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Ideas from Zariski Topology in the Study of Cubical Homology

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Abstract. Cubical sets and their homology have been used in dynamical systems as well as in digital imaging. We take a fresh look at this topic, following Zariski ideas from algebraic geometry. The cubical topology is defined to be a topology in $\mathbb{R}^d$ in which a set is closed if and only if it is cubical. This concept is a convenient frame for describing a variety of important features of cubical sets. Separation axioms which, in general, are not satisfied here, characterize exactly those pairs of points which we want to distinguish. The noetherian property guarantees the correctness of the algorithms. Moreover, maps between cubical sets which are continuous and closed with respect to the cubical topology are precisely those for whom the homology map can be defined and computed without grid subdivisions. A combinatorial version of the Vietoris–Begle theorem is derived. This theorem plays the central role in an algorithm computing homology of maps which are continuous with respect to the Euclidean topology.

1 Introduction

Representable sets, in particular, cubical sets, and their homology have proved to be useful geometric structures in a variety of applications from the Conley index in dynamical systems [9, 13, 16, 17] to image and pattern recognition in digital imaging [2, 3, 9][1]. We take a fresh look at this topic, following Zariski ideas from algebraic geometry. Recall from [4, 8] that the *Zariski topology* in the Euclidean space $\mathbb{R}^d$ is defined by declaring that a proper subset of $\mathbb{R}^d$ is *closed* if and only if it is algebraic. The *cubical topology* is a topology in $\mathbb{R}^d$ in which a proper subset is closed if and only if it is cubical.

It seems foolish at first to abandon the standard Euclidean topology and introduce one which is not only not metrizable, but which does not even satisfy any separation axiom. Nevertheless, the points which we want to distinguish in a cubical set are exactly those which belong to different cells or different elementary cubes, *i.e.*, the points separated by the cubical topology. In digital imaging, computer scientists seem to have a hard time deciding if they prefer to interpret pixels as unit size squares or as isolated points in a square grid. The cubical topology permits these two interpretations to co-exist on mathematical grounds.

Cubical topology has some more interesting features. A crucial property of the Zariski topology used in algebraic geometry is that it is *noetherian*, that is, every decreasing sequence of closed sets eventually becomes constant. That the cubical topol-

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[1]The homology used in the cited papers is cubical but it differs from the classical singular cubical homology [5, 10] in the same way that the singular homology differs from the simplicial homology.

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ogy is noetherian is quite obvious, but the simplicity of this observation does not diminish its importance. In fact, all algorithms which construct isolating neighborhoods and index pairs in dynamics are based on this property. Also, *irreducible* closed sets are precisely elementary cubes. Moreover, maps $f: X \to Y$ between cubical sets which are continuous and closed with respect to the cubical topology are precisely those for which the homology map $H_*(f)$ can be defined and computed without grid rescaling (the concept of rescaling is defined in [9]) or equivalently, without grid subdivisions. Although this class of maps, called cubical maps, seems somewhat restrictive, its study leads to algorithms for constructing homology of maps which are continuous with respect to the Euclidean topology.

This paper is organized as follows. In Section 2, definitions and properties of cubical sets, cubical chain complexes, and representable sets are recalled from [9]. Note that the cubical chain complex studied here is a combinatorial concept, in contrast with a well known, but less suitable for algorithms, concept of *singular cubical complex* presented for instance in [10, 5]. In Section 3, definition and properties of cubical topology are presented. For some routine proofs we refer to [18]. In Section 4, we define the class of cubical maps as the class of maps on cubical sets which are continuous and closed with respect to the relative cubical topology. We discuss the relation of this definition to the one given in [6], and give an explicit formula for a cubical map in terms of its coordinate functions. Using that formula, the homomorphism induced in homology by a cubical map is constructed.

In Section 4.3 we present a combinatorial version of the Vietoris–Begle theorem with a direct elementary proof. The classical version of that theorem has motivated extensions of homology theory to various classes of multivalued maps (see [7] and references therein) and is also used as a justification of a combinatorial procedure for computing homology of single-valued maps via their multivalued enclosures [1]. This approach has been further developed in [9]. The Vietoris–Begle theorem is at the heart of the explicit algorithm presented in [11]. Unfortunately, the role of that theorem in the construction presented there is hidden by many technicalities needed for the efficiency of the algorithm. Also, that paper uses the classical version of the theorem, but the elementary version presented in this paper is sufficient. For this reason, in Section 5 we provide a brief review of [11] in terms of our combinatorial Vietoris–Begle theorem. The reader interested in details is referred to [11].

