J. Appl. Prob. Spec. Vol. **48A,** 51–63 (2011)

NEW FRONTIERS IN APPLIED PROBABILITY

A Festschrift for SØREN ASMUSSEN
Edited by P. GLYNN, T. MIKOSCH and T. ROLSKI

Part 1. Risk theory

EXIT TIMES FOR A CLASS OF RANDOM WALKS: EXACT DISTRIBUTION RESULTS

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APPLIED PROBABILITY TRUST
AUGUST 2011

https://doi.org/10.1239/jap/1318940455 Published online by Cambridge University Press

EXIT TIMES FOR A CLASS OF RANDOM WALKS: EXACT DISTRIBUTION RESULTS

BY MARTIN JACOBSEN

Abstract

For a random walk with both downward and upward jumps (increments), the joint distribution of the exit time across a given level and the undershoot or overshoot at crossing is determined through its generating function, when assuming that the distribution of the jump in the direction making the exit possible has a Laplace transform which is a rational function. The expected exit time is also determined and the paper concludes with exact distribution results concerning exits from bounded intervals. The proofs use simple martingale techniques together with some classical expansions of polynomials and Rouché's theorem from complex function theory.

Keywords: One-sided exit; mean exit time; two-sided exit; partial eigenfunction; overshoot

2010 Mathematics Subject Classification: Primary 60G50; 60G40
Secondary 60G42; 60J05

1. Introduction

The object of study in this paper is nonlattice, one-dimensional random walks $(X_n)_{n\geq 0}$ in discrete time and their exits from an interval. The main result characterises the joint distribution of the exit time and the overshoot at exit in terms of a suitable generating function $\mathcal{G}$. For any given values of the arguments of $\mathcal{G}$, the results are also given as a function of the initial state $X_0 \equiv x$ of the random walk, and the emphasis is on models where this dependence on x is particularly simple, viz. a finite linear combination of exponentials $x \mapsto \mathrm{e}^{\gamma x}$. This requires that the distribution of the increments $X_n - X_{n-1}$ in the direction of the exit has a special form: below it is assumed that either the Laplace transform is a rational function (class $\mathcal{R}$) or the density is a *linear* combination (class $\mathcal{LE}$) of exponentials. The finitely many γ-values are found as solutions to a Cramér–Lundberg equation, while the coefficients for the terms $\mathrm{e}^{\gamma x}$ are either determined explicitly or found as solutions to a system of linear equations.

The method used is that developed in [5], [6], [7], and [8] for much more complicated continuous-time models. A main purpose here is to show that the method applies also in discrete time and that, for random walks, it is possible to obtain explicit formulae that appear to be simpler than those found elsewhere, and may well in part be new—certainly they are not easy to locate in the literature.

Much of the existing literature on random walks deals with asymptotic results: a major reference is [2]. As for exact results concerning exit problems, early contributions can be found in [11, Sections 9, 10, 11], [12], and [17, Sections 17, 21, 24, 25] (only walks on integer lattices are treated in [17]). There is a host of contributions by Soviet/Russian/Ukrainian authors, classical (Rogozin, Pechenskii, A. A. Borovkov, Zolotarev, and others) as well as new. Some recent references, notably on two-sided exit problems, include [3], [4], [9], and [10] with results quite different in form from those presented here.

It is natural to compare random walks in discrete time with Lévy processes in continuous time, where there is a huge literature on exit problems. A much favoured case for obtaining simple explicit results is that of spectrally one-sided Lévy processes (one-sided jumps), where at exit the process will hit the exit level precisely (exit by creeping). For nonlattice random walks, there is no analogue of this so the problem of dealing with the overshoot is forced upon you. For Lévy processes where the exit is caused by a jump across the boundary, Wiener–Hopf factorisations are a favourite tool, often combined with an assumption that the jumps causing the exit be of phase type (class $\mathcal{PH}$); see, e.g. [1] and [16].

The three classes of jump distributions, $\mathcal{R}$, $\mathcal{LE}$, and $\mathcal{PH}$, are all dense in the class of all distributions on $\mathbb{R}_+$ with $\mathcal{R}$ containing the other two and $\mathcal{LE}$ containing the nondense class of hyper-exponential distributions (i.e. *mixtures* of exponentials). The results presented below depend on the explicit form of the distribution considered, i.e. the polynomials determining the Laplace transform of a class $\mathcal{R}$ distribution, the coefficients, and exponential parameters determining a class $\mathcal{LE}$ density. In general, for a given class $\mathcal{PH}$ distribution, the explicit class $\mathcal{R}$ or class $\mathcal{LE}$ representation is not readily available and, indeed, the existing class $\mathcal{PH}$ results are of a different form from those given here (see, e.g. [1, Lemma 1]).

For existing class $\mathcal{R}$ results for Lévy processes, see, e.g. [15] (a paper closely related to [5] and [6]) and [14]. For random walks, in [3], [4], [9], and [10] the authors used the concept of semicontinuity, i.e. the case where the downward or upward jumps have an exponential distribution (geometric distribution in the lattice case). Recently, in their work [13] on meromorphic Lévy processes the authors used jump distributions that are *infinite mixtures* of exponentials.

It should be noted that Wiener–Hopf factorisations are not used anywhere below. In fact, as shown in Section 2, the 'partial eigenfunction' method used below applies to *any* Markov process (Jacobsen and Jensen [8] provided a non-Lévy continuous-time example) and any exit region, and in that generality Wiener–Hopf methods are of course meaningless. What then makes the random walk case simple and tractable is the fact that in the setup used here, the partial eigenfunctions have a simple form.

The layout of the paper is as follows: the general result is given in Section 2 and in Section 3 we discuss the one-sided exit problem for crossing a level below the initial state using one-sided class $\mathcal{R}$ increments; in Section 4 we determine the expected exit time and in Section 5 we treat two-sided exits using two-sided class $\mathcal{LE}$ increments.

