\mathcal{I}) \cap \mathrm{H}^*_{alg}(X) = \emptyset. \tag{3.8}$$

Proof. The ‘if’ part follows directly from Lemma 3.2. For the ‘only if’ part, suppose there exists some stability condition σ and an integer point $w \in \Omega_v(\sigma)$. Then, we can find σ-stable objects F_1 and F_2 such that $v(F_1) = w$ and $v(F_2) = v - w$, and σ will be lying on a wall of $E := F_1 \oplus F_2$ from the construction. For the strictly semistable case, one just notes that the Mukai vectors of all the factors are lying on $\mathrm{L}_{(O,v)}$. □

According to Proposition 3.4, we will say a Mukai vector $v \in \mathrm{H}^*_{\mathrm{alg}}(X)$ **admits no wall in** $\mathcal{I}$ if Equation (3.7) holds and **admits no strictly semistable condition** if Equation (3.8) holds.

Note that from the definition, one automatically has $\Omega_v(\sigma) = \Omega_v(\mathrm{L}_{(\sigma',\sigma'')})$. This motivates us to find a subregion of $V(X)$ with regular boundary. A candidate is

$$\Gamma = \left\{ (y,x) \in \mathbb{R}^2 \,\middle|\, x > gy^2, x < \sqrt{2/H^2} \text{ when } y = 0 \right\} \subseteq V(X) \tag{3.9}$$

which is used in [12]. As a consequence, if v admits no wall in $I \subseteq \mathrm{L}_{(o,o')}$, then it admits no wall in $\Delta_{\pi_v}(I) \cap \Gamma$ as well.

Remark 3.5. Comparing Proposition 3.1 with Proposition 3.4, one can conclude that for $v^2 = 0$ and $\mathcal{I}$ being a line segment, the condition (3.1) implies Equation (3.7). Actually, if $\Omega_v^+(\sigma)$ contains any integer point δ, then π_δ is a root lying in $\mathrm{L}_{(\pi_v,\sigma]}$. This suggests that the condition (3.1) can be replaced by $\Delta_{\pi_v}(I) \subseteq V'(X)$, where

$$V'(X) = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x > \frac{H^2 y^2}{2} \right\} \setminus \bigcup_{\delta \in \mathrm{R}(X)} \{\pi_\delta\}.$$

4. The restriction map to Brill–Noether locus

Given a positive primitive vector $v = (r,c,s) \in \mathrm{H}^*_{\mathrm{alg}}(X)$, let $\mathbf{M}(v)$ be the moduli space of H-Gieseker semistable sheaves on the surface X with Mukai vector v. In this section, we always assume $r,c,s > 0$ and

$$\gcd(r,c) = 1 \text{ and } r > \frac{v^2}{2} + 1.$$

Then $\mathbf{M}(v)$ is a smooth variety consisting of μ_H-stable locally free sheaves (cf. [21, Remark 3.2]). The main result is:

Theorem 4.1. *For any irreducible curve $C \in |mH|$, the restriction map is an injective morphism*

$$\psi : \mathbf{M}(v) \to \mathbf{BN}_C(v),$$
$$E \mapsto E|_C$$

with stable image (i.e., $E|_C$ is stable) if the following conditions hold

(i) $(mr - c)s > rc$;

(ii) *v admits no wall in $\mathrm{L}_{(o,\sigma_v]}$, where $\sigma_v = (\frac{rc}{(mr-c)s}, 0)$;*

(iii) *$v(-m) = \big(r, c - mr, s + (g-1)m(mr-2c)\big)$ admits no wall in $\Delta_{\pi_{v(-m)}}(\mathrm{L}_{(o,o']}) \cap \Gamma$, where Γ is defined in Equation (3.9).*

Proof. It suffices to prove that for any $E \in \mathbf{M}(v)$, the restriction $E|_C$ is slope stable with $h^0(C, E|_C) \geq r + s$ and $E|_C$ uniquely determines E.

Firstly, we show that $E|_C$ is slope semistable. Note that condition (i) ensures that σ_v lies in $\mathrm{L}_{(o,o')}$. By [11, Lemma 2.13 (b)], it suffices to show that $i_*(E|_C)$ is σ_v-semistable. Consider the exact sequence

$$0 \to E(-C) \to E \to i_*(E|_C) \to 0, \tag{4.1}$$

we have $\pi_{i_*(E|_C)} = (\frac{r}{(g-1)(2c-mr)}, 0)$ is lying on $\mathrm{L}_{\pi_E, \pi_{E(-C)}}$. Since E is slope stable, according to [11, Lemma 2.15], E is σ-stable for any $\sigma \in \mathrm{L}_{(o,\pi_v)} \cap V(X)$. Choose $\sigma_1 \in \mathrm{L}_{(o,\pi_v)}$ sufficiently close to o. We have

$$\mathbf{P}_{o\sigma_v\sigma_1} \setminus \{o\} \subseteq \Gamma \subseteq V(X)$$

as in Figure 6. Note that for any line L passing through π_E, the intersection $\mathbf{P}_{o\sigma_v\sigma_1} \setminus \{o\} \cap$ L is connected. By our assumption 4.1, v admits no wall in $\mathrm{L}_{(o,\sigma_v]}$. This implies it also admits no wall in $\mathbf{P}_{o\sigma_v\sigma_1} \setminus \{o\}$. Hence, E is σ_v-stable as E is σ_1-stable. Similarly, we have

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

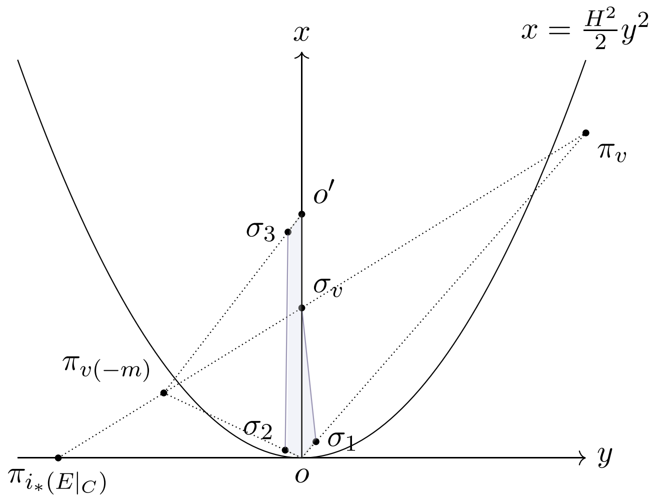


Figure 6. Any point in the colored region is a stability condition.

$E(-C)$ is also σ_v-stable by using the assumption (iii). As in the proof of Proposition 2.2, E and $E(-C)$ are of the same σ_v-phase since $\sigma_v \in \mathrm{L}_{\pi_E,\pi_{E(-C)}}$. Hence, the restriction $i_*(E|_C)$ is σ_v-semistable with E and $E(-C)[1]$ as its JH-factors.

Secondly, we show that $E|_C$ is slope stable. By using [11, Lemma 2.13 (b)], we are reduced to prove $i_*(E|_C)$ is σ-stable for some $\sigma \in \mathrm{L}_{(o,\sigma_v)}$. Moreover, due to [11, Lemma 2.13 (a)], $i_*(E|_C)$ is semistable for any stability condition lying in a line segment $\mathrm{L}_{(o,a)} \subseteq \mathrm{L}_{(o,\sigma_v)}$. Suppose that $i_*(E|_C)$ is strictly semistable for all stability conditions in $\mathrm{L}_{(o,a)}$. Then for any $\sigma_0 \in \mathrm{L}_{(o,a)}$ and any destabilising sequence

$$F_1 \hookrightarrow i_*(E|_C) \twoheadrightarrow F_2 \in \mathrm{Coh}^{\beta=0}(X)$$

such that F_1, F_2 are σ_0-semistable with the same σ_0-phase as $i_*(E|_C)$, we have $\pi_{F_1} = \pi_{i_*(E|_C)}$. This gives $\phi_{\sigma_v}(F_1) = \phi_{\sigma_v}(i_*(E|_C))$, which implies that F_1 is σ_v-semistable. However, this contradicts to the uniqueness of JH-factors of $i_*(E|_C)$ with respect to σ_v. Thus, $i_*(E|_C)$ is σ-stable for some $\sigma \in \mathrm{L}_{(o,a)}$.

Next, we show that $h^0(C,E|_C) = h^0(X,E) \geq r+s$. Let us consider the long exact sequence of cohomology induced by (4.1)

$$0 \to \mathrm{H}^0(X,E(-C)) \to \mathrm{H}^0(X,E) \to \mathrm{H}^0(C,E|_C)) \to \mathrm{H}^1(X,E(-C)) \to \dots$$

As $E(-C)$ is μ_H-stable and $\mu_H(E(-C)) < 0$, we have

$$\mathrm{H}^0(X,E(-C)) = \mathrm{Hom}_X(\mathcal{O}_X, E(-C)) = 0.$$

Then we choose $\sigma_2 \in \mathrm{L}_{(\pi_{v(-m)},o)}$ sufficiently close to o and $\sigma_3 \in \mathrm{L}_{(\pi_{v(-m)},o')}$ sufficiently close to o' so that $\mathbf{P}_{o\sigma_2\sigma_3 o'} \setminus \{o,o'\} \subseteq \Gamma$; see Figure 6. As shown above, $E(-C)$ is σ-stable

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

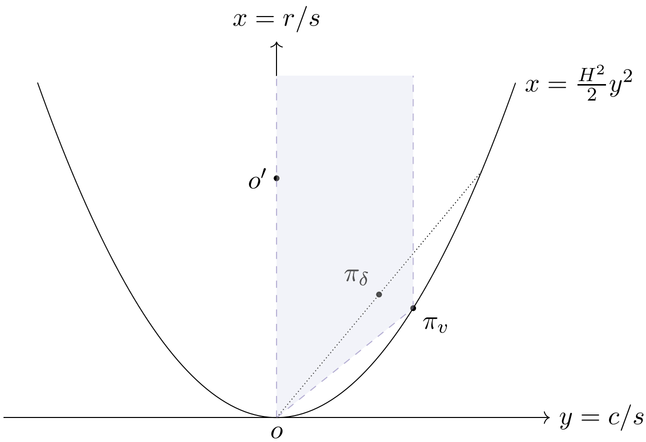


Figure 7. π_δ in the interior of $\mathbf{P}^\circ_{o\pi_v\infty}$ (colored area).

for any $\sigma \in \mathbf{P}_{o\sigma_2\sigma_3 o'} \setminus \{o,o'\}$. In particular, $E(-C)$ is σ_3-stable. According to [11, Lemma 2.15], we have

$$\mathbf{P}_{o'\sigma_3\sigma_v} \setminus \{o'\} \subseteq V(X)$$

and $\mathcal{O}_X$ is also σ_v-stable. Note that $\pi_{\mathcal{O}_X} = o'$. By Proposition 3.1 and Proposition 2.2, we know that $\mathcal{O}_X$ is σ_3-stable and $\phi_{\sigma_3}(E(-C)) = \phi_{\sigma_3}(\mathcal{O}_X)$. Then we have

$$\mathrm{H}^1(X,E(-C)) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X,E(-C)[1]) = 0$$

where $\mathcal{A} = \mathrm{Coh}^{\beta(\sigma_3)}(X)$. Therefore, we get an isomorphism $\mathrm{H}^0(X,E) \xrightarrow{\simeq} \mathrm{H}^0(C,E|_C)$. By Serre duality and the stability of E, we have $\mathrm{H}^2(X,E) \cong \mathrm{Hom}_X(E,\omega_X) \cong \mathrm{Hom}_X(E,\mathcal{O}_X) = 0$. It follows that

$$h^0(C,E|_C) = h^0(X,E) \geq \chi(E) = r+s. \tag{4.2}$$

This proves our claim.