2 Preliminaries

We recall here basic terminology related to cubical sets, cubical chain complexes, and representable sets [9]. The proofs of all statements of this section can be found in [9], except for Proposition 2.1 which is proved in [18].

2.1 Cubical Sets

An *elementary cube* is a finite product of intervals

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d, \tag{1}$$

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where I_i is either a unit interval $[l_i, l_i + 1]$ or a point (*degenerated interval*) $[l_i, l_i] = [l_i] := \{l_i\}$, and $l_i \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. The set of all elementary cubes is denoted by $\mathcal{K}$, and the set of those which are in $\mathbb{R}^d$ for a specific d is denoted by $\mathcal{K}^d$. The number d in (1) is called the *embedding number* of Q and is denoted by $\operatorname{emb} Q$. The *dimension* of Q is the number of non-degenerated intervals I_i of the form $[l_i, l_i + 1]$ in (1) and is denoted by $\dim Q$. We put

$$\mathcal{K}_k := \{Q \in \mathcal{K} \mid \dim Q = k\}, \quad \mathcal{K}_k^d := \mathcal{K}^d \cap \mathcal{K}_k.$$

Let $Q, P \in \mathcal{K}$. If $Q \subset P$, then Q is a *face* of P. If $Q \subset P$ and $Q \neq P$, then Q is a *proper face* of P.

Proposition 2.1 *Let $Q \in \mathcal{K}_k^d$. Then Q has 3^k faces.*

A set $X \subset \mathbb{R}^d$ is *cubical* if X can be written as a finite union of elementary cubes. Given a cubical set $X \subset \mathbb{R}^d$, we denote by $\mathcal{K}(X)$, respectively $\mathcal{K}_k(X)$, the set of those $Q \in \mathcal{K}^d$, respectively $Q \in \mathcal{K}_k^d$, such that $Q \subset X$. If $Q \in \mathcal{K}(X)$ is not a proper face of some $P \in \mathcal{K}(X)$, then it is called a *maximal face* in X. The set of maximal faces in X is denoted by $\mathcal{K}_{\max}(X)$.

2.2 Cubical Chain Complex

The group C_k^d of *k-dimensional chains* of $\mathbb{R}^d$ (k-chains for short) is the free abelian group generated by $\mathcal{K}_k^d$. By definition, the elements of C_k^d are functions $c\colon \mathcal{K}_k^d \to \mathbb{Z}$ such that $c(Q) = 0$ for all but a finite number of $Q \in \mathcal{K}_k^d$. We distinguish between the geometric objects, elementary cubes $Q \in \mathcal{K}_k^d$, and the corresponding algebraic objects, their *duals* $\widehat{Q}\colon \mathcal{K}_k^d \to \mathbb{Z}$, defined on any $P \in \mathcal{K}^d$ by

$$\widehat{Q}(P) := \begin{cases} 1 & \text{if } P = Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The set $\{\widehat{Q} \mid Q \in \mathcal{K}_k^d\}$ is the *canonical basis* for C_k^d. We put $C_k^d = 0$ if $k > d$, $k < 0$, or $d < 0$. In order to define the chain complex structure for the collection of groups $\{C_k^d\}_{k \in \mathbb{Z}}$, we first need the following auxiliary operation.

Given $P \in \mathcal{K}_k^d$ and $Q \in \mathcal{K}_{k'}^{d'}$, we have $P \times Q \in \mathcal{K}_{k+k'}^{d+d'}$. Set $\widehat{P} \diamond \widehat{Q} := \widehat{P \times Q}$. This definition extends to arbitrary chains $c_1 \in C_k^d$ and $c_2 \in C_{k'}^{d'}$ by

$$c_1 \diamond c_2 := \sum_{\substack{P \in \mathcal{K}_k \\ Q \in \mathcal{K}_{k'}}} c_1(P) c_2(Q) \widehat{P \times Q}.$$

The chain $c_1 \diamond c_2 \in C_{k+k'}^{d+d'}$ is called the *cubical product* of c_1 and c_2.

Given $k \in \mathbb{Z}$, the *cubical boundary map* $\partial_k\colon C_k^d \to C_{k-1}^d$ is a homomorphism defined on generators $\widehat{Q}$, where $Q \in \mathcal{K}_k^d$, by induction on the embedding number d as follows.