2. A general result on exit times for Markov chains

Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain in discrete time with state space $(E, \mathcal{E})$. Write $p(x, \mathrm{d}y)$ for the transition probability and $\mathcal{P}$ for the transition operator of the chain,

$$\mathcal{P} f(x) = \int_E f(y)\, p(x, \mathrm{d}y) = \mathrm{E}_x f(X_1),$$

acting on the space of bounded and measurable $f\colon E \to \mathbb{R}$.

For a given $A \in \mathcal{E}$ with $A \neq \varnothing$ and $A \neq E$, consider the hitting time $\tau_A = \tau_A(X_1, X_2, \dots)$ given by

$$\tau_A = \inf\{n \geq 1\colon X_n \in A\}$$

with $\inf \varnothing = \infty$. For $g\colon A \to \mathbb{R}$ bounded and measurable and $0 < t < 1$, we have the

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following result characterising the function $f_0\colon \mathbb{R} \to \mathbb{R}$ given by

$$f_0(x) = \mathrm{E}_x[t^{\tau_A} g(X_{\tau_A}); \tau_A < \infty], \quad x \in A^{\mathrm{c}}, \qquad f_0(x) = g(x), \quad x \in A.$$

Theorem 1. *Suppose that $\phi\colon E \to \mathbb{R}$ is a bounded and measurable function satisfying*

$$\mathcal{P}\phi(x) = t^{-1}\phi(x), \quad x \in A^{\mathrm{c}}, \qquad \phi(x) = g(x), \quad x \in A. \tag{1}$$

Then $\phi = f_0$. Conversely, $\phi = f_0$ satisfies (1).

Proof. For any $n \in \mathbb{N}$, write

$$\begin{aligned} t^{\tau_A \wedge n}\phi(X_{\tau_A \wedge n}) = \phi(X_0) &+ \sum_{k=1}^{n} \mathbf{1}_{\{\tau_A \geq k\}}\, t^k(\phi(X_k) - \mathrm{E}_x[\phi(X_k) \mid X_{k-1}]) \\ &+ \sum_{k=1}^{n} \mathbf{1}_{\{\tau_A \geq k\}}\, t^k(\mathcal{P}\phi(X_{k-1}) - t^{-1}\phi(X_{k-1})). \end{aligned} \tag{2}$$

The first sum corresponds to a martingale (in the filtration $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$) and in the second sum each term vanishes because of (1). Taking expectations then gives $\mathrm{E}_x t^{\tau_A \wedge n}\phi(X_{\tau_A \wedge n}) = \phi(x)$, and letting $n \to \infty$, using dominated convergence and the definition of $\phi(x)$ for $x \in A$, $\phi = f_0$ follows.

For the converse, define $\tilde{\tau}_A = \tau_A(X_2, X_3, \dots)$ and note that $\tau_A = \tilde{\tau}_A + 1$ on $(X_1 \in A^{\mathrm{c}})$. Therefore,

$$f_0(x) = t\mathrm{E}_x[g(X_1); X_1 \in A] + t\mathrm{E}_x[t^{\tilde{\tau}_A}; X_1 \in A^{\mathrm{c}}] = t\mathcal{P} f_0(x),$$

since, by the Markov property,

$$\mathrm{E}_x[t^{\tilde{\tau}_A}; X_1 \in A^{\mathrm{c}}] = \mathrm{E}_x[f_0(X_1); X_1 \in A^{\mathrm{c}}].$$

In the terminology of Jacobsen [6], [7], a ϕ satisfying the first condition in (1) is a *partial eigenfunction for* $\mathcal{P}$ corresponding to the eigenvalue t^{-1}.

3. One-sided exit times for random walks

Now let $(X_n)_{n\geq 0}$ denote a random walk with a fixed initial value $X_0 \equiv x \geq 0$ and, for $n \geq 0$,

$$X_{n+1} = x - Y_1 - \cdots - Y_n, \qquad X_0 \equiv x,$$

with $(Y_n)_{n\geq 1}$ an independent and identically distributed sequence of $\mathbb{R}$-valued random variables with distribution F. Define the one-sided *exit time*

$$\tau_0 = \inf\{n \geq 1\colon X_n < 0\}$$

(with $\inf \varnothing = \infty$) and the *overshoot*

$$Z_0 := -X_{\tau_0} = |X_{\tau_0}|,$$

defined only on the set $(\tau_0 < \infty)$.

In order to make the exit of the random walk below the level 0 possible, it is assumed throughout the paper that

$$p := \mathrm{P}(Y_n > 0) > 0.$$

By imposing a suitable structure on the distribution of the downward jumps of the random walks, i.e. the distribution of the strictly positive Y_n, we shall be able to derive explicit expressions for the joint transform

$$\mathrm{E}_x[t^{\tau_0}\mathrm{e}^{-\zeta Z_0};\, \tau_0 < \infty], \qquad 0 < t \le 1,\ \zeta \ge 0. \tag{3}$$

For $\zeta = 0$ and $0 < t < 1$, this yields the moment generating function

$$\mathrm{E}_x t^{\tau_0} = \sum_{n=1}^{\infty} t^n \mathrm{P}_x(\tau_0 = n).$$

For $\zeta = 0$ and $t = 1$, we obtain the *exit probability* $\mathrm{P}_x(\tau_0 < \infty)$ and, for $\zeta > 0$ and $t = 1$, the marginal (possibly deficient) Laplace transform for the distribution of the overshoot Z_0 appears.