In the end, the uniqueness of E follows from the fact that the JH factors of $i_*(E|_C)$ are unique with respect to σ_v. □

A numerical criterion

As in [11], we would like to find a purely numerical condition for Theorem 4.1 to hold. An elementary result is

Lemma 4.2. *Let $\mathbf{P}_{o\pi_v\infty}$ be the trapezoidal region bounded by $\mathrm{L}_{[o,\pi_v]}$, the (positive) half x-axis $\mathrm{L}_{[o,\infty)}$ and the vertical ray $\mathrm{L}_{[\pi_v,\infty)}$ in Figure 7. Then v admits no wall in $\mathbf{P}_{o\pi_v\infty} \cap \Gamma$ if one of the following conditions holds*

(i) $v^2 = 0$ *and* $r/\gcd(r,c) \leq g-1$.

(ii) $s = \lfloor \frac{(g-1)c^2+1}{r} \rfloor$ *and* $g-1 \geq \max\{\frac{r^2}{c}, r+1\}$.

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Proof. (i). By Proposition 3.1, it will be sufficient to show that

$$\mathbf{P}^{\circ}_{o\pi_v\infty} \subseteq V(X).$$

Due to the explicit description of $V(X)$ in (2.1), this is equivalent to showing that there is no projection of root lying in $\mathbf{P}^{\circ}_{o\pi_v\infty}$. Suppose there exists a root $\delta = (r',c',s') \in \mathrm{R}(X)$ with $\pi_\delta \in \mathbf{P}^{\circ}_{o\pi_v\infty}$. Then we have

$$\frac{c'}{r'} < \frac{c}{r} \quad \text{and} \quad \frac{c'}{s'} < \frac{c}{s}, \tag{4.3}$$

see Figure 7. Note that $2rs = c^2(2g-2)$ and $2r's' = (c')^2(2g-2)+2$, one can plug into (4.3) to get

$$\frac{r}{\gcd(r,c)}c' < \frac{c}{\gcd(r,c)}r' < \frac{r}{\gcd(r,c)}c' + \frac{r}{\gcd(r,c)(g-1)c'} \leq \frac{r}{\gcd(r,c)}c' + 1. \tag{4.4}$$

which is not possible.

(ii). According to Proposition 3.4, we just need to show that $\Omega_v^+(\sigma) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset$ for any $\sigma \in \mathbf{P}_{o\pi_v\infty} \cap \Gamma$. Suppose there is an integer point $(x,y,z) \in \Omega_v^+(\sigma_0)$ for some $\sigma_0 \in \mathbf{P}_{o\pi_v\infty}$. By the construction of $\Omega_v^+(\mathbf{P}_{o\pi_v\infty} \cap \Gamma)$, we have $0 < y \leq c$ and the point (x,y,z) is lying in the interior of the triangle $\mathbf{P}_{u_1u_2u_3}$ with vertices

$$u_1 = (\frac{ry}{c}, y, \frac{gcy}{r}), u_2 = (\frac{ry}{c}, y, \frac{sy}{c}) \text{ and } u_3 = (\frac{gcy}{s}, y, \frac{sy}{c}).$$

As $y^2(g-1)+1 \geq xz$ and $z \geq \frac{sy}{c}$, one has

$$\frac{ry}{c} < x < \frac{y^2(g-1)+1}{sy/c}.$$

Note that $c^2(g-1) - \frac{v^2}{2} = rs$, the condition $s = \lfloor \frac{(g-1)c^2+1}{r} \rfloor$ is equivalent to $r > \frac{v^2}{2}+1$. Then we have

$$\begin{aligned} 0 < \frac{y^2(g-1)+1}{sy/c} - \frac{ry}{c} &< \max\left\{\frac{gc}{s} - \frac{r}{c}, \frac{c^2(g-1)+1}{s} - r\right\} \\ &= \max\left\{\frac{r(c^2+\frac{v^2}{2})}{c(c^2(g-1)-\frac{v^2}{2})}, \frac{r(\frac{v^2}{2}+1)}{c^2(g-1)-\frac{v^2}{2}}\right\} \\ &\leq \max\left\{\frac{r(c^2+r-2)}{c(c^2(g-1)-r+2)}, \frac{r^2-r}{c^2(g-1)-r+2}\right\} \\ &\leq \frac{1}{c}. \end{aligned} \tag{4.5}$$

Here, the last inequality follows from our assumption $g-1 \geq \max\{\frac{r^2}{c}, r+1\}$. This means $0 < x - \frac{ry}{c} < \frac{1}{c}$ which contradicts to the fact x is an integer. □

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Injectivity condition

Let us summarise our numerical criterion in short. We may say $v=(r,c,s)\in \mathrm{H}^*_{\mathrm{alg}}(X)$ satisfying the injectivity condition (✶) if the following inequalities holds

$$r>\max\left\{\frac{v^2}{2}+1,\frac{c}{m}\right\},\quad c>0,\quad s>\frac{rc}{mr-c},\quad \gcd(r,c)=1 \tag{✶–1}$$

and

$$g-1\geq\begin{cases} r, & \text{if } v^2=0\\ \max\{\frac{r^2}{c},\frac{r^2}{mr-c},r+1\}, & \text{if } v^2>0.\end{cases} \tag{✶–2}$$

Then we have

Corollary 4.3. *The restriction map $\psi:\mathbf{M}(v)\to\mathbf{BN}_C(v)$ is an injective morphism with stable image if v satisfies the condition:*

Proof. The condition $mr>c>0$ ensures π_v lies in the first quadrant while $\pi_{v(-m)}$ lies in the second quadrant, and the condition $s(mr-c)>rc$ ensures σ_v is below o'. The assertion then follows from the direct computation that $\mathrm{L}_{(o,\sigma_v]}\subseteq\Delta_{\pi_w}(\mathrm{L}_{[o,o']})\cap\Gamma\subseteq\mathbf{P}_{o\pi_w\infty}\cap\Gamma$ for $w=v$ or $v(-m)$. □

Remark 4.4. Under the assumption $r>c$, the conditions in Corollary 4.3 can be easily reduced to Equation (1.4).

5. Surjectivity of the restriction map

Throughout this section, we let $v=(r,c,s)\in\mathrm{H}^*_{\mathrm{alg}}(X)$ be a positive vector satisfying the injectivity condition (✶). Due to Corollary 4.3, the restriction map

$$\psi:\mathbf{M}(v)\to\mathbf{BN}_C(v)$$

is an injective morphism with stable image. Following the ideas in [11, 13], we give sufficient conditions such that ψ is surjective.

The first wall

As in [11], we first describe the wall that bounds the Gieseker chamber of i_*F for $F\in\mathbf{BN}_C(v)$. The following result is an extension of [11, Proposition 4.2].

Theorem 5.1. *For any $F\in\mathbf{BN}_C(v)$, the wall that bounds the Gieseker chamber of i_*F is not below the line $\mathrm{L}_{\pi_v,\pi_{v(-m)}}$, and they coincide if and only if $F=E|_C$ for some $E\in\mathbf{M}(v)$.*

Proof. The argument is essentially the same as the primitive case proved in [11, Proposition 4.2]. Here, we provide the details for completeness.

We first show that for any v satisfying Equation (✶–1), if both v and $v(-m)$ admit no wall in $(o,\sigma_v]$, then so does

$$v|_C:=v-v(-m).$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Let $\mathcal{W}_{i_*F}$ be the first wall, and let $\sigma_{\alpha',0} \in \mathcal{W}_{i_*F}$ be a stability condition. Suppose $\mathcal{W}_{i_*F}$ is below or on the line $\mathrm{L}_{\pi_v,\pi_{v(-m)}}$. Then for any destabilising sequence

$$F_1 \hookrightarrow i_*F \twoheadrightarrow F_2 \tag{5.1}$$

in $\mathrm{Coh}^{\beta=0}(X)$ such that F_1, F_2 are $\sigma_{\alpha',0}$-semistable, and

$$\phi_{\alpha,0}(F_1) > \phi_{\alpha,0}(i_*F) \quad \text{for} \quad \alpha < \alpha'. \tag{5.2}$$

Taking the cohomology of Equation (5.1) gives a long exact sequence of sheaves

$$0 \to \mathrm{H}^{-1}(F_2) \to F_1 \xrightarrow{d_0} i_*F \xrightarrow{d_1} \mathrm{H}^0(F_2) \to 0. \tag{5.3}$$

Set $v(F_1) = (r',c',s')$, then we have $r' > 0$ by Equation (5.2). Let T be the maximal torsion subsheaf of F_1, and we can write $v(T) = (0,\hat{c},\hat{s})$ for some $\hat{c},\hat{s} \in \mathbb{Z}$. Consider the inclusions $T \hookrightarrow F_1 \hookrightarrow i_*F$ and take the cohomology, one can get

$$0 \to \mathrm{H}^{-1}(\mathrm{cok}) \to T \to i_*F \to \mathrm{H}^0(\mathrm{cok}) \to 0.$$

Since $\mathrm{H}^{-1}(\mathrm{cok})$ is torsion-free, it must be zero. It follows that T is a subsheaf of i_*F and $\mathrm{rk}(i^*T) = \frac{\hat{c}}{m}$. If we let $v(\mathrm{H}^0(F_2)) = (0,c'',s'')$, by restricting (5.3) to the curve C, one can get

$$\begin{aligned} r' + \frac{\hat{c}}{m} &= \mathrm{rk}(F_1/T) + \mathrm{rk}(i^*T) \geq \mathrm{rk}(i^*F_1) \\ &\geq \mathrm{rk}(i^*\ker d_1) \geq \mathrm{rk}(i^*F) - \mathrm{rk}(i^*\mathrm{H}^0(F_2)) = r - \frac{c''}{m}. \end{aligned}$$

In other words,

$$\mu(F_1/T) - \mu(\mathrm{H}^{-1}(F_2)) = \frac{c'-\hat{c}}{r'} - \frac{c'+c''-mr}{r'} \leq m. \tag{5.4}$$

Using Lemma 5.2 below, we can take the destabilising sequence (5.1) satisfying

$$\mu_H^-(F_1/T) \geq \frac{c}{r} \text{ and } \mu_H^+(H^{-1}(F_2)) \leq \frac{c-mr}{r}. \tag{5.5}$$

This gives

$$\mu_H^-(F_1/T) - \mu_H^+(H^{-1}(F_2)) \geq m. \tag{5.6}$$

Combining Equations (5.4) and (5.6), we get $mr - c'' - \hat{c} = mr'$, thus

$$\mu_H(F_1/T) = \frac{c'-\hat{c}}{r'} = \frac{c'-\hat{c}}{r - \frac{c''+\hat{c}}{m}} = \frac{c}{r},$$

and both F_1/T and $\mathrm{H}^{-1}(F_2)$ are μ_H-semistable. Since $\gcd(r,c) = 1$ and i_*F does not contain any skyscraper sheaf, we have $\hat{c} = c'' = 0$ and $\hat{s} = 0$. This shows $T = 0$, and hence $v(F_1) = (r,c,s')$. Note that by our assumption, we have $\pi_{v(F_1)} \in \mathrm{L}_{(o,\pi_v)}$, which means $s \leq s'$. If $s < s'$; however, as $v^2 < 2r-2$ by Equation (✶-1), this gives

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

$$
\begin{aligned}
v(F_1)^2 &= c^2(2g-2) - 2rs' \\
&= v^2 + 2r(s-s') \\
&< 2r(s-s'+1) - 2 \leq -2
\end{aligned}
$$

which contradicts to the fact that F_1 is μ_H-stable. This forces $s = s'$ and $\mathcal{W}_{i_*\mathcal{F}} \subseteq \mathrm{L}_{\pi_v,\pi_{v(-m)}}$.