First let $d = 1$. Then $Q = [l] \in \mathcal{K}_0^1$ or $Q = [l, l+1] \in \mathcal{K}_1^1$ for some $l \in \mathbb{Z}$. Define

$$\partial_k \widehat{Q} := \begin{cases} 0 & \text{if } Q = [l], \\ \widehat{[l+1]} - \widehat{[l]} & \text{if } Q = [l, l+1]. \end{cases}$$

Let $d > 1$ and $Q = \prod_{i=1}^{d} I_i$, where I_i are intervals (some possibly degenerate) in $\mathbb{R}$. Put $I = I_1$ and $P = \prod_{i=2}^{d} I_i$. Then $\widehat{Q} = \widehat{I} \diamond \widehat{P}$. Define

$$\partial_k \widehat{Q} := \partial_{k_1} \widehat{I} \diamond \widehat{P} + (-1)^{\dim I} \widehat{I} \diamond \partial_{k_2} \widehat{P}, \tag{3}$$

where $k_1 = \dim I$ and $k_2 = \dim P$. Finally, we extend the definition to all chains by linearity. It is shown in [9] that cubical boundary maps satisfy the algebraic condition for a boundary map in an arbitrary chain complex, that is,

$$\partial_k \circ \partial_{k+1} = 0, \tag{4}$$

for all $k \in \mathbb{Z}$. Thus $\mathcal{C} := \{C_k, \partial_k\}_{k \in \mathbb{Z}}$ is a chain complex. We shall now localize this chain complex to cubical sets. The *support* of a chain $c \in C_k^d$ is the cubical set

$$|c| := \bigcup \{Q \in \mathcal{K}_k^d \mid c(Q) \neq 0\}.$$

Given a cubical set $X \subset \mathbb{R}^d$, we define

$$C_k(X) = \left\{ c \in C_k^d \;\middle|\; |c| \subset X \right\}. \tag{5}$$

$C_k(X)$ is a finitely generated free abelian group and the set $\{\widehat{Q} \mid Q \in \mathcal{K}_k(X)\}$ is a basis called its *canonical basis*. We also have $\partial_k(C_k(X)) \subset C_{k-1}(X)$. Hence, the restricted boundary map $\partial_k^X : C_k(X) \to C_{k-1}(X)$ is well defined and $\mathcal{C}(X) := \left\{C_k(X), \partial_k^X\right\}_{k \in \mathbb{Z}}$ is a chain complex called the *cubical chain complex* of X. When X is clear from the context, we will use the notation ∂_k for the restricted map ∂_k^X.

The *homology* of X is the collection $H_*(X) = \{H_k(X)\}_{k \in \mathbb{Z}}$ of quotient groups $H_k(X) := Z_k(X)/B_k(X)$, where $Z_k(X) := \ker \partial_k^X$ is the group of k-*cycles* of X and $B_k(X) := \operatorname{im} \partial_{k+1}^X$ is the group of k-*boundaries* of X.

2.3 Representable Sets

Note that any cubical set $X \subset \mathbb{R}^d$ is closed and bounded. Intersections and finite unions of cubical sets are cubical. We want to obtain a larger class of sets, closed under the subtraction $X \setminus Y$.

Given any elementary cube $Q = I_1 \times I_2 \times \cdots \times I_d$, the corresponding *elementary cell*

$$\mathring{Q} = \mathring{I}_1 \times \mathring{I}_2 \times \cdots \times \mathring{I}_d$$

is the set obtained by replacing all non-degenerate closed intervals $I_i = [l_i, l_i + 1]$ in the expression for Q by the open ones $\mathring{I}_i = (l_i, l_i + 1)$, while $\mathring{I}_i := I_i$ if $I_i = [l_i]$.