The transition operator $\mathcal{P}$ for the random walk is given by

$$\mathcal{P} f(x) = \int f(x-y)\, F(\mathrm{d}y),$$

and, from Theorem 1, it follows directly that, for $0 < t < 1$ and $\zeta \ge 0$, the joint transform (3) is determined by finding the unique bounded and measurable function ϕ satisfying

$$\mathcal{P}\phi(x) = t^{-1}\phi(x), \quad x \ge 0, \qquad \phi(x) = \mathrm{e}^{\zeta x}, \quad x < 0. \tag{4}$$

The $t = 1$ case is special and will be discussed below.

Consider the distribution F of the Y_n and write it in the form

$$F(\mathrm{d}y) = pF_+(\mathrm{d}y) + qF_-(\mathrm{d}y),$$

where $0 < p \le 1$, $q = 1 - p$, $F_+(\mathrm{d}y)$ is a probability on $\mathbb{R}_{++} = (0, \infty)$ (the strictly negative jumps of the random walk), and F_- is a probability on $\mathbb{R}_- = (-\infty, 0]$ (the nonnegative jumps for the random walk). Introduce the Laplace transforms

$$L_+(\nu) = \int_{(0,\infty)} \mathrm{e}^{-\nu y} F_+(\mathrm{d}y), \qquad \nu \in \mathbb{C},\ \mathrm{Re}\,\nu \ge 0, \tag{5}$$
$$L_-(\nu) = \int_{(-\infty,0]} \mathrm{e}^{-\nu y} F_-(\mathrm{d}y), \qquad \nu \in \mathbb{C},\ \mathrm{Re}\,\nu \le 0.$$

Of course, for the relevant values of ν,

$$L_+(\nu) = \mathrm{E}[\mathrm{e}^{-\nu Y_n} \mid Y_n > 0], \qquad L_-(\nu) = \mathrm{E}[\mathrm{e}^{-\nu Y_n} \mid Y_n \le 0],$$

with the qualification that if $p = 1$ (the random walk is strictly decreasing), F_- is irrelevant.

The main assumption we shall make concerning the structure of F_+ is that L_+ *be a rational function* (class $\mathcal{R}$ from the introduction),

$$L_+(\nu) = \frac{P_+(\nu)}{R_+(\nu)}, \qquad \mathrm{Re}\,\nu \ge 0, \tag{6}$$

with P_+ and R_+ polynomials, standardised so that they have no common roots. We write $m \ge 1$ for the degree of R_+ and note that then P_+ is of degree less than m.

It is assumption (6) that will ensure that the partial eigenfunctions $\phi(x)$ from (4) have a simple form as linear combinations of m exponentials in x for $x \geq 0$; see (10) below.

The form of (6) immediately permits a well-defined analytic extension,

$$\bar{L}_+(\nu) = \frac{P_+(\nu)}{R_+(\nu)}, \qquad \mathrm{Re}\nu < 0,$$

to the negative half of the complex plane, except for the finitely many singularities ν_s matching the roots of R_+, all of which necessarily satisfy $\mathrm{Re}\nu_s < 0$. Note that representation (5) of $\bar{L}_+(\nu)$ is invalid unless $\mathrm{Re}\nu < 0$ is sufficiently close to 0.

The distributions with Laplace transforms as in (6) all have Lebesgue densities (see, e.g. Equation (1.1) of [14], where structure (6) was also used). An important subclass (the class $\mathcal{LE}$ from the introduction) that we shall use later is given by

$$F_+(\mathrm{d}x) = \sum_{j=1}^{m} \alpha_j \mu_j \mathrm{e}^{-\mu_j x}\, \mathrm{d}x, \qquad x > 0, \tag{7}$$

with, e.g. $0 < \mu_1 < \cdots < \mu_m$, all $\alpha_j \in \mathbb{R}$, and $\alpha_j \neq 0$ (but not necessarily $\alpha_j > 0$, corresponding to mixtures of exponentials) with $\sum_j \alpha_j = 1$, and, of course, it must hold that the density is greater than or equal to 0 for all $x > 0$.

The fact that F_+ always has a density implies in particular that we are disregarding all random walks on lattices such as $\mathbb{Z}$. Also, note that $\tau_0 = \inf\{n \geq 1 \colon X_n \leq 0\}$, P_x–almost surely (P_x-a.s.) for all $x \geq 0$: from time 1 onwards, it is impossible for the random walk to hit the level 0 exactly.

We shall also make one assumption concerning the distribution F_- of the upward jumps: the Laplace transform $L_-(\nu)$ is assumed to have an analytic extension from $\mathbb{C}_- = \{z \in \mathbb{C} \colon \mathrm{Re}z \leq 0\}$ to an open set $S \supset \mathbb{C}_-$. Formally, this is required for the proof of Lemma 1 below, but the conclusions of the lemma and Theorem 2 may well be valid in great generality without this assumption. The assumption implies in particular that F_- is light tailed in the sense that all moments of F_- are finite.

Let $0 < t \leq 1$, and consider the *Cramér–Lundberg equation*

$$p\bar{L}_+(\gamma) + qL_-(\gamma) = t^{-1}, \qquad \gamma \in \mathbb{C},\ \mathrm{Re}\gamma \leq 0. \tag{8}$$

Lemma 1. *For $0 < t < 1$, the Cramér–Lundberg equation (8) has precisely m solutions γ, counted with multiplicity, with $\mathrm{Re}\gamma < 0$.*