In the case that $\mathcal{W}_{i_*\mathcal{F}} \subseteq \mathrm{L}_{\pi_v,\pi_{v(-m)}}$, we have

$$\mu_H^-(F_1/T) - \mu_H^+(H^{-1}(F_2)) = m$$

and F_1 is a stable sheaf. Note that the map $d_0 : F_1 \to i_*F$ factors through $d_0' \colon i_*(F_1|_C) \to i_*F$ and $\mu_H(i_*(F_1|_C)) = \mu_H(i_*F)$. Applying Theorem 4.1 to F_1, we know that $i_*(F_1|_C)$ is stable as well. It follows that d_0' is an isomorphism. □

Lemma 5.2. *With notations and assumptions as above, one can find a destabilising sequence (5.1) such that F_i satisfies*

$$\mu_H^-(F_1/T) \geq \frac{c}{r} \quad \textit{and} \quad \mu_H^+(\mathrm{H}^{-1}(F_2)) \leq \frac{c-mr}{r}. \tag{5.7}$$

Proof. Denote $\sigma_1 = \mathcal{W}_{i_*F} \cap \mathrm{L}_{(o,\sigma_v]}$. By Remark 3.3, we can take the destabilising sequence

$$F_1 \hookrightarrow i_*F \twoheadrightarrow F_2$$

satisfying $v(F_1) \in \Omega^+_{v|_C}(\sigma_1) \subseteq \Omega^+_{v|_C}(\mathrm{L}_{(o,\sigma_v]})$ and $v(F_2) \in \Omega^-_{v|_C}(\sigma_1) \subseteq \Omega^-_{v|_C}(\mathrm{L}_{(o,\sigma_v]})$. We divide the proof into three steps. □

Step 1. We show that for any point $u = (x_0, y_0, z_0)$ with $u^2 \geq -2$ and $x_0 > 0$, u is lying in $\Omega^+_v(\mathrm{L}_{(o,\sigma_v]})$ if $x_0 \leq r$ or $z_0 \leq s$, and $\pi_u \in \mathbf{P}^\circ_{o\sigma_v\pi_v}$. By its definition, we know that $u \in \Omega^+_v(\mathrm{L}_{(o,\sigma_v]})$ if

$$u \in \mathbf{P}^\circ_{Ovv^+_\sigma} \quad \text{and} \quad (u-v)^2 \geq -2 \tag{5.8}$$

for some $\sigma \in \mathrm{L}_{(o,\sigma_v]}$.

As $\pi_u = (\frac{x_0}{z_0}, \frac{y_0}{z_0})$ is lying in the interior of the triangle $\mathbf{P}_{o\sigma_v\pi_v}$, we have

$$\frac{y_0}{z_0} < \frac{c}{s} \quad \text{and} \quad \frac{x_0/z_0}{y_0/z_0} = \frac{x_0}{y_0} > \frac{r}{c}. \tag{5.9}$$

The line L_{π_u,π_v} will meet the open edge $\mathrm{L}_{(o,\sigma_v)}$. Denote by σ the intersection point $\mathrm{L}_{(o,\sigma_v]} \cap \mathrm{L}^+_{\pi_v,\pi_u}$. From the construction, we know that u is coplanar to v, v^+_σ and O. Indeed, it is lying in the planar cone bounded by the two rays $\mathrm{L}^+_{O,v}$ and $\mathrm{L}^+_{O,v^+_\sigma}$. The condition $x_0 \leq r$ or $z_0 \leq s$ will ensure that $u \in \mathbf{P}^\circ_{Ovv^+_\sigma}$.

Moreover, when $x_0 \leq r, z_0 \geq s$ or $x_0 \geq r, z_0 \leq s$, we have $(u-v)^2 \geq (g-1)(y_0-c)^2 > 0$. When $x_0 \leq r$ and $z_0 \leq s$, then we have

$$(u-v)^2 \geq \frac{(c-y_0)^2}{c^2} v^2 > 0,$$

by Equation (5.9).

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Step 2. Set $v(F_1)=(r',c',s')$ and $v(F_2)=(-r',mr-c',s-\tilde{s}-s')$ with $0<c'<mr$ and $r'>0$. We claim that

$$\mu_H(F_1)\geq \frac{c}{r} \quad \text{and} \quad \mu_H(F_2)\leq \frac{c-mr}{r}. \tag{5.10}$$

Firstly, we must have either $r'\leq r$ or $s'\leq s$. Otherwise, one will have

$$v(F_1)^2<(g-1)c^2-r(s+1)\leq -2$$

or

$$(v(F_1)-v|_C)^2<(g-1)(mr-c)^2-r(\tilde{s}+1)\leq -2.$$

Both of them are impossible as $v(F_1)\in \Omega^+_{v|_C}(\mathrm{L}_{(o,\sigma_v]})$.

Now, suppose $\mu_H(F_1)<\frac{c}{r}$. Then we have $v(F_1)\in \mathbf{P}^\circ_{o\sigma_v\pi_v}$ as $\phi_{\sigma_v}(F_1)\geq \phi_{\sigma_v}(v)$. According to Step 1, we get

$$v(F_1)\in \Omega^+_v(\mathrm{L}_{(o,\sigma_v]})$$

which contradicts to the assumption $\Omega^+_v(\mathrm{L}_{(o,\sigma_v]})\cap \mathrm{H}^*_{\mathrm{alg}}(X)=\emptyset$. Similarly, we have $\mu_H(F_2)\leq \frac{c-mr}{r}$ as there is no integer point in $\Omega^+_{v(-m)}(\mathrm{L}_{(o,\sigma_v]})$. This proves the claim. As a consequence, we get

$$\frac{mr'}{r}=\mu_H(F_1)-\mu_H(F_2)\geq \frac{c}{r}-\frac{c-mr}{r}=m$$

which implies $r'\leq r$.

Step 3. Let $(F_1)_{\min}$ be the last μ_H-HN factor of F_1, hence also of F_1/T. According to [4, Proposition 14.2], for σ sufficiently close to o, we always have

- $(F_1)_{\min}$ is σ-semistable,
- $v(G)$ is proportional to $v((F_1)_{\min})$ for any σ-stable factor G of $(F_1)_{\min}$.

As $(F_1)_{\min}$ is a quotient sheaf of F_1, it is also a quotient of F_1 in $\mathrm{Coh}^{\beta=0}(X)$. Since F_1 is σ_1-semistable, we have

$$\phi_{\sigma_1}(F_1)\leq \phi_{\sigma_1}((F_1)_{\min}).$$

Combined with the fact $\mu_H(F_1)\geq \mu_H((F_1)_{\min})$, we have $\pi_G=\pi_{(F_1)_{\min}}\in \mathbf{P}_{o\sigma_1\pi_{F_1}}$. As the triangle $\mathbf{P}_{o\sigma_1\pi_{F_1}}$ is lying below the ray $\mathrm{L}^+_{\sigma_v,\pi_v}$, we get $\pi_G\in \mathbf{P}^\circ_{o\sigma_v\pi_v}$ if $\mu_H(G)<\frac{c}{r}$. Note that $\mathrm{rk}(G)\leq \mathrm{rk}(F_1)=r$. We must have $\mu_H(G)\geq \frac{c}{r}$ otherwise one will get $\pi_G\in \Omega^+_v(\mathrm{L}_{(o,\sigma_v]})$ by the same argument in Step 2. It follows that

$$\mu^-_H(F_1/T)=\mu_H((F_1)_{\min})=\mu_H(G)\geq \frac{c}{r}.$$

A similar argument shows $\mu^+_H(\mathrm{H}^{-1}(F_2))\leq \frac{c-mr}{r}$. This finishes the proof.

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

HN-polygon

Let $\sigma_{\alpha,0}$ be a stability condition with α close to $\sqrt{2/H^2}$. By [11, Proposition 3.4], for fixed E, the HN filtration of $\sigma_{\alpha,0}$ will stay the same for $\sqrt{2/H^2}+\epsilon > \alpha > \sqrt{2/H^2}$. Denote by $\overline{\sigma}$ the limit of $\sigma_{\alpha,0}$. The 'stability function' can be written as

$$\overline{Z}(E) = r - s + c\sqrt{-1}$$

if $v(E) = (r,c,s)$. Let $\mathbf{P}_{i_*F}$ be the HN polygon (Here, our definition of HN polygon is slightly different from [11, Definition 3.3]. We drop off the part on the right-hand side of the line segment $\mathrm{L}_{[0,\overline{Z}(i_*F)]}$) for i_*F with respect to $\overline{\sigma}$. For $E \in \mathbf{M}(v)$, we have $\mathbf{P}_{i_*(E|_C)} = \mathbf{P}_{0z_1z_2}$, where

$$z_1 = r - s + c\sqrt{-1} \quad \text{and} \quad z_2 = m(g-1)(mr-2c) + mr\sqrt{-1}.$$

As the polygon $\mathbf{P}_{i_*(E|_C)}$ only depends on v, we may simply write it as $\mathbf{P}_v$.

Theorem 5.3. *For any $F \in \mathbf{BN}_C(v)$, we have $\mathbf{P}_{i_*F} \subseteq \mathbf{P}_v$. Moreover, they coincide if and only if $F = E|_C$ for some $E \in \mathbf{M}(v)$.*

Proof. When $v^2 = 0$, this is essentially proved in [11, Lemma 4.3]. Let us give a slightly different argument which also works for $v^2 > 0$. Suppose the HN-filtration of i_*F for $\bar{\sigma} = (\mathrm{Coh}^{\beta=0}(X), \overline{Z})$ is given by

$$0 = \widetilde{E}_0 \subset \widetilde{E}_1 \subset \cdots \subset \widetilde{E}_{l-1} \subset \widetilde{E}_l = i_*F \tag{5.11}$$

with $E_i := \widetilde{E}_i/\widetilde{E}_{i-1}$ the semistable HN-factors. To show $\mathbf{P}_{i_*F} \subseteq \mathbf{P}_v$, it suffices to show that

$$\phi_{\bar{\sigma}}(v) \geq \phi_{\bar{\sigma}}(E_1) \quad \text{and} \quad \phi_{\bar{\sigma}}(E_l) \geq \phi_{\bar{\sigma}}(v(-m)) \tag{5.12}$$

(see Figure 8b) since $\mathbf{P}_{i_*F}$ is convex. According to the proof of Proposition 2.2, for any object in $\mathrm{Coh}^0(X)$, the angle $\phi_{\bar{\sigma}}$ in Figure 8b is an increasing function of the angle $\theta_{o'}$ in Figure 8a. (They are actually equal in this case, as $\cot\theta_{o'} = \frac{y-1}{x} = \frac{r-s}{c} = \frac{\mathrm{Re}\,\overline{Z}}{\mathrm{Im}\,\overline{Z}} = \cot\phi_{\bar{\sigma}}$). Therefore, it is equivalent to show

$$\theta_{o'}(\pi_v) \geq \theta_{o'}(\pi_{E_1}) \quad \text{and} \quad \theta_{o'}(\pi_{E_l}) \geq \theta_{o'}(\pi_{v(-m)}) \tag{5.13}$$

in Figure 8a.