Proposition 2.2 *Elementary cells have the following properties:*

(i) $\mathbb{R}^d = \bigcup\{\overset{\circ}{Q} \mid Q \in \mathcal{K}^d\}$.
(ii) *If $A \subset \mathbb{R}^d$ is bounded, then the set $\{Q \in \mathcal{K}^d \mid \overset{\circ}{Q} \cap A \neq \varnothing\}$ is finite.*
(iii) *If $P, Q \in \mathcal{K}^d$, then $\overset{\circ}{P} \cap \overset{\circ}{Q} = \varnothing$ or $P = Q$.*
(iv) *For every $Q \in \mathcal{K}$, $\operatorname{cl} \overset{\circ}{Q} = Q$.*
(v) *$Q \in \mathcal{K}^d$ implies that $Q = \bigcup\{\overset{\circ}{P} \mid P \in \mathcal{K}^d$ such that $\overset{\circ}{P} \subset Q\}$.*
(vi) *If X is a cubical set and $\overset{\circ}{Q} \cap X \neq \varnothing$ for some elementary cube Q, then $Q \subset X$.*

A set $Y \subset \mathbb{R}^d$ is called *representable* if it is a finite union of elementary cells. The family of representable sets in $\mathbb{R}^d$ is denoted by $\mathcal{R}^d$.

Proposition 2.3 *Representable sets have the following properties:*

(i) *Every elementary cube is representable.*
(ii) *If $A, B \in \mathcal{R}^d$, then $A \cup B, A \cap B, A \setminus B \in \mathcal{R}^d$.*
(iii) *A set $X \subset \mathbb{R}^d$ is cubical if and only if it is closed and representable.*
(iv) *A bounded set $A \subset \mathbb{R}^d$ is representable if and only if for every $Q \in \mathcal{K}^d$, $\overset{\circ}{Q} \cap A \neq \varnothing$ implies $\overset{\circ}{Q} \subset A$.*

Let $A \subset \mathbb{R}^d$ be a bounded set. Then the *open hull* of A is

$$\operatorname{oh}(A) := \bigcup\{\overset{\circ}{Q} \mid Q \in \mathcal{K}, Q \cap A \neq \varnothing\}, \tag{6}$$

and the *closed hull* of A is

$$\operatorname{ch}(A) := \bigcup\{Q \mid Q \in \mathcal{K}, \overset{\circ}{Q} \cap A \neq \varnothing\}. \tag{7}$$

Proposition 2.4 *Assume $A \subset \mathbb{R}^d$. Then*

(i) $\operatorname{oh}(A) = \bigcap\{U \in \mathcal{R}^d \mid U$ *is open and* $A \subset U\}$.
(ii) $\operatorname{ch}(A) = \bigcap\{B \in \mathcal{R}^d \mid B$ *is closed and* $A \subset B\}$.

3 Cubical Topology

It is obvious that a union of a finite family of cubical sets in $\mathbb{R}^d$ is a cubical set and easy to show that the intersection of any family of cubical sets is a cubical set. Thus the following definition makes sense.

Definition 3.1 The *cubical topology* in $\mathbb{R}^d$ is defined by the family $\mathcal{V}^d$ of *closed sets* given by $\mathcal{V}^d := \{X \subseteq \mathbb{R}^d \mid X \text{ is a cubical set}\} \cup \{\varnothing, X\}$. More precisely, the family $\mathcal{T}^d$ of *open sets* called the *cubical topology* in $\mathbb{R}^d$ is given by $U \in \mathcal{T}^d$ if and only if $\mathbb{R}^d \setminus U \in \mathcal{V}^d$.

Note that open sets, the complements of cubical sets, are unbounded. In particular, representable sets which are open with respect to Euclidean topology are not open in cubical topology. This slight inconvenience may be avoided by restricting the topology to a fixed cubical set $X \subset \mathbb{R}^d$, which is always done in practical applications. Let $\mathcal{T}_{|X} := \{U \cap X \mid U \in \mathcal{T}^d\}$ be the relative cubical topology of X. The following is easily verified.

Proposition 3.2 *Let X be a cubical set. A set $U \subset X$ is in $\mathcal{T}_{|X}$ if and only if U is representable and open in the relative Euclidean topology of X.*

It is easy to see that the cubical topology does not satisfy any separation axioms. For example, points in the open interval $(1, 2) \subset \mathbb{R}$ cannot be separated in the sense of any axiom. We introduce the following refinement to two axioms of interest to us.

Definition 3.3 Let $(X, \mathcal{T})$ be a topological space and $x, y \in X$.

(i) The points x and y are *T_0-separable* if there exists $U \in \mathcal{T}$ which contains exactly one of those two points.

(ii) The points x and y are *T_1-separable* if there exist $U, W \in \mathcal{T}$ such that U contains x and not y and W contains y and not x.

(iii) The points x and y are *T_2-separable* or *Hausdorff-separable* if there exist $U, W \in \mathcal{T}$ such that $U \cap W = \varnothing$, $x \in U$ and $y \in W$.