Proof. Rewrite (8) as $tpP_+(\gamma) + tqR_+(\gamma) - R_+(\gamma) = 0$. Because of the analyticity of L_- just assumed, the result follows from Rouché's theorem from complex function theory by showing that, for sufficiently large $K > 0$ and $\gamma \in \mathbb{C}$ such that either $\mathrm{Re}\gamma < 0$, $|\gamma| = K$ or $\mathrm{Re}\gamma = 0$, $|\mathrm{Im}\gamma| \leq K$, it holds that

$$|tpP_+(\gamma) + tqR_+(\gamma)L_-(\gamma)| < |R_+(\gamma)|, \tag{9}$$

since then $tpP_+(\gamma) + tqR_+(\gamma) - R_+(\gamma)$ will have the same number of roots in the domain $\{\gamma \in \mathbb{C} \colon \mathrm{Re}\gamma < 0, |\gamma| < K\}$ as $R_+(\gamma)$, i.e. m roots for large enough K. Because $t < 1$, (9) follows easily because $|L_-(\gamma)| \leq 1$, while $|P_+(\gamma)| < |R_+(\gamma)|$ for sufficiently large $|\gamma|$ and, using (5), we have $|L_+(\gamma)| \leq 1$, i.e. $|P_+(\gamma)| \leq |R_+(\gamma)|$ if $\mathrm{Re}\gamma = 0$.

Theorem 2. *For any $0 < t < 1$ such that the m t-dependent solutions $\gamma_1, \dots, \gamma_m$ to the Cramér–Lundberg equation (8) are distinct, it holds, for all $x \geq 0$ and $\zeta \geq 0$, that the joint transform (3) is given by*

$$\mathrm{E}_x[t^{\tau_0} \mathrm{e}^{-\zeta Z_0}; \tau_0 < \infty] = \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x}, \qquad x \geq 0, \tag{10}$$

where

$$c_i = \frac{R_+(\gamma_i)}{R_+(\zeta)} \frac{\prod_{j \neq i}(\zeta - \gamma_j)}{\prod_{j \neq i}(\gamma_i - \gamma_j)}. \tag{11}$$

Remark 1. If in (8), γ_i with $\mathrm{Re}\gamma_i < 0$ is a root of multiplicity $r > 1$, in (10) we need r terms $\sum_{j=1}^{r} c_{ij} x^{j-1} \mathrm{e}^{\gamma_i x}$, where the coefficients are found using a generalisation of Lemma 2 below.

Remark 2. In Lemma 1 of [1] the authors considered a (continuous-time) Lévy process with downward jumps of phase type. Their formula (16) is closely related to (10) above, but a notable qualitative difference is that the formulae in the lemma rely explicitly on the relevant Wiener–Hopf factorisation.

The proof uses the following elementary polynomial expansions, also used in [5] and [6], where a polynomial is identified through its values at sufficiently many given points.

Lemma 2. *Let $r \in \mathbb{N}$, and let Π denote a polynomial of degree less than or equal to r. Then, for any choice of $r + 1$ distinct complex numbers $\delta_1, \dots, \delta_{r+1}$ and all $z \in \mathbb{C}$,*

$$\Pi(z) = \sum_{i=1}^{r+1} \frac{\Pi(\delta_i)}{\rho_{\setminus i}(\delta_i)} \rho_{\setminus i}(z), \tag{12}$$

where $\rho_{\setminus i}(z) = \prod_{1 \leq j \leq r+1,\, j \neq i}(z - \delta_j)$.

We shall refer to expansion (12) as the expansion of Π using the *centres* δ_j.

Proof of Theorem 2. Define, cf. (10) and (11),

$$\phi(x) = \begin{cases} \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x}, & x \geq 0, \\ \mathrm{e}^{\zeta x}, & x < 0. \end{cases}$$

The result follows if we show that ϕ is a partial eigenfunction, see (4) and (3). We find that, for $x \geq 0$,

$$\begin{aligned} \mathcal{P}\phi(x) = q \int_{(-\infty,0]} \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i (x-y)} \, F_-(\mathrm{d}y) + p \int_{(0,x]} \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i (x-y)} \, F_+(\mathrm{d}y) \\ + p \int_{(x,\infty)} \mathrm{e}^{\zeta(x-y)} \, F_+(\mathrm{d}y), \end{aligned} \tag{13}$$

which is required to equal $t^{-1} \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x}$. The first term on the right-hand side of (13) equals

$$q \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x} L_-(\gamma_i) = \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x} (t^{-1} - p\bar{L}_+(\gamma_i))$$

using (8); hence, the identity $\mathcal{P}\phi(x) = \phi(x)/t$ becomes

$$-\sum_{i=1}^{m} c_i e^{\gamma_i x} \bar{L}_+(\gamma_i) + \int_{(0,x]} \sum_{i=1}^{m} c_i e^{\gamma_i (x-y)}\, F_+(dy) + \int_{(x,\infty)} e^{\zeta(x-y)}\, F_+(dy) = 0 \tag{14}$$

for $x \geq 0$. We show this by computing the Laplace transform: let $\theta \geq 0$, multiply by $e^{-\theta x}$, and Lebesgue integrate x from 0 to ∞. Then (14) becomes

$$-\sum_{i=1}^{m} c_i \bar{L}_+(\gamma_i) \frac{1}{\theta - \gamma_i} + L_+(\theta) \sum_{i=1}^{m} c_i \frac{1}{\theta - \gamma_i} + \frac{1}{\theta - \zeta}(L_+(\zeta) - L_+(\theta)) = 0 \quad \text{for } \theta \geq 0.$$

Solving for $L_+(\theta)$ gives

$$\begin{aligned} L_+(\theta) &= \frac{\sum_{i=1}^{m} c_i \bar{L}_+(\gamma_i)/(\theta - \gamma_i) - L_+(\zeta)/(\theta - \zeta)}{\sum_{i=1}^{m} c_i/(\theta - \gamma_i) - 1/(\theta - \zeta)} \\ &= \frac{\sum_{i=1}^{m} c_i \bar{L}_+(\gamma_i)(\theta - \zeta)\pi_{\setminus i}(\theta) - \pi(\theta) L_+(\zeta)}{\sum_{i=1}^{m} c_i (\theta - \zeta)\pi_{\setminus i}(\theta) - \pi(\theta)}, \end{aligned} \tag{15}$$

where, for $z \in \mathbb{C}$ and $1 \leq i \leq m$,

$$\pi_{\setminus i}(z) = \prod_{1 \leq i' \leq m,\, i' \neq i} (z - \gamma_{i'}), \qquad \pi(z) = (z - \gamma_i)\pi_{\setminus i}(z) = \prod_{i'=1}^{m} (z - \gamma_{i'}).$$