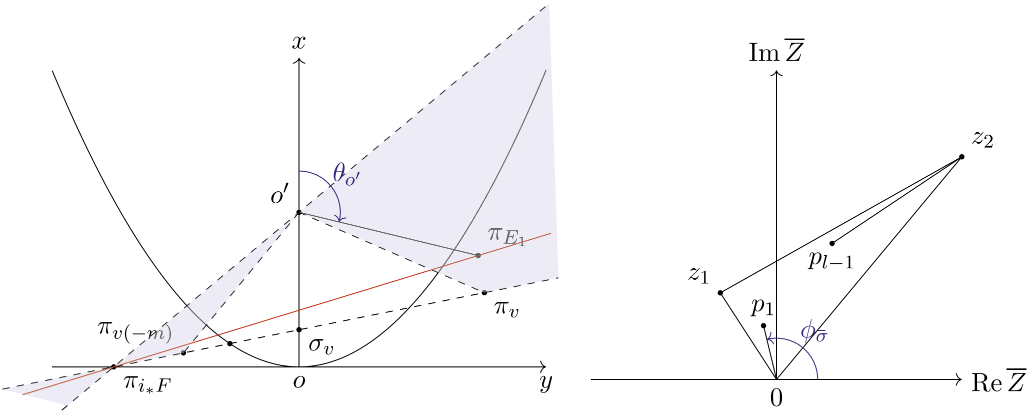
To prove Equation (5.13), consider the sequence

$$0 \to \widetilde{E}_n \xrightarrow{f_n} i_*F \to \mathrm{cok}(f_n) \to 0 \tag{5.14}$$

for each $\widetilde{E}_n$. Since the first wall is not below $\mathrm{L}_{\pi_{i_*F},\pi_v}$, we have $\phi_{\sigma_v}(\widetilde{E}_n) \leq \phi_{\sigma_v}(i_*F)$. As $\phi_{\bar{\sigma}}(\widetilde{E}_n) \geq \phi_{\bar{\sigma}}(i_*F)$, there exists some stability condition $\sigma \in \mathrm{L}_{(o',\sigma_v]}$ such that the objects in Equation (5.14) have the same σ-phase. As a consequence, we have

$$\pi_{\widetilde{E}_i} \in \bigcup_{\sigma \in \mathrm{L}_{[\sigma_v,o')}} \mathrm{L}_{\pi_{i_*F,\sigma}}. \tag{5.15}$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press



(A) Any point π_{E_1} in the colored region satisfies $\theta_{o'}(\pi_{v(-m)}) \le \theta_{o'}(\pi_{E_1}) \le \theta_{o'}(\pi_v)$

(B) HN-factors, where $p_i = \overline{Z}(\widetilde{E}_i)$

Figure 8. The angle $\theta_{o'}$ is equal to the angle $\phi_{\bar{\sigma}}$.

Take $n=1$, and set $v(E_1) = (r',c',s')$. We claim that $\pi_{E_1} \notin \mathbf{P}_{o'\pi_v\sigma_v} \setminus \{\pi_v\}$ which yields $\theta_{o'}(\pi_{E_1}) \le \theta_{o'}(\pi_v)$. This can be proved by cases as follows:

Case (1) If $v^2 = 0$, $\pi_{E_1} \notin \mathbf{P}_{o'\pi_v\sigma_v} \setminus \{\pi_v\}$ automatically holds. This is because $\mathbf{P}_{o'\pi_v\sigma_v} \setminus \{\pi_v, o'\} \subseteq V(X)$ and Equation (2.2).

Case (2) If $v^2 > 0$ and $r' \le r$ or $s' \le s$, as E_1 is $\bar{\sigma}$-semistable, we may assume $v(E_1)^2 \ge -2$ otherwise we may replace E_1 by its first JH-factor. According to Step 1 in Lemma 5.2, we have

$$v(E_1) \in \Omega_v^+(\mathrm{L}_{(o,\,o')})$$

which contradicts to the assumption $\Omega_v^+(\mathrm{L}_{(o,\,o')}) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset$.

Case (3) If $r' > r$ and $s' > s$, we claim that $r' < r+1$. Choose a stability condition $\sigma \in \mathrm{L}_{[\sigma_v, o')}$ such that $\phi_\sigma(i_*F) = \phi_\sigma(E_1)$. Then $v(E_1) \notin \{O, v|_c\}$ is lying in the triangle $\mathbf{P}_{Ov|_c(v|_c)^+_\sigma}$. This means we have $0 < c' < mr$ and

$$g(c')^2 - r's' \ge 0, \quad g(c'-mr)^2 - r'(s' + (g-1)m(mr-2c)) \ge 0.$$

After reduction, we know that $r' < r+1$ as $gc^2 - (r+1)s \le 0$ and $g(c-mr)^2 - (r+1)\tilde{s} \le 0$ by (✶-2).

Similarly, take $n = l-1$ and use the $\bar{\sigma}$-semistability of $\mathrm{cok}(f_{l-1}) = E_l$, one can prove the second inequality of Equation (5.13).

Finally, if $\mathbf{P}_{i_*F} = \mathbf{P}_v$, the first wall will coincide with the line $\mathrm{L}_{\pi_{v(-m)},\pi_v}$ and the last assertion follows from Theorem 5.1. □

Remark 5.4. The discussion above can be much more simplified if the following is true: For any σ on a wall of i_*F, there exists a JH-filtration of i_*F which is convex (i.e., the polygon with vertices $v(\widetilde{E}_i)$ is convex in the plane of $\mathbf{P}_{Ovv^+_\sigma}$).

Now, we provide a numerical criterion for verifying $\mathbf{P}_v = \mathbf{P}_{i_*F}$ via Euclidean geometry. The key ingredient is the upper bound on the number of global sections of an object $E \in \mathrm{D}^b(X)$ established by Feyzbakhsh in [11, 13]. Recall that for any $x,y \in \mathbb{Z}$, there is a function

$$\ell(x+\sqrt{-1}y) := \sqrt{x^2+2H^2y^2+4(\gcd(x,y))^2}$$

and one can define $\ell(E) := \sum_i \ell(\overline{Z}(E_i))$, where E_i's are the $\overline{\sigma}$-semistable factors of E. Moreover, we have a metric function given by

$$\|x+\sqrt{-1}y\| := \sqrt{x^2+(2H^2+4)y^2},$$

and we set $\|E\| := \sum_i \|\overline{Z}(E_i)\|$. Clearly, one has $\|E\| \geq \ell(E)$ once the y-coordinates are nonzero.

Proposition 5.5 [13, Proposition 3.3 and Remark 3.4]. *Suppose $E \in \mathrm{Coh}^0(X)$ which has no subobject $F \subsetneq E$ in $\mathrm{Coh}^0(X)$ with $c_1(F)=0$, we have*

$$h^0(X,E) \leq \sum_i \left\lfloor \frac{\ell(E_i)+\chi(E_i)}{2} \right\rfloor = \sum_i \left\lfloor \frac{\ell(E_i)-\mathrm{Re}\,\overline{Z}(E_i)}{2} \right\rfloor, \tag{5.16}$$

where E_i's are semistable factors with respect to $\overline{\sigma}$. In particular,

$$h^0(X,E) \leq \frac{\|E\|+\chi(E)}{2}. \tag{5.17}$$

Following [11], we can give a criterion for the surjectivity of ψ.

Theorem 5.6. *With the notation as in §5.2: Let $z_1^{+1} = r-s+1+c\sqrt{-1}$, $z_1' = r-s-\frac{r-s}{c}+(c-1)\sqrt{-1}$ and $z_2' = r-s-\frac{r-\gamma^2 s}{\gamma c}+(c+1)\sqrt{-1}$, where $\gamma = \frac{mr}{c}-1$. Assume that*

(i) $\frac{s-r}{c}+\frac{s-r-\chi}{mr-c} \geq 2$

(ii) $\|z_1-z_1'\|-\|z_1'-z_1^{+1}\|+\|z_1-z_2'\|-\|z_2'-z_1^{+1}\| \geq \frac{2c^2}{r+s}+\frac{2(mr-c)^2}{r+s-\chi}$,

where $\chi = \chi(i_*F) = m(g-1)(2c-mr)$. Then the restriction map ψ will be surjective.

Proof. Suppose we have $\mathbf{P}_v \neq \mathbf{P}_{i_*F}$ for some $F \in \mathbf{BN}_C(v)$. By Proposition 5.5 and the convexity, we have

$$r+s \leq h^0(C,F) = h^0(X,i_*F) \leq \frac{\|i_*F\|+\chi}{2} \leq \frac{\hbar+\chi}{2},$$

where $\hbar = \sqrt{(r+s-\chi)^2+4(mr-c)^2}+\sqrt{(r+s)^2+4c^2}$. Then we get

$$\frac{\hbar+\chi}{2}-(r+s) \geq \frac{\hbar+\chi}{2}-\frac{\|i_*F\|+\chi}{2} = \frac{\hbar-\|i_*F\|}{2}. \tag{5.18}$$

However, note that the polygon $\mathbf{P}_{0z_1'z_1^{+1}z_2'z_2}$ is convex under the assumption (i), we have

$$\hbar-\|i_*F\| \geq \|z_1-z_1'\|-\|z_1'-z_1^{+1}\|+\|z_1-z_2'\|-\|z_2'-z_1^{+1}\|. \tag{5.19}$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Combined with assumption (ii), we get

$$\begin{aligned}\hbar+\chi-2(r+s) &= \sqrt{(r+s)^2+4c^2}-(r+s)\\ &\quad+\sqrt{(r+s-\chi)^2+4(mr-c)^2}-(r+s-\chi)\\ &< \frac{2c^2}{r+s}+\frac{2(mr-c)^2}{r+s-\chi}. \qquad (5.20)\\ &\le \hbar-\|i_*F\| \qquad (5.21)\end{aligned}$$

which contradicts Equation (5.18). This proves the assertion. □

Surjectivity condition

As an application, we get an explicit criterion for ψ being surjective for $v^2\ge 0$.

Corollary 5.7. *The restriction map $\psi:\mathbf{M}(v)\to\mathbf{BN}_C(v)$ is bijective if we further have the inequality*

$$g\ge 4r^2+1. \qquad (\star\star)$$

We may call it a surjectivity condition.

Proof. As $r>1+\frac{v^2}{2}$, we have

$$s=\lfloor\frac{(g-1)c^2+1}{r}\rfloor\ge 4rc^2.$$

This gives

$$\begin{aligned}s-r&\ge 4rc^2-r\ge 3c\\ s-r-\chi&=\frac{(g-1)(mr-c)^2}{r}-\frac{v^2}{2r}-r\ge 4r(mr-c)^2-r\ge 3(mr-c),\end{aligned} \qquad (5.22)$$

as $mr-c>0$. Moreover, one can compute that

$$\frac{2s-2r-c}{2s+2r+c}\ge\frac{8rc^2-2r-c}{8rc^2+2r+c}\ge\frac{6r-1}{10r+1}>\frac{1}{r}\ge\frac{4c^2}{s}.$$

It follows that

$$\begin{aligned}\|z_1-z_1'\|-\|z_1'-z_1^{+1}\| &= \sqrt{(\frac{s-r}{c})^2+4g}-\sqrt{(\frac{s-r}{c}-1)^2+4g}\\ &> \frac{\frac{s-r}{c}-\frac{1}{2}}{\sqrt{(\frac{s+r}{c})^2+4(1+\frac{n}{c^2})}}\\ &> \frac{2s-2r-c}{2s+2r+c}>\frac{2c^2}{s}+\frac{2c^2}{r+s}\end{aligned}$$

and $\|z_1-z_2'\|-\|z_2'-z_1^{+1}\|=\sqrt{(\frac{s-r-\chi}{mr-c})^2+4g}-\sqrt{(\frac{s-r-\chi}{mr-c}-1)^2+4g}\ge 0$. The assertion can be concluded from Theorem 5.6. □

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Remark 5.8. For g sufficiently large, it is not hard to find Mukai vectors satisfying the conditions in Theorem 5.6. For instance, when $v^2=0$ and $g>84$, the Mukai vectors given in [11] and [13] will automatically satisfy the conditions for any $m\geq 1$. However, when g is small, it becomes impossible to find such Mukai vectors.

6. Surjectivity for special Mukai vectors

According to Remark 5.8, Theorem 5.6 does not work well for small g. In this section, we develop a way to improve the estimate in §5 for special Mukai vectors of square zero. Let us first introduce the sharpness of the polygon $\mathbf{P}_v$.

Definition 6.1. Denote by z_1^{+d} the point $r-s+d+c\sqrt{-1}$. Let z_1',z_2' be the points as in the Theorem 5.6. We say the polygon $\mathbf{P}_v$ is d-sharp if for any $\mathbf{P}_{i_*F}\neq \mathbf{P}_v$, one of the following is true:

(i) $\mathbf{P}_{i_*F}$ is contained in the polygon $\mathbf{P}_{0z_1'z_1^{+d}z_2'z_2}$.

(ii) z_1^{+j} is a vertex of $\mathbf{P}_{i_*F}$ for some $1\leq j\leq d-1$.

There is a simple numerical criterion for the d-sharpness of $\mathbf{P}_v$.