Proposition 3.4 *Consider the cubical topology $\mathcal{T}^d$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.*

(i) *The points x and y are T_0-separable if and only if they are in distinct elementary cells $\mathring{P}$, $\mathring{Q}$.*

(ii) *The points x and y are T_1-separable if and only if they are in distinct elementary cells $\mathring{P}$, $\mathring{Q}$, such that neither $P \subset Q$ nor $Q \subset P$.*

(iii) *Let X be a cubical set with the restricted cubical topology $\mathcal{T}_{|X}$ and let $x, y \in X$. The points x and y are T_2-separable in X if and only if* $\operatorname{oh}(x) \cap \operatorname{oh}(y) = \varnothing$.

Proof (i) Suppose that x and y are T_0 separable, and let $U \in \mathcal{T}$ be a set with $x \in U$, $y \notin U$. Then $Y = \mathbb{R}^d \setminus U$ is a cubical set containing y. By Proposition 2.2(i) and Proposition 2.3(iii), both U and Y are unions of elementary cells, hence there exist $P, Q \in \mathcal{K}$ such that $x \in \mathring{P} \subset U$ and $y \in \mathring{Q} \subset Y$. Since U and Y are disjoint, so are $\mathring{P}$ and $\mathring{Q}$.

Now suppose that there exist distinct cells $\mathring{P}$ and $\mathring{Q}$ such that $x \in \mathring{P}$ and $y \in \mathring{Q}$. If $x \notin Q$, then we may take $U = \mathbb{R}^d \setminus Q$. Then $x \in U$ and $y \notin U$. If $y \notin P$, then we may take $U = \mathbb{R}^d \setminus P$, and the conclusion follows the same way. If neither of these assumptions hold, then $x \in Q \cap \mathring{P}$ and $y \in P \cap \mathring{Q}$. Then Proposition 2.3(iv) implies that $\mathring{P} \subset Q$ and $\mathring{Q} \subset P$. By Proposition 2.2(iv), $P = Q$, a contradiction.

(ii) Suppose that x and y are T_1 separable; let $U \in \mathcal{T}$ be a set with $x \in U$, $y \notin U$ and $W \in \mathcal{T}$ be a set with $y \in W$, $x \notin W$. Then $x \in U \setminus W$ and $y \in W \setminus U$. Since U and W are both unions of elementary cells, there are cells $\mathring{P}$ and $\mathring{Q}$ such that $x \in \mathring{P} \subset U \setminus W$ and $y \in \mathring{Q} \subset W \setminus U$. It remains to show that $P \not\subset Q$ and $Q \not\subset P$. We have $\mathring{Q} \cap U = \varnothing$. Since U is open in $\mathcal{T}$, it is also open in the Euclidean topology, and since $Q = \operatorname{cl} \mathring{Q}$, it follows that $Q \cap U = \varnothing$. If $P \subset Q$, we get a contradiction to $\mathring{P} \subset U$. The argument for $Q \not\subset P$ is analogous.

Now suppose that there exist distinct cells $\mathring{P}$ and $\mathring{Q}$ such that $x \in \mathring{P}$, $y \in \mathring{Q}$, $P \not\subset Q$, and $Q \not\subset P$. If $x \notin Q$ and $y \notin P$, then we may take $U = \mathbb{R}^d \setminus Q$, $W = \mathbb{R}^d \setminus P$ and the conclusion follows as in the proof of (i). If one of these assumptions fails, for example $x \in Q$, then we show, as in (i), that $\mathring{P} \subset Q$, so $P = \operatorname{cl} \mathring{P} \subset Q$, a contradiction.

(iii) By Proposition 2.4(i), $\mathrm{oh}(x)$ and $\mathrm{oh}(y)$ are the smallest open (in the Euclidean topology) representable sets containing, respectively, x and y. Therefore the conclusion follows from Proposition 3.2. ■

Cubical topology has analogous properties to Zariski topology. This analogy is exhibited in the following definitions and propositions.

Definition 3.5 A topological space $(X, \mathcal{T})$ is called *noetherian* if, given any decreasing family $V_1 \supset V_2 \supset V_3 \supset \cdots$ of closed sets, there exists an integer $n \geq 1$ such that $V_n = V_{n+j}$ for all $j \in \mathbb{N}$.

Proposition 3.6 *The space* $(\mathbb{R}^d, \mathcal{T}^d)$ *is noetherian.*

Proof