We shall show that expression (15) for $L_+(\theta)$ holds by using (12) to argue that

$$P_+(\theta) = K \sum_{i=1}^{m} c_i \bar{L}_+(\gamma_i)(\theta - \zeta)\pi_{\setminus i}(\theta) - K\pi(\theta) L_+(\zeta), \tag{16}$$

$$R_+(\theta) = K \sum_{i=1}^{m} c_i (\theta - \zeta)\pi_{\setminus i}(\theta) - K\pi(\theta), \tag{17}$$

with $K = -R_+(\zeta)/\pi(\zeta)$, and the proof of the theorem will then be complete.

With the polynomial R_+ being of degree m, consider expansion (12) of R_+ with centres $\gamma_1, \dots, \gamma_m, \zeta$, i.e.

$$R_+(\theta) = \sum_{i=1}^{m} \frac{R_+(\gamma_i)}{(\gamma_i - \zeta)\pi_{\setminus i}(\gamma_i)} (\theta - \zeta)\pi_{\setminus i}(\theta) + \frac{R_+(\zeta)}{\pi(\zeta)} \pi(\theta).$$

Referring to the value of K and (11), it is immediately verified that this tallies with (17). Using the same centres for the expansion of P_+, it is also clear that the expansion agrees with (16) when recalling that $R_+(\zeta)L_+(\zeta) = P_+(\zeta)$ and $R_+(\gamma_i)\bar{L}_+(\gamma_i) = P_+(\gamma_i)$.

Example 1. An instructive case is when $q = 0$ and $x = 0$: trivially, then $\tau_0 = 1$, P_0-a.s., so referring to (10) we should have $\sum_{i=1}^{m} c_i = t$ with the c_i given by (11) with $\zeta = 0$. But, by (8), $\bar{L}_+(\gamma_i) = t^{-1}$ for all i; hence, $R_+(\gamma_i) = t P_+(\gamma_i)$ so

$$\sum_{i=1}^{m} c_i = \frac{t}{R_+(0)} \sum_{i=1}^{m} \frac{P_+(\gamma_i)}{\pi_{\setminus i}(\gamma_i)} \pi_{\setminus i}(0) = t \frac{P_+(0)}{R_+(0)} = t,$$

since we recognise the sum in the middle as the expansion of $P_+(0)$ using the centres $\gamma_1, \dots, \gamma_m$.

Remark 3. Suppose that F_+ is given by (7). We may then use (14) directly to find the partial eigenfunction ϕ, assuming of course that the m solutions γ to (8) with $\mathrm{Re}\gamma < 0$ are distinct, and find that the c_i in (10) solve the linear equation system

$$\sum_{i=1}^{m} c_i \frac{1}{\mu_j + \gamma_i} = \frac{1}{\mu_j + \zeta}, \qquad 1 \le j \le m. \tag{18}$$

To see that these solutions agree with the c_i from (11), observe first that

$$L_+(\theta) = \sum_{i=1}^{m} \frac{\alpha_i \mu_i}{\mu_i + \theta}$$

so that we may take

$$R_+(z) = \prod_{i=1}^{m} (\mu_i + z).$$

Inserting expression (11) for the c_i into (18), we find that

$$\sum_{i=1}^{m} \frac{R_+(\gamma_i)}{\pi_{\setminus i}(\gamma_i)} \frac{\mu_j + \zeta}{\mu_j + \gamma_i} \pi_{\setminus i}(\zeta) = R_+(\zeta).$$

Using the fact that $R_+(-\mu_j) = 0$, a quick check shows that this is the expansion of $R_+(\zeta)$ using the centres $\gamma_1, \dots, \gamma_m, -\mu_j$.

Consider now the problem of determining

$$\mathrm{E}_x[\mathrm{e}^{-\zeta Z_0}; \tau_0],$$

corresponding to taking $t = 1$ in (3). If ϕ is bounded and satisfies (4) with $t = 1$, from $\mathrm{E}_x \phi(X_{\tau_0 \wedge n}) = \phi(x)$ for $x \ge 0$, it follows by dominated convergence that

$$\mathrm{E}_x[\mathrm{e}^{-\zeta Z_0}; \tau_0 < \infty] + \lim_{n \to \infty} \mathrm{E}_x[\phi(X_n); \tau_0 > n] = \phi(x).$$

If $\xi := \mathrm{E} Y_n \ge 0$, we have $\mathrm{P}_x(\tau_0 < \infty) = 1$; hence,

$$\mathrm{E}_x[\mathrm{e}^{-\zeta Z_0}; \tau_0 < \infty] = \phi(x), \qquad \zeta \ge 0,\ x \ge 0. \tag{19}$$

If $\xi < 0$, the same identity holds provided that $\lim_{x \to \infty} \phi(x) = 0$, since then $\lim_{n \to \infty} \phi(X_n) = 0$, P_x-a.s.