Lemma 6.2. *With the notations as before, suppose that*

$$\frac{s-r}{c}+\frac{\gamma^2 s-r}{\gamma c}\geq 2d, \tag{6.1}$$

where $\gamma=\frac{mr}{c}-1$, *the polygon* $\mathbf{P}_v$ *will be* d*-sharp.*

Proof. From the definition of two polygons, one observes that the interior of $\mathbf{P}_v-\mathbf{P}_{0z_1'z_1^{+d}z_2'z_2}$ only contains z_1^{+j} $(1\leq j\leq d-1)$ as integer points. If $\mathbf{P}_{0z_1'z_1^{+d}z_2'z_2}$ is convex, then either $\mathbf{P}_{i_*F}$ is contained in $\mathbf{P}_{0z_1'z_1^{+d}z_2'z_2}$ or z_1^{+j} is a vertex of $\mathbf{P}_{i_*F}$. A little writing reveals the convexity of this polygon literally means Equation (6.1). □

Surjectivity condition for special Mukai vectors

The following is an enhancement of Theorem 5.6 for special Mukai vectors.

Theorem 6.3. *Suppose* $g\geq 3$. *Let* $v=(g-1,k,k^2)\in \mathrm{H}^*_{\mathrm{alg}}(X)$ *be a primitive Mukai vector with* $\gcd(g-1,k)=1$. *Assume that* (m,k) *satisfies the conditions*

$$g<\min\left\{2k,2(mg-m-k)\right\},\ g\neq k \text{ and } g\neq m(g-1)-k, \tag{$\star\star^\dagger$}$$
$$\text{either } k\nmid g+1 \text{ or } m(g-1)-k\nmid g+1.$$

Then the restriction map $\psi:\mathbf{M}(v)\to \mathbf{BN}_C(v)$ *is surjective.*

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Proof. By Lemma 6.2, if there is $\mathbf{P}_{i_*F} \neq \mathbf{P}_v$ for some F, the polygon $\mathbf{P}_v$ will be at least 3-sharp. Therefore, one of the following is true:

(i) $\mathbf{P}_{i_*F}$ is contained in the polygon $\mathbf{P}_{0z_1'z_1^{+3}z_2'z_2}$.

(ii) $z_1^{+1} = g - k^2 + k\sqrt{-1}$ is a vertex of $\mathbf{P}_{i_*F}$

(iii) $z_1^{+2} = g + 1 - k^2 + k\sqrt{-1}$ is a vertex of $\mathbf{P}_{i_*F}$

We will analyse them by cases. Let us first show that case (i) is impossible if $(g,k,m) \neq (5,3,3)$. By Equation (5.18), it suffices to show that

$$\hbar - \|i_*F\| > \hbar + \chi - 2(g - 1 + k^2). \tag{6.2}$$

when $\mathbf{P}_{i_*F} \subseteq \mathbf{P}_{0z_1'z_1^{+3}z_2'z_2}$. Set $\tilde{k} = m(g-1) - k$. As in the proof of Theorem 5.6, from the convexity and a direct computation, one can get

$$\begin{aligned}
\hbar - \|i_*F\| &> \|z_1 - z_1'\| - \|z_1' - z_1^{+3}\| + \|z_1 - z_2'\| - \|z_2' - z_1^{+3}\| \\
&= \sqrt{(\frac{k^2-g+1}{k})^2 + 4g} - \sqrt{(\frac{k^2-g+1}{k} - 3)^2 + 4g} \\
&\quad + \sqrt{(\frac{\tilde{k}^2-g+1}{\tilde{k}})^2 + 4g} - \sqrt{(\frac{\tilde{k}^2-g+1}{\tilde{k}} - 3)^2 + 4g} \\
&\geq \frac{4k^2}{\sqrt{(g-1+k^2)^2 + 4k^2} + (g-1+k^2)} + \frac{4\tilde{k}^2}{\sqrt{(g-1+\tilde{k}^2)^2 + 4\tilde{k}^2} + (g-1+\tilde{k}^2)} \\
&= \hbar + \chi - 2(g-1+k^2)
\end{aligned} \tag{6.3}$$

whenever $(g,k,m) \notin \left\{(5,3,m),(6,4,3),(8,5,2)\right\}$ satisfies our assumption.

In the case $(g,k,m) = (6,4,3)$, $(8,5,2)$ or $(5,3,m)$ with $m \geq 4$, though the inequality (6.3) fails, one can give an improvement of the estimate (6.3) by considering the convex hull of integer points in $\mathbf{P}_{0z_1'z_1^{+3}z_2'z_2}$. In those cases, the convex hull is a convex polygon with vertices $z_1, z_1^{+3}, z_1', z_2'$ and z_3, where z_3 is given as below:

- $(g,k,m) = (5,3,m)$, $z_3 = -3 + 2\sqrt{-1}$;
- $(g,k,m) = (6,4,3)$, $z_3 = -8 + 3\sqrt{-1}$;
- $(g,k,m) = (8,5,2)$, $z_3 = -14 + 4\sqrt{-1}$.

Then one can get

$$\hbar - \|i_*F\| > \|z_1\| - \|z_3\| - \|z_3 - z_1^{+3}\| + \|z_1 - z_2'\| - \|z_2' - z_1^{+3}\|. \tag{6.4}$$

A computer calculation of their values shows that Equation (6.2) still holds.

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

In case (ii) and (iii), if z_1^{+1} or z_1^{+2} is a vertex of $\mathbf{P}_{i_*F}$, there exists $\widetilde{E}_j \subset i_*F$ in the HN-filtration (5.11) such that $\overline{Z}(\widetilde{E}_j) = z_1^{+1}$ (respectively, z_1^{+2}). Then we have

$$\begin{aligned} h^0(X, i_*F) &\leq \sum_i \left\lfloor \frac{l(E_i)+\chi(E_i)}{2} \right\rfloor \\ &\leq \sum_{i\leq j}\left(\left\lfloor \frac{l(E_i)+\chi(E_i)}{2} \right\rfloor - \frac{\chi(E_i)}{2}\right) + \sum_{i>j}\left(\left\lfloor \frac{l(E_i)+\chi(E_i)}{2} \right\rfloor - \frac{\chi(E_i)}{2}\right) + \frac{\chi}{2} \\ &\leq \frac{\hbar+\chi}{2}. \end{aligned}$$

For simplicity, we may use $\frac{\hbar_1}{2}$ and $\frac{\hbar_2}{2}$ to denote the first two terms in the second row. As $h^0(X, i_*F) \geq g-1+k^2$, following the argument in Theorem 5.6, it suffices to prove the inequality

$$\frac{\hbar+\chi}{2} - (g-1+k^2) < \frac{\hbar-\hbar_1-\hbar_2}{2},$$

or equivalently,

$$\hbar_1+\hbar_2 < 2(g-1+k^2) - \chi = \|z_1^{+1}\| + \|z_1^{+1} - z_2\| - 2. \tag{6.5}$$

For case ii), a direct computation shows

$$\begin{aligned} \|z_1^{+1}\| - \hbar_1 &= \sum_{i\leq j}\|E_i\| - \hbar_1 - \Big(\sum_{i\leq j}\|E_i\| - \|z_1^{+1}\|\Big) \\ &\geq \sum_{i\leq j}(\|E_i\| - \ell(E_i)) - \Big(\sum_{i\leq j}\|E_i\| - \|z_1^{+1}\|\Big) \\ &\geq (\|E_1\| - \ell(E_1)) - \Big(\sum_{i\leq j}\|E_i\| - \|z_1^{+1}\|\Big) \\ &\geq \frac{3}{\sqrt{k^2+4g-3}} - \Big(\sum_{i\leq j}\|E_i\| - \|z_1^{+1}\|\Big) & (6.6) \\ &\geq \frac{3}{\sqrt{k^2+4g-3}} - (\|z_1'\| + \|z_1' - z_1^{+1}\| - \|z_1^{+1}\|) & (6.7) \\ &> 0. & (6.8) \end{aligned}$$

Let us explain why the inequality (6.6) holds. Note that $\overline{Z}(E_1) = x + y\sqrt{-1}$ satisfies

$$k^2 - g \leq -\frac{x}{y} \leq k^2 - g + 1.$$

Then we have $-x < ky$ and $y \nmid x$ by our assumption $g-1 \neq k$, $g \neq k$, and $g < 2k$. This will give

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

$$\begin{aligned}\|E_1\| - \ell(E_1) &\geq \sqrt{x^2+4gy^2} - \sqrt{x^2+4(g-1)y^2+y^2} \quad (\text{since } y\nmid x)\\ &\geq \frac{3y^2}{2\sqrt{x^2+(4g-3)y^2}}\\ &\geq \frac{3y^2}{2\sqrt{k^2y^2+(4g-3)y^2}}\\ &> \frac{3}{\sqrt{k^2+4g-3}} \quad (\text{since } y\geq 2, \text{ which is a consequence of } y\nmid x).\end{aligned}$$

The inequality (6.8) holds because

$$\begin{aligned}\|z_1^{+1}-z_1'\|+\|z_1'\|-\|z_1^{+1}\| &= \sqrt{(\frac{k^2-(g-1)}{k}-1)^2+4g}+(k-1)\sqrt{(\frac{k^2-(g-1)}{k})^2+4g}-(k^2+g)\\ &= \sqrt{(\frac{k^2-(g-1)}{k}-1)^2+4g}-\left(k-1+\frac{g+1}{k}\right)\\ &\quad +(k-1)\sqrt{(\frac{k^2-(g-1)}{k})^2+4g}-\left((k^2+g)-(k-1+\frac{g+1}{k})\right)\\ &< \frac{2g(k-1)}{k^3-k^2+(g+1)k}+\frac{-2g(k-1)^2}{k^4-k^3+(g+1)k^2-(g+1)k}\\ &= \frac{2g(k-1)}{(k^2+g+1)(k^2-k+g+1)}\\ &\leq \frac{3}{\sqrt{k^2+4g-3}}\end{aligned}$$

when $k\geq \frac{g+1}{2}\geq 2$. Note that $\|z_1^{+1}\|-\hbar_1 = k^2+g-2\sum_{i\leq j}\left\lfloor \frac{l(E_i)+\chi(E_i)}{2}\right\rfloor - \sum_{i\leq j}\chi(E_i)$ is an even number, this yields $\|z_1^{+1}\|-\hbar_1\geq 2$.

Next, recall that $\widetilde{E}_l = i_*F$ in the HN filtration (5.11), we can get

$$\begin{aligned}\|z_1^{+1}-z_2\|-\hbar_2 &\geq \sum_{i>j}(\|E_i\|-\ell(E_i))-(\sum_{i>j}\|E_i\|-\|z_1^{+1}-z_2\|)\\ &\geq (\|E_l\|-\ell(E_l))-(\|z_1^{+1}-z_2'\|+\|z_2'-z_2\|-\|z_1^{+1}-z_2\|)\\ &> \frac{3}{\sqrt{\tilde{k}^2+4g-3}}-\frac{2g(\tilde{k}-1)}{(\tilde{k}^2+g+1)(\tilde{k}^2-\tilde{k}+g+1)}>0\end{aligned}$$

as $\tilde{k}>\frac{g+1}{2}$. Combining them together, we can obtain (6.5).