In order to determine ϕ, it is an obvious idea to consider the Cramér–Lundberg equation (8) for $t = 1$, i.e.

$$p\bar{L}_+(\gamma) + qL_-(\gamma) = 1, \qquad \gamma \in \mathbb{C},\ \mathrm{Re}\gamma \le 0, \tag{20}$$

which always has the solution $\gamma = 0$. Without going into the details of the proof we then have the following results corresponding to the two cases $\xi > 0$ and $\xi < 0$:

(i) if $\xi > 0$, (20) has the solution $\gamma_m = 0$ and precisely $m - 1$ solutions $\gamma_1, \dots, \gamma_{m-1}$ (counted with multiplicity) satisfying $\mathrm{Re}\gamma_i < 0$,

(ii) if $\xi < 0$, (20) has precisely m solutions $\gamma_1, \dots, \gamma_m$ (counted with multiplicity) satisfying $\mathrm{Re}\gamma_i < 0$.

The $\xi = 0$ case is deliberately left out; however, although then $\gamma = 0$ is a root in (20) of multiplicity 2, we believe that (21) below is still valid in the form corresponding to case (i).

Theorem 3. *Let $\xi \neq 0$, and let $\gamma_1, \ldots, \gamma_m$ be the solutions to (20) described above for cases (i) and (ii). If these solutions are distinct, it holds, for all $\zeta \geq 0$, that*

$$\mathrm{E}_x[\mathrm{e}^{-\zeta Z_0}; \tau_0 < \infty] = \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x}, \qquad x \geq 0, \tag{21}$$

where

$$c_i = \frac{R_+(\gamma_i)}{R_+(\zeta)} \frac{\prod_{\{j:\, j \neq i\}}(\zeta - \gamma_j)}{\prod_{\{j:\, j \neq i\}}(\gamma_i - \gamma_j)}.$$

Note that, if $\xi < 0$, the candidate $\sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x}$ for the partial eigenfunction ϕ satisfies $\lim_{x \to \infty} \phi(x) = 0$, as it should in order that (19) be valid.

Taking $\zeta = 0$ in (21) we obtain $\mathrm{P}_x(\tau_0 < \infty)$. For case (ii), we obtain

$$\mathrm{P}_x(\tau_0 < \infty) = \sum_{i=1}^{m} \frac{R_+(\gamma_i)}{R_+(\zeta)} \frac{\prod_{\{j:\, j \neq i\}}(-\gamma_j)}{\prod_{\{j:\, j \neq i\}}(\gamma_i - \gamma_j)} \mathrm{e}^{\gamma_i x},$$

while, for case (i), since the c_i for $i = 1, \ldots, m-1$ contain the factor $-\gamma_m = 0$ and, therefore, are equal to 0, we simply obtain $\mathrm{P}_x(\tau_0 < \infty) = c_m = 1$, as we should!

4. The expected exit time

In this section we shall determine the expected exit time $\mathrm{E}_x \tau_0$ when $\xi = \mathrm{E} Y_n > 0$. With $\xi > 0$, it is standard that $\mathrm{E}_x \tau_0 < \infty$ and if $\mathrm{E} Y_n^2 < \infty$ (automatic with the assumptions we have made about F_+ and F_-), it also holds that

$$\lim_{n \to \infty} \mathrm{E}_x[X_n; \tau_0 > n] = 0, \qquad x \geq 0, \tag{22}$$

a fact that will be used below. (A quick proof of (22) is as follows. Write

$$\mathrm{E}_x[X_n; \tau_0 > n] = \mathrm{E}_x[X_n - n\xi - x; \tau_0 > n] + (n\xi + x)\mathrm{P}_x(\tau_0 > n).$$

Because $\mathrm{E}_x \tau_0 < \infty$, the last term on the right-hand side tends to 0 as $n \to \infty$. By the Cauchy–Schwarz inequality, the first term is less than or equal to $(n \operatorname{var} Y_1 \, \mathrm{P}_x(\tau_0 > n))^{1/2} \to 0$.)

Rather than treating all F_+ with a rational Laplace transform, we shall assume that F_+ is of the form (7); see Remark 3.

Theorem 4. *Suppose that $\xi > 0$ and that F_+ is given by (7). Also, assume that the solutions $\gamma_1, \ldots, \gamma_{m-1}$ to (20) with $\mathrm{Re}\gamma_i < 0$ are distinct. Then, for $x \geq 0$, the expected exit time is given by*

$$\mathrm{E}_x \tau_0 = \frac{x}{\xi} + B + \sum_{i=1}^{m-1} c_i^{\circ} \mathrm{e}^{\gamma_i x}, \tag{23}$$

where $B, c_1^{\circ}, \ldots, c_{m-1}^{\circ}$ are the solutions to the linear system of equations

$$\sum_{i=1}^{m-1} c_i^{\circ} \frac{\mu_j}{\mu_j + \gamma_i} + B = \frac{1}{\xi \mu_j}, \qquad 1 \leq j \leq m. \tag{24}$$

https://doi.org/10.1239/jap/1318940455 Published online by Cambridge University Press

Proof. Define

$$\phi(x) = \begin{cases} \dfrac{1}{\xi}x + B + \displaystyle\sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We claim that

$$\mathcal{P}\phi(x) = \phi(x) - 1, \qquad x \geq 0. \tag{25}$$

If true, from

$$\phi(X_{\tau_0 \wedge n}) = \phi(x) + \sum_{k=1}^{n} \mathbf{1}_{\{\tau_0 \geq k\}}(\phi(X_k) - \mathrm{E}_x[\phi(X_k) \mid X_{k-1}])$$
$$+ \sum_{k=1}^{n} \mathbf{1}_{\{\tau_0 \geq k\}}(\mathcal{P}\phi(X_{k-1}) - \phi(X_{k-1}))$$

(cf. (2)), it follows that

$$\mathrm{E}_x\phi(X_{\tau_0 \wedge n}) = \phi(x) - \mathrm{E}_x\tau_0 \wedge n.$$

Since, for $y \geq 0$, $\phi(y) = C^\circ y$ plus a bounded function, by (22), $\mathrm{E}_x[\phi(X_n); \tau_0 > n]$ converges to 0 and since $\mathrm{E}_x\tau_0 \wedge n \uparrow \mathrm{E}_x\tau_0$ by monotone convergence and $\phi(X_{\tau_0}) = 0$ on $(\tau_0 < \infty)$ (i.e. P_x-a.s.), we conclude that $0 = \phi(x) - \mathrm{E}_x\tau_0$, proving (23).

It remains to establish (25), which is easily done by direct calculation: introducing $\xi_- = \int_{(-\infty,0]} y\, F_-(\mathrm{d}y)$ and $\bar{F}_+(y) = F_+((y,\infty))$, we find that

$$\mathcal{P}\phi(x) = q\xi^{-1}x - q\xi^{-1}\xi_- + qB + q\sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x} L_-(\gamma_i) + p\xi^{-1}x(1 - \bar{F}_+(x))$$
$$- p\xi^{-1}\int_{(0,x]} y\, F_+(\mathrm{d}y) + pB(1 - \bar{F}_+(x)) + p\sum_{i=1}^{m-1} c_i^\circ \int_{(0,x]} \mathrm{e}^{\gamma_i(x-y)}\, F_+(\mathrm{d}y).$$

By partial integration, $\int_{]0,x]} y\, F_+(\mathrm{d}y) = -x\bar{F}_+(x) + \int_0^x \bar{F}_+(y)\,\mathrm{d}y$, and, by (20),

$$q\sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x} L_-(\gamma_i) = \sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x}(1 - p\bar{L}_+(\gamma_i)).$$

Using these two facts together with the form of F_+, which in particular gives $\bar{L}_+(z) = \sum_{j=1}^m \alpha_j\mu_j/(\mu_j + z)$, it is seen that (25) is equivalent to the equation

$$-1 = -q\xi^{-1}\xi_- - p\sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x}\sum_{j=1}^{m} \alpha_j \frac{\mu_j}{\mu_j + \gamma_i} - p\xi^{-1}\sum_{j=1}^{m} \frac{\alpha_j}{\mu_j}(1 - \mathrm{e}^{-\mu_j x})$$
$$- pB\sum_{j=1}^{m} \alpha_j \mathrm{e}^{-\mu_j x} + p\sum_{i=1}^{m-1} c_i^\circ \mathrm{e}^{\gamma_i x}\sum_{j=1}^{m} \alpha_j \frac{\mu_j}{\mu_j + \gamma_i}(1 - \mathrm{e}^{-(\mu_j + \gamma_i)x}).$$

On the right-hand side, the terms with some $\mathrm{e}^{\gamma_i x}$ cancel out, the constant terms yield $-q\xi^{-1}\xi_- - p\xi^{-1}\sum_{j=1}^m \alpha_j/\mu_j = -1$, and the exponential $\mathrm{e}^{-\mu_j x}$ appears with the coefficient

$$p\xi^{-1}\frac{\alpha_j}{\mu_j} - pB\alpha_j - p\sum_{i=1}^{m-1} c_i^\circ \alpha_j \frac{\mu_j}{\mu_j + \gamma_i},$$

which is equal to 0 because of (24).

When the downward jumps for the random walk are exponential, the result is particularly simple.

Corollary 1. *Suppose that* $\xi > 0$, *and let* $F_+(\mathrm{d}x) = \mu \mathrm{e}^{-\mu x}\,\mathrm{d}x$ *for* $x > 0$, *where* $\mu > 0$. *Then*

$$\mathrm{E}_x \tau_0 = \frac{x}{\xi} + \frac{1}{\mu\xi}.$$

5. Two-sided exits

Throughout this section, it is assumed that $0 < p < 1$, i.e. it is possible for the random walk to jump down as well as up.

Let $a < b \in \mathbb{R}$ with $a \le 0 \le b$, and start the random walk from $x \in [a, b]$. Consider the exit time

$$\tau_{ab} = \inf\{n \ge 1 \colon X_n < a \text{ or } X_n > b\},$$

which is finite P_x-a.s. Also, define the events

$$G_a = (X_{\tau_{ab}} < a), \qquad G_b = (X_{\tau_{ab}} > b),$$

corresponding to the random walk exiting from the interval $[a, b]$ by jumping across the lower boundary a or the upper boundary b. Finally, define the downward overshoot at exit and the upward overshoot at exit by

$$Z_a = (a - X_{\tau_{ab}})\,\mathbf{1}_{G_a}, \qquad W_b = (X_{\tau_{ab}} - b)\,\mathbf{1}_{G_b}\,.$$

In this section we shall determine the joint transforms of τ_{ab} and the downward/upward overshoot. This requires not only that F_+ is, e.g. of class $\mathcal{R}$ but that the same is also true for F_-. In order to simplify, we assume in the remainder of the section that F_+ is given by (7) with F_- also of class $\mathcal{LE}$,

$$F_-(\mathrm{d}x) = \sum_{j'=1}^{l} \beta_{j'} \nu_{j'} \mathrm{e}^{\nu_{j'} x}\,\mathrm{d}x, \qquad x < 0,$$

with all $\nu_{j'} > 0$ and distinct, and all $\beta_{j'} \ne 0$ with $\sum_{i'} \beta_{j'} = 1$, and the density of course required to be greater than or equal to 0 for $x < 0$.