For case (iii), if $k\nmid g+1$, we have

$$\begin{aligned}\|z_1^{+1}\|-\hbar_1 &= \sum_{i\leq j}\|E_i\|-\hbar_1+\|z_1^{+1}\|-\sum_{i\leq j}\|E_i\|\\ &\geq \|E_1\|-\ell(E_1)+\|z_1^{+1}\|-\|z_1'\|-\|z_1'-z_1^{+2}\|\\ &\geq \frac{3}{\sqrt{k^2+4g-3}}+\|z_1^{+1}\|-\|z_1'\|-\|z_1'-z_1^{+2}\| \qquad (6.9)\\ &> 1. \qquad (6.10)\end{aligned}$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Here, the inequality (6.10) holds because

$$\begin{aligned}\|z_1^{+2}-z_1'\|+\|z_1'\|-\|z_1^{+1}\| &= \sqrt{(\frac{k^2-(g-1)}{k}-2)^2+4g}+(k-1)\sqrt{(\frac{k^2-(g-1)}{k})^2+4g}-(k^2+g)\\ &= \sqrt{(\frac{k^2-(g-1)}{k}-2)^2+4g}-(k-2+\frac{g+1}{k})\\ &\quad +(k-1)\sqrt{(\frac{k^2-(g-1)}{k})^2+4g}-(k^2-k+(g+1)-\frac{g+1}{k})-1\\ &\le \frac{2g(2k-1)}{k(g+(k-1)^2)}-\frac{2g(k-1)}{k(g+k^2+1)}-1\\ &= \frac{2g\left(k^2+2k+g-1\right)}{(k^2+g+1)(k^2-2k+g+1)}-1\\ &< \frac{3}{\sqrt{k^2+4g-3}}-1.\end{aligned}$$

Note that $\|z_1^{+1}\|-\hbar_1$ is an odd number, this yields $\|z_1^{+1}\|-\hbar_1\ge 3$. Similarly, one can get

$$\|z_1^{+1}-z_2\|-\hbar_2\ge 3$$

under the assumption $m(g-1)-k\nmid g+1$. Since both of them are at least positive under our assumption, we get Equation (6.5) as well. This finishes the proof for $(g,k,m)\neq(5,3,3)$.

For the remaining case $(g,k,m)=(5,3,3)$, we have to make use of the 4-sharpness of $\mathbf{P}_v$. We just need to verify $\mathbf{P}_{i_*F}$ is not contained in $\mathbf{P}_{0z_1'z_1^{+4}z_2'z_2}$ and $z_1^{+3}=-2+3\sqrt{-1}$ is not a vertex of $\mathbf{P}_{i_*F}$. As above, by using the convex hull of integer points in $\mathbf{P}_{0z_1'z_1^{+4}z_2'z_2}$, we have

$$\begin{aligned}\hbar-\|i_*F\| \ge& \|z_1\|-\|-3+2\sqrt{-1}\|-\|-3+2\sqrt{-1}-z_1^{+4}\|+\|z_1-z_2'\|-\|z_2'-z_1^{+4}\|\\ &= -2\sqrt{6}-\sqrt{89}+\sqrt{205}+\frac{\sqrt{7549}}{9}-\frac{\sqrt{3301}}{9}\\ &> \hbar+\chi-2(g-1+k^2)\end{aligned}$$

which show that $\mathbf{P}_{i_*F}$ cannot lie in $\mathbf{P}_{0z_1'z_1^{+4}z_2'z_2}$. Moreover, a similar estimate of $\|E_1\|-\ell(E_1)$ and $\|E_l\|-\ell(E_l)$ in (ii) and (iii) shows that z_1^{+3} is not a vertex of $\mathbf{P}_{i_*F}$. □

Remark 6.4. One can also directly check the small genera cases by running the computer program in [13, Section 4] .

7. Surjectivity of the tangent map

In this section, we adapt Feyzbakhsh's approach to study the surjectivity of the tangent map and obtain a sufficient condition for ψ being an isomorphism.

Theorem 7.1. *Let $v=(r,c,s)\in \mathrm{H}^*_{alg}(X)$ be a Mukai vector satisfying the injectivity condition (✶). The morphism*

$$\psi:\mathbf{M}(v)\to \mathbf{BN}_C(v)$$

is an isomorphism whenever the following conditions hold

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

(i) *ψ is surjective;*

(ii) *$h^0(X,E) = r+s$ for any $E \in \mathbf{M}(v)$;*

(iii) *there exists $\sigma \in \mathrm{L}_{(\pi_{v(-m)},\pi_{v_K})} \cap V(X)$ such that*

$$\Omega^+_{v(-m)}(\mathrm{L}_{(o,\sigma]}) \cap \mathrm{H}^*_{alg}(X) = \Omega^+_{v_K}(\mathrm{L}_{(o,\sigma]}) \cap \mathrm{H}^*_{alg}(X) = \emptyset, \tag{7.1}$$

where $v_K = (s,-c,r)$;

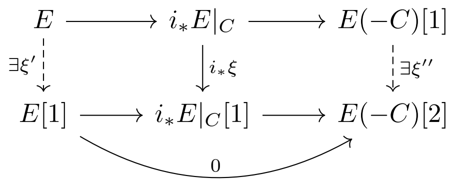
(iv) *$2s > v^2 + 2c^2$, or $2s > v^2 + 2$ and $\gcd(c,s) = 1$.*

Proof. As ψ is bijective, it suffices to show the tangent map $\mathrm{d}\psi$ is surjective. The argument is similar as [11, §6]. For the convenience of readers, we sketch the proof as below. For any $E \in \mathbf{M}(v)$, the differential map $\mathrm{d}\psi : T_{[E]}\mathbf{M}(v) \to T_{[E|_C]}\mathbf{BN}_C(v)$ at $[E]$ can be identified as the map

$$\mathrm{d}\psi : \mathrm{Hom}(E,E[1]) \to \ker\left(\mathrm{Hom}_C(E|_C, E|_C[1]) \xrightarrow{\mathrm{H}^0} \mathrm{Hom}(\mathrm{H}^0(C,E|_C), \mathrm{H}^1(C,E|_C))\right)$$

sending $(E \to E[1])$ to $(E|_C \to E|_C[1])$.

Let $\xi\colon E|_C \to E|_C[1]$ be a tangent vector in $T_{[E|_C]}\mathbf{BN}_C(v)$. Then Feyzbakhsh has shown in [11, §6] that there exist morphisms ξ' and ξ'' such that the following commutative diagram holds



provided that

$$K_E = M[1] \quad \text{and} \quad \mathrm{Hom}_X(M, E(-C)[1]) = 0, \tag{7.2}$$

where K_E is the cone of the evaluation map $\mathcal{O}_X^{h^0(X,E)} \to E \to K_E$ in $\mathrm{D}^{\mathrm{b}}(X)$. Note that $\mathrm{d}\psi(\xi') = \xi$, we are therefore reduced to check (7.2) holds for every E.

Note that $v(K_E) = -v_K$ and $\pi_{v_K} = \pi_{K_E}$. We can choose the stability condition $\sigma_1 \in \mathrm{L}_{(\pi_{v_K},o')}$ sufficiently close to o' and $\sigma_2 \in \mathrm{L}_{(\pi_K,o)}$ sufficiently close to o so that

$$\mathbf{P}_{o\sigma_2\sigma_1 o'} \setminus \{o,o'\} \subseteq \mathbf{P}_{o\pi_{v_K}\infty} \cap \Gamma \subseteq V(X);$$

see Figure 9. As in the proof of Theorem 4.1, we have $\mathcal{O}_X$ and E are σ_1-semistable of the same phase. Then as the quotient of E by $\mathcal{O}_X^{h^0(X,E)}$, K_E is also σ_1-semistable of the same σ_1-phase. Note that Lemma 4.2 still holds if we exchange r and s in Mukai vector v. Then we get

$$\Omega^+_{v_K}(\mathbf{P}_{o\pi_{v_K}\infty} \cap \Gamma) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset.$$

Since v_K is primitive, we have $\Omega_{v_K}(\mathbf{P}_{o\pi_{v_K}\infty} \cap \Gamma) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset$. By Proposition 3.4, v_K admits no strictly semistable stability conditions in $\mathbf{P}_{o\sigma_2\sigma_1 o'} \setminus \{o,o'\}$. Therefore, K_E is

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

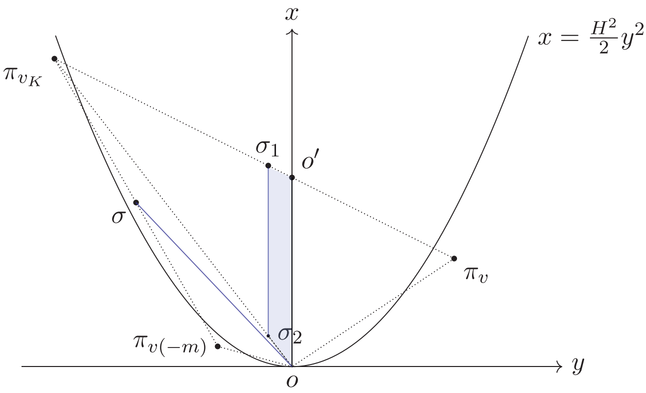


Figure 9. Any point in the colored region is a stability condition.

stable for any $\tau \in \mathbf{P}_{o\sigma_2\sigma_1 o'} \setminus \{o,o'\}$. This implies that K_E is $\sigma_{\alpha,0}$-stable for $\alpha > \sqrt{\frac{2}{H^2}}$. By [15, Lemma 6.18], we have $\mathrm{H}^{-1}(K_E)$ is a μ_H-semistable torsion-free sheaf and $\mathrm{H}^0(K_E)$ is a torsion sheaf supported in dimension zero. So we can set $v(\mathrm{H}^0(K_E)) = (0,0,a)$ and $v(\mathrm{H}^{-1}(K_E)) = (s,-c,r+a)$ for some $a \geq 0$. By [11, Lemma 3.1], we have

$$-2c^2 \leq v(\mathrm{H}^{-1}(K_E))^2 = v^2 - 2sa. \tag{7.3}$$

When $\gcd(c,s) = 1$, we have $\mathrm{H}^{-1}(K_E)$ is slope stable and $v(\mathrm{H}^{-1}(K_E))^2 \geq -2$. Then by condition (iv), we have $a = 0$ and $\mathrm{H}^0(K_E) = 0$. So we obtain $K_E = M[1]$, where $M = \mathrm{H}^{-1}(K_E)$ is a μ_H-semistable torsion-free sheaf.

Since $\Omega^+_{v_K}(\mathrm{L}_{(o,\sigma]}) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset$, $v(M)$ admits no strictly semistable condition in $\mathrm{L}_{(o,\sigma]}$. It follows that M is σ-stable as it is σ_2-stable. Similarly, we have $E(-C)[1]$ is also σ-stable. Since M and $E(-C)[1]$ are σ-stable of the same phase, one must have $\mathrm{Hom}_X(M,E(-C)[1]) = 0$. This proves the assertion. ☐

Conditions for reconstructing K3 surfaces

As a first application, we obtain a numerical criterion for Mukai's program of reconstructing K3 surfaces, that is, the case $v^2 = 0$.

Theorem 7.2. *Assume $g > 2$. Let $v = (r,c,ck) \in \mathrm{H}^*_{alg}(X)$ be a primitive Mukai vector with $v^2 = 0$. Suppose it satisfies the condition*

$$r \mid g-1,\ k \nmid g,\ 0 < k \leq 3g-3 \text{ and } m > 1 + \frac{ck}{r(k-1)}. \tag{$\star\star\star$}$$

The restriction map $\psi : \mathbf{M}(v) \to \mathbf{BN}_C(v)$ is an isomorphism if it is a surjective morphism.

Proof. Let us check that the conditions (ii)–(iv) in Theorem 7.1 are satisfied. By our assumption, we know that $\gcd(r-s,c) = 1$. According to [13, Lemma 3.1], one has

$$h^0(X,E) \leq r+s,$$

which forces $h^0(X,E) = r+s$ by Equation (4.2). This verifies the condition (ii).

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

For the condition (iv), note that we have $s=\frac{c^2(g-1)}{r}\geq c^2$ where the equality holds when $r=g-1$. If $r=g-1$, the inequality in Equation (7.3) will be equality. By [11, lemma 3.1], we have $c\mid(g-1)$ which is a contradiction. Thus, we only need to verify the condition (iii). By Remark 3.5, it suffices to show

$$\mathbf{P}_{o\pi_{v_K}\pi_{v(-m)}}\setminus\{\text{vertices}\}\subseteq V(X).$$

To make the computation easier, we may consider the action of tensoring the invertible sheaf $\mathcal{O}_X(H)$ which sends the triangle $\mathbf{P}_{o\pi_{v_K}\pi_{v(-m)}}$ to the triangle $\mathbf{P}_{op_1p_2}$, where $p_1=\pi_{v_K(1)}$ and $p_2=\pi_{v(1-m)}$. Then it is equivalent to show there are no projection of roots in $\mathbf{P}_{op_1p_2}-\{\text{vertices}\}$.

Firstly, we show that there is no projection of root on the two edges joining o. By definition, we have

$$p_1=(\frac{kc}{(k-1)^2r},\frac{c}{(k-1)r})\text{ and }p_2=(\frac{cr}{k((m-1)r-c)^2},\frac{c}{k(c-(m-1)r)}).$$

Then two open edges $\mathrm{L}_{(o,p_1)}$ and $\mathrm{L}_{(o,p_2)}$ do not contain any projection of roots by Observation (B).

Next, since $(m-1)r>\frac{ck}{(k-1)}>c$, we know that p_1 is lying in the first quadrant while projection p_2 is lying in the second quadrant. So the region

$$\mathbf{P}_{op_1p_2}\setminus(\mathrm{L}_{[o,p_1]}\cup\mathrm{L}_{[o,p_2]})$$

is contained in the union of two trapezoidal regions $\mathbf{P}^\circ_{op_1\infty}$, $\mathbf{P}^\circ_{op_2\infty}$ and the x-axis. As $r|(g-1)$ and $\gcd(r,c)=1$, we have an inclusion

$$\mathbf{P}^\circ_{op_2\infty}\subseteq V(X)$$

from Lemma 4.2 (i). Moreover, if there is a root $\delta=(r',c',s')\in\mathrm{R}(X)$ with $r'>0$ whose projection π_δ is lying in $\mathbf{P}^\circ_{op_1\infty}$, one can follow the computation in Lemma 4.2 to get inequalities

$$kc'<(k-1)r'<kc'+\frac{k}{(g-1)c'}. \tag{7.4}$$

However, one can directly check that there are no such integers (r',c',s') satisfying Equation (7.4) under the assumption $k\leq g-1$ or $3<k\leq 3g-3$.

It remains to show that $\mathbf{P}_{op_1p_2}\cap y\text{-axis}\subseteq V(X)$. Note that

$$\mathrm{L}_{[p_1,p_2]}\cap x\text{-axis}=(\frac{c/(k-1)}{(m-1)r-c},0)$$

which is below o'. It follows that $\mathbf{P}_{op_1p_2}\cap y\text{-axis}\subseteq\mathrm{L}_{(o,o')}\subseteq V(X)$. □

Conditions for reconstructing hyper-Kähler

Now, we reconstruct hyper-Kähler varieties as Brill–Noether locus for Mukai vectors given in Corollary 5.7.

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Theorem 7.3. *Assume $v \in \mathrm{H}^*_{alg}(X)$ satisfying conditions (✶) and (✶✶). The restriction map $\psi : \mathbf{M}(v) \to \mathbf{BN}_C(v)$ is an isomorphism.*

Proof. By Corollary 5.7, ψ is a bijective morphism. We only need to verify the conditions (ii)–(iv) in Theorem 7.1. We will check them one by one.

(1) Let us first verify that $h^0(X,E) = r+s$ for any $E \in \mathbf{M}(v)$. By [13, Proposition 3.1], it suffices to show that

$$\frac{\sqrt{(r-s)^2+(2g+2)c^2}}{2} < \frac{r+s}{2}+1. \tag{7.5}$$

After simplification, one can find that Equation (7.5) is equivalent to

$$\frac{(g+1)c^2-1}{2r+2}-1 < s.$$

This holds when $(g-1)c^2 - rs < r$ and $g > 4r^2+1$.

(2) For condition (ii), we claim that

$$\mathrm{L}_{(\pi_{v(-m)},\pi_{v_K})} \cap \Gamma \neq \emptyset,$$

and hence $\mathrm{L}_{(\pi_{v(-m)},\pi_{v_K})} \cap V(X) \neq \emptyset$. Let us write $v(-m) = (r,\tilde{c},\tilde{s})$ and $v_K = (s,-c,r)$ with $\tilde{c} = c - mr$ and $\tilde{s} = \lfloor \frac{(g-1)\tilde{c}^2+1}{r} \rfloor$. Then we only need to show that the quadratic equation

$$g((1-t)t\tilde{c}+t(-c))^2 = ((1-t)r+ts)((1-t)\tilde{s}+tr) \tag{7.6}$$

has roots for $0 < t < 1$. By calculating the discriminant of (7.6), we know it has a solution t_0 satisfying

$$0 < t_0 < \frac{\tilde{c}^2g - r\tilde{s}}{s\tilde{s}+r^2+2\tilde{c}cg+2(\tilde{c}^2g-r\tilde{s})} < 1. \tag{7.7}$$

(3) Choose $\sigma \in \mathrm{L}_{(\pi_{v(-m)},\pi_{v_K})} \cap \Gamma$. We first verify that

$$\Omega^+_{v(-m)}(\mathrm{L}_{(o,\sigma]}) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset.$$

Suppose there is an integer point $p_0 = (x_0,y_0,z_0) \in \Omega^+_{v(-m)}(\mathrm{L}_{[\sigma,o)})$. By Lemma 7.4 below, we have

$$\tilde{c}-1 < y_0 < 0.$$

Moreover, one may observe that p_0 is lying in a (closed) planer region enclosed by the conic

$$Q = \left\{y = y_0,\, (g-1)y^2+1 = xz\right\}$$

and two lines

$$L_1 = \left\{y = y_0,\, z = \frac{y_0\tilde{s}}{\tilde{c}}\right\};\ L_2 = \left\{(1-t)\frac{y_0}{\tilde{c}}(r,\tilde{c},\tilde{s}) - \frac{ty_0}{c}(s,-c,r), t \in \mathbb{R}\right\}.$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

It has three vertices given by the intersection points $L_1 \cap L_2$, $L_1 \cap Q$ and $L_2 \cap Q$. This yields

$$\frac{y_0 r}{\tilde{c}} \le x_0 \le (1-t')\frac{y_0 r}{\tilde{c}} - \frac{t' y_0 s}{c}, \tag{7.8}$$

where t' is the smaller root of the quadratic equation

$$(g-1)y_0^2 + 1 = [(1-t)\frac{y_0 r}{\tilde{c}} - \frac{t y_0 s}{c}][(1-t)\frac{y_0 \tilde{s}}{\tilde{c}} - \frac{t y_0 r}{c}].$$

Solving the equation, one can get

$$t' \le 2\frac{(g-1)y_0^2 + 1 - \frac{r\tilde{s}y_0^2}{\tilde{c}^2}}{-\frac{\tilde{c}\tilde{s}s}{c} - 2r\tilde{s}} \le \frac{2c(y_0^2 r + \tilde{c}^2)}{(-\tilde{c}\tilde{s}s - 2rc\tilde{s})\tilde{c}^2} \tag{7.9}$$

as $(g-1)\tilde{c}^2 - r\tilde{s} < r$. Plugging Equation (7.9) into (7.8), we get

$$\begin{aligned} 0 < x_0 - \frac{y_0 r}{\tilde{c}} &\le \frac{2cy_0(y_0^2 r + \tilde{c}^2)}{(\tilde{c}\tilde{s}s + 2rc\tilde{s})\tilde{c}^2}(\frac{s\tilde{c} + rc}{c\tilde{c}}) \\ &= (\frac{y_0}{\tilde{c}})(\frac{y_0^2 r + \tilde{c}^2}{\tilde{s}\tilde{c}^2})(\frac{s\tilde{c} + rc}{s\tilde{c} + 2rc}) \\ &\le \frac{3(r+1)}{\tilde{s}} \\ &< \frac{3r(r+1)}{(g-1)\tilde{c}^2 - r} \\ &< -\frac{1}{\tilde{c}}, \end{aligned}$$

where the last inequality holds because $g - 1 \ge 4r^2$. This contradicts to $x_0, y_0 \in \mathbb{Z}$. A similar computation shows that $\Omega^+_{v_K}(\mathrm{L}_{(o,\sigma]}) \cap \mathrm{H}^*_{\mathrm{alg}}(X) = \emptyset$ as well.

(4) Condition (iv) holds since our assumption $g > 4r^2 + 1$ ensures that

$$2s > 2r - 2 + 2c^2 > v^2 + 2c^2.$$

□

Lemma 7.4. *For any integer point* $(x_0,\ y_0,\ z_0) \in \Omega^+_{v(-m)}(\mathrm{L}_{(o,\sigma]})$ *in Theorem 7.3, we have*

$$\tilde{c} - 1 < y_0 < 0.$$

Proof. Set $v_t = (1-t)v(-m) + tv_K$. Let $0 < t_0 < t_1 < 1$ be the roots of Equation (7.6). We set

$$w = (x', y', z') = \mathrm{L}_{O, v_{t_0}} \cap \mathrm{L}_{v(-m), v(-m) + v_{t_1}}.$$

Then $\Omega^+_{v(-m)}(\mathrm{L}_{(o,\sigma]})$ is contained in the tetrahedron $\mathbf{T}_{Ov(-m)\varpi w}$ with four vertices $O, v(-m), w$ and $\varpi = (r, \tilde{c}, \frac{g\tilde{c}^2}{r})$. This gives $y' < y_0 < 0$. Hence, we only need to estimate the lower bound of y'.

Set $v_{t_0} = (r_{t_0}, c_{t_0}, s_{t_0})$, then we have $w = \frac{y'}{c_{t_0}} v_{t_0} \in \mathrm{L}_{o, v_{t_0}}$. Note that $w - v(-m) \in \mathrm{L}_{O,\, v_{t_1}}$ is lying on the hyperboloid

$$\left\{(x,y,z) \in \mathbb{R}^3 |\ gy^2 - xz = 0\right\}.$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

Then one can see that $y' < \tilde{c} - 1$ if

$$\frac{\tilde{c}-1}{c_{t_0}} v_{t_0} - v(-m) \in \Big\{ (x,y,z) \in \mathbb{R}^3 | \; gy^2 - xz < 0 \Big\}, \tag{7.10}$$

that is,

$$\Big(\frac{\tilde{c}-1}{c_{t_0}} r_{t_0} - r\Big)\Big(\frac{\tilde{c}-1}{c_{t_0}} s_{t_0} - \tilde{s}\Big) > g. \tag{7.11}$$

Plugging the coordinates of v_{t_0} into Equation (7.11) and simplify all the terms, one can obtain a quadratic inequality of t_0 and one can easily see that Equation (7.11) holds if

$$t_0 < \frac{(2-\tilde{c})(\tilde{c}^2 g - r\tilde{s})}{\tilde{c}[r\tilde{s} - r^2 - s\tilde{s} + (2-\tilde{c})g(c+\tilde{c})] + r^2 + s\tilde{s} - (c+2)r\tilde{s}}.$$

Using the upper bound of t_0 given in Equation (7.7), we are reduced to check

$$\tilde{c}(r\tilde{s} - r^2 - s\tilde{s} + (2-\tilde{c})g(c+\tilde{c})) + r^2 + s\tilde{s} - (c+2)r\tilde{s} < (2-\tilde{c})(s\tilde{s} + r^2 - 2r\tilde{s} + 2\tilde{c}g(\tilde{c}+c)).$$

After further simplification and reduction, the inequality above becomes

$$0 < s\tilde{s} + r^2 + 2\tilde{c}g + c - (\tilde{c}+c-2)(\tilde{c}^2 g - r\tilde{s}). \tag{7.12}$$

The right-hand side can be estimated as below

$$\begin{aligned} \mathrm{RHS} &> r^2 + s\tilde{s} + 2\tilde{c}g - (\tilde{c}+c-2)(\tilde{c}^2 g - r\tilde{s}) \\ &= r^2 + s\tilde{s} + 2\tilde{c}g + (mr + 2 - 2c)(\tilde{c}^2 + \frac{v^2}{2}) \\ &> r^2 + s\tilde{s} + 2\tilde{c}g - r(\tilde{c}^2 + r) \\ &\geq \Big(\frac{(g-1)c^2}{r} - 1\Big)\Big(\frac{(g-1)\tilde{c}^2}{r} - 1\Big) + 2\tilde{c}g - r\tilde{c}^2 \\ &> 0. \end{aligned} \tag{7.13}$$

□

8. Proof of the main theorems

We now prove our main theorems by finding suitable Mukai vectors $v \in \mathrm{H}^*_{\mathrm{alg}}(X)$ satisfying the conditions in Theorem 7.2 and Theorem 7.3, respectively.