We now have

$$\bar{L}_+(z) = \sum_{i'=1}^{m} \alpha_{i'} \frac{\mu_{i'}}{\mu_{i'} + z}, \qquad \bar{L}_-(z) = \sum_{j'=1}^{l} \beta_{j'} \frac{\nu_{j'}}{\nu_{j'} - z},$$

and note that, for $0 < t < 1$, the Cramér–Lundberg equation (8),

$$p \sum_{j=1}^{m} \alpha_{i'} \frac{\mu_{i'}}{\mu_{i'} + z} + q \sum_{j'=1}^{l} \beta_{j'} \frac{\nu_{j'}}{\nu_{j'} - z} = \frac{1}{t}, \tag{26}$$

has precisely m solutions $\gamma_1, \dots, \gamma_m$ (counted with multiplicity) with $\mathrm{Re}\gamma_i < 0$ and precisely l solutions $\delta_1, \dots, \delta_l$ (counted with multiplicity) with $\mathrm{Re}\delta_j > 0$. If $t = 1$ and $\xi > 0$, (20) has precisely $m - 1$ solutions $\gamma_1, \dots, \gamma_{m-1}$ with $\mathrm{Re}\gamma_i < 0$ and precisely l solutions $\delta_1, \dots, \delta_l$ with $\mathrm{Re}\delta_j > 0$, and we write $\gamma_m = 0$ for the remaining solution 0. If $t = 1$ and $\xi < 0$, (20) has precisely m solutions $\gamma_1, \dots, \gamma_m$ with $\mathrm{Re}\gamma_i < 0$ and precisely $l - 1$ solutions $\delta_1, \dots, \delta_{l-1}$ with $\mathrm{Re}\delta_j > 0$, and we write $\delta_l = 0$ for the remaining solution 0.

Theorem 5. *Let $0 < t \leq 1$, $\zeta \geq 0$, $\rho \geq 0$, and $K, L \in \mathbb{R}$. Define $\gamma_1, \ldots, \gamma_m$ and $\delta_1, \ldots, \delta_l$ as the solutions to the Cramér–Lundberg equation (26) in the manner just described. Assume that all the solutions are distinct. Then, for $a \leq x \leq b$,*

$$\mathrm{E}_x[K t^{\tau_{ab}} \mathrm{e}^{-\zeta Z_a}; G_a] + \mathrm{E}_x[L t^{\tau_{ab}} \mathrm{e}^{-\rho W_b}; G_b] = \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x} + \sum_{j'=1}^{l} d_j \mathrm{e}^{\delta_j x}, \tag{27}$$

where the c_i and d_j are the solutions to the linear equation system

$$\sum_{i=1}^{m} c_i \frac{\mathrm{e}^{\gamma_i a}}{\mu_{i'} + \gamma_i} + \sum_{j=1}^{l} d_j \frac{\mathrm{e}^{\delta_j a}}{\mu_{i'} + \delta_j} = \frac{K}{\mu_{i'} + \zeta}, \qquad 1 \leq i' \leq m,$$

$$\sum_{i=1}^{m} c_i \frac{\mathrm{e}^{\gamma_i b}}{\nu_{j'} - \gamma_i} + \sum_{j=1}^{l} d_j \frac{\mathrm{e}^{\delta_j b}}{\nu_{j'} - \delta_j} = \frac{L}{\nu_{j'} + \rho}, \qquad 1 \leq j' \leq l.$$

Proof. We just outline the proof of (27). Referring to the technique from Section 2, the task is to find a function ϕ such that with

$$\phi(x) = \begin{cases} L\mathrm{e}^{-\rho(x-b)}, & x > b, \\ \sum_{i=1}^{m} c_i \mathrm{e}^{\gamma_i x} + \sum_{j=1}^{l} d_j \mathrm{e}^{\delta_j x}, & a \leq x \leq b, \\ K\mathrm{e}^{-\zeta(a-x)}, & x < a, \end{cases}$$

we have

$$\mathcal{P}\phi(x) = t^{-1}\phi(x)$$

for $a \leq x \leq b$, since then

$$\mathrm{E}_x t^{\tau_{ab}} \phi(X_{\tau_{ab}}) = \phi(x), \tag{28}$$

as desired. (It should be noted that ϕ is automatically bounded and that, as $n \to \infty$, $\mathrm{E}_x[\phi(X_n); \tau_{ab} > n] \to 0$ since $\tau_{ab} < \infty$, P_x-a.s. The latter is needed for obtaining (28) when $t = 1$.)

Computing $\mathcal{P}\phi(x)$ explicitly (tedious but easy) for $a \leq x \leq b$ yields an expression which is a linear combination of the exponential functions $\mathrm{e}^{\gamma_i x}$, $\mathrm{e}^{\delta_j x}$ and $\mathrm{e}^{-\mu_{i'} x}$, $\mathrm{e}^{-\delta_{j'} x}$. Referring to (26), it is verified that the linear combination of $\mathrm{e}^{\gamma_i x}$ and $\mathrm{e}^{\delta_{ji} x}$ is precisely $t^{-1}\phi(x)$. Because of (27), it is also verified that the coefficient to each of the exponentials $\mathrm{e}^{-\mu_{i'} x}$ and $\mathrm{e}^{-\delta_{j'} x}$ equals 0.

Remark 4. It suffices to take $(K, L) = (1, 0)$ in order to find $\mathrm{E}_x[t^{\tau_{ab}} \mathrm{e}^{-\zeta Z_a}; G_a]$ and $(K, L) = (0, 1)$ in order to find $\mathrm{E}_x[t^{\tau_{ab}} \mathrm{e}^{-\rho W_b}; G_b]$. Having both these transforms of course immediately gives (27).

Acknowledgements

This work was supported by the Danish Research Council FNU grant 09-072331, Point Process Modelling and Statistical Inference. Many thanks also to Alex Novikov at the University of Technology, Sydney, and Kostya Borovkov at the University of Melbourne for sponsoring the author's visit to Australia in February–March 2010, and giving him the opportunity to present and discuss this and related work. Thanks are also due to the anonymous referee for suggesting relevant references on exits for Lévy processes and whose comments inspired the discussion of the different jump distributions presented in the introduction.

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