Proof of Theorem 1.1

As the case of $m = 1$ is already known, we may always assume $m > 1$. There will be two cases:

(i) If $(g,m) \neq (7,2)$, we can choose the Mukai vector $v = (g-1, k, k^2)$ with k given in the Table 1.

Note that when $g \geq 8$, we have

$$k = \min\Big\{ k_0 \mid k_0 > \frac{g}{2}, \gcd(g-1, k_0) = 1 \Big\} < g-2.$$

By a direct computation, one can easily see that the values of k and m in Table 1 satisfy the special surjectivity condition $(\ast\ast^{\dagger})$ given in Theorem 6.3. This

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

TABLE 1. Choices of Mukai vectors.

Values of g	Values of k	Range of m
3	$k=5$	$m\geq 5$
4	$k=5$	$m\geq 4$
5	$k=3$	$m\geq 3$
6	$k=4$	$m\geq 3$
7	$k=5$	$m\geq 3$
≥ 8	$k=\min\{k_0 \mid k_0>\frac{g}{2}, \gcd(g-1,k_0)=1\}$	$m\geq 2$

ensure the restriction map $\psi:\mathbf{M}(v)\to\mathbf{BN}_C(v)$ is surjective. The assertion follows from Theorem 7.2 as it satisfies the condition (✶✶✶). Indeed, the only nontrivial condition one needs to check is

$$m>1+\frac{k^2}{(g-1)(k-1)}.$$

(ii) If $(g,m)=(7,2)$, Theorem 6.3 cannot be applied because primitive Mukai vectors of the form $(6,k,k^2)$ do not satisfy the assumptions in Theorem 6.3. However, we can choose $v=(2,1,3)$, and the assertion can be concluded by the following result.

Proposition 8.1. *Suppose $g=7$. The restriction map $\psi:\mathbf{M}(2,1,3)\to\mathbf{BN}_C(2,1,3)$ is an isomorphism for any irreducible curve $C\in|2H|$.*

Proof. Note that v satisfies the injectivity condition (✶), ψ is an injective morphism with stable image. It also satisfies the condition (✶✶✶). Due to Theorem 7.2, it suffices to show that ψ is surjective. The idea is to use Theorem 5.3. Suppose one has

$$\mathbf{P}_v\neq\mathbf{P}_{i_*F}$$

for some $F\in\mathbf{BN}_C(v)$. A direct computation shows $\mathbf{P}_v$ is at least 2-sharp. Then either $\mathbf{P}_{i_*F}$ lies inside the polygon $\mathbf{P}_{0z_1^{+2}z_2'z_2}$, or it has z_1^{+1} as a vertex. For the first case, one has

$$\begin{aligned}\hbar-\|i_*F\| &> \|z_1-z_1'\|-\|z_1'-z_1^{+2}\|+\|z_1-z_2'\|-\|z_2'-z_1^{+2}\|\\ &=\frac{\sqrt{877}}{3}-\frac{\sqrt{613}}{3}\\ &>\sqrt{29}+\sqrt{877}-34=\hbar+\chi-10\end{aligned}$$

which contradicts Equation (5.18). For the second case, it forces $\overline{Z}(\widetilde{E}_1)=z_1^{+1}$ and hence

$$5\leq h^0(C,F)=h^0(X,i_*F)\leq\lfloor\frac{\ell(z_1^{+1})}{2}\rfloor+\frac{\|z_1^{+1}-z_2'\|+\|z_2'-z_2\|+\chi}{2}\leq\frac{\hbar+\chi}{2}.$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

However, we have

$$\begin{aligned}\frac{\hbar-(\|z_1^{+1}-z_2'\|+\|z_2'-z_2\|)}{2}-\lfloor\frac{\ell(z_1^{+1})}{2}\rfloor &= \frac{1}{6}\left(\sqrt{877}-4\sqrt{46}\right)+\frac{\sqrt{7}}{2}-1\\ &> \frac{1}{2}\left(\sqrt{29}+\sqrt{877}-34\right)\\ &= \frac{\hbar+\chi}{2}-5\end{aligned}$$

which is impossible. It follows from Theorem 5.3 that ψ is surjective. □

Proof of Theorem 1.2

By Corollary 5.7, for each $n>0$, we need to find a positive Mukai vectors $v=(r,c,s)$ with $v^2=c^2(2g-2)-2rs=2n$ satisfying conditions (✶) and (✶✶), that is,

$$g\geq 4r^2+1,\ r>\max\left\{n+1,\frac{c}{m}\right\},\ s>\frac{rc}{mr-c}\ \text{and}\ \gcd(r,c)=1.$$

A key tool is

Lemma 8.2. *For each n, there is an integer $N=N(n)$ such that for $g>N$, one can find a prime number p satisfying that*

(i) $n+1<p<\frac{\sqrt{g-1}}{2}$ *and* $\gcd(p,8(g-1)n)=1$,
(ii) *the equation* $x^2\equiv(g-1)n \bmod p$ *has a solution.*

Proof. The idea is to use the bound for prime character nonresidues. In [19, Theorem 1.3], it has been proved that there exists an integer m_0 with the property: if $j>j_0$ and χ is a quadratic character modulo j, there are at least $\log(j)$ primes $\ell\leq\sqrt[3]{j}$ with $\chi(\ell)=1$. Choose N to be the minimal integer satisfying

- $8(N-1)n\geq j_0$,
- the $\lfloor\log(8(N-1)n)\rfloor$-th prime number $>n+1$,
- $\sqrt[3]{8(N-1)n}\leq\frac{\sqrt{N-1}}{2}$.

Clearly, N only depends on n. For $g>N$, we write

$$(g-1)n=a^2\prod_{i=1}^{k}q_i,$$

where q_i are distinct primes. Let χ_i be the character defined by

$$\chi_i(d)=\left(\frac{d}{q_i}\right)(-1)^{\frac{(d-1)(q_i-1)}{4}}$$

if q_i is odd and $\chi_i(d)=(-1)^{\frac{d^2-1}{8}}$ if $q_i=2$. Consider the quadratic character

$$\chi(d)=\prod_{i=1}^{k}\chi_i(d)$$

https://doi.org/10.1017/S1474748024000495 Published online by Cambridge University Press

modulo $8(g-1)n$. As $8(g-1)n > N \geq j_0$, there exists a prime p such that $\chi(p) = 1$ and

$$n+1 < p < \sqrt[3]{8(g-1)n} \leq \frac{\sqrt{g-1}}{2}.$$

Moreover, one can compute the Jacobi symbol

$$\left(\frac{n(g-1)}{p}\right) = \prod_{i=1}^{k} \left(\frac{q_i}{p}\right) = \prod_{i=1}^{k} \chi_i(p) = \chi(p) = 1$$

by the law of reciprocity. It follows that $x^2 = (g-1)n \bmod p$ has a solution. $\square$

Due to Lemma 8.2, when $g > N(n)$, we can find an odd prime p and an integer $0 < c < p$ satisfying

$$n+1 < p < \frac{\sqrt{g-1}}{2} \text{ and } p \text{ divides } c^2(g-1)-n.$$

Here, $c(g-1)$ is actually a solution of the equation of $x^2 \equiv (g-1)n \bmod p$. Choose the Mukai vector $v = (p, c, \frac{c^2(g-1)-n}{p})$, then we have

$$p > \max\left\{n+1, c\right\},\ g \geq 4p^2+1,$$

and

$$\frac{c^2(g-1)-n}{p} > \frac{4c^2p^2-p}{p} = 4c^2p-1 > pc,$$

by Lemma 8.2 (i). The assertion then follows immediately.

Acknowledgements. The authors want to thank Yifeng Liu for very helpful discussions. The authors were supported by NSFC Grant (No. 12121001, 12171090 and 12425105) and Shanghai Pilot Program for Basic Research (No. 21TQ00). The first author was also supported by NKRD Program of China (No. 2020YFA0713200) and LNMS. This project was initiated from the workshop 'Curves on K3 surfaces' in June 2021 supported by NSFC General Program (No. 12171090).

References

[1] Arbarello E, Bruno A and Sernesi E (2014) Mukai's program for curves on a K3 surface. *Algebr. Geom.* **1**(5), 532–557.

[2] Bayer A (2018) Wall-crossing implies Brill–Noether: Applications of stability conditions on surfaces. In *Algebraic Geometry: Salt Lake City 2015*, vol. 97.1 Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 3–27.

[3] Bridgeland T (2007) Stability conditions on triangulated categories. *Ann. of Math. (2)* **166** (2), 317–345.

[4] Bridgeland T (2008) Stability conditions on $K3$ surfaces. *Duke Math. J.* **141**(2), 241–291.

[5] Ciliberto C and Dedieu T (2022) $K3$ curves with index $k > 1$. *Boll. Unione Mat. Ital.* **15**(1-2), 87–115.

[6] Ciliberto C, Dedieu T and Sernesi E (2020) Wahl maps and extensions of canonical curves and $K3$ surfaces. *J. Reine Angew. Math.* **761**, 219–245.

[7] Ciliberto C, Flamini F, Galati C and Knutsen AL (2017) Moduli of nodal curves on $K3$ surfaces. *Adv. Math.* **309**, 624–654.

[8] Ciliberto C, Lopez A and Miranda R (1993) Projective degenerations of $K3$ surfaces, Gaussian maps, and Fano threefolds. *Invent. Math.* **114**(3), 641–667.

[9] Ciliberto C, Lopez AF and Miranda R (1998) Classification of varieties with canonical curve section via Gaussian maps on canonical curves. *Amer. J. Math.* **120**(1), 1–21.

[10] Dutta Y and Huybrechts D (2022) Maximal variation of curves on K3 surfaces. *Tunis. J. Math.* **4**(3), 443–464.

[11] Feyzbakhsh S (2020) Mukai's program (reconstructing a K3 surface from a curve) via wall-crossing. *J. Reine Angew. Math.* **765**, 101–137.

[12] Feyzbakhsh S (2022) Hyperkähler varieties as Brill–Noether loci on curves. arXiv: 2205.00681.

[13] Feyzbakhsh S (to appear) Mukai's program (reconstructing a K3 surface from a curve) via wall-crossing, II. *Pure Appl. Math. Q.* 22020.

[14] Kemeny M (2015) The moduli of singular curves on K3 surfaces. *J. Math. Pures Appl. (9)* **104**(5), 882–920.

[15] Macrì E and Schmidt B (2017) Lectures on Bridgeland stability. In *Moduli of Curves*, vol. 21, Lect. Notes Unione Mat. Ital. Cham: Springer, 139–211.

[16] Mukai S (1988) Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10. In *Algebraic Geometry and Commutative Algebra*, vol. I. Tokyo: Kinokuniya, 357–377.

[17] Mukai S (1996) Curves and K3 surfaces of genus eleven. *Lecture Notes in Pure and Applied Mathematics*, 189–198.

[18] Mukai S (2001) Non-abelian Brill–Noether theory and Fano 3-folds [translation of Sūgaku 49 (1997), no. 1, 1–24; MR1478148 (99b:14012)], Sugaku Expositions, vol. 14, 125–153.

[19] Pollack P (2017) Bounds for the first several prime character nonresidues. *Proc. Amer. Math. Soc.* **145**(7), 2815–2826.

[20] Yoshioka K (1999) Irreducibility of moduli spaces of vector bundles on K3 surfaces. arXiv:9907001.

[21] Yoshioka K (2001) Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.* **321**(4), 817–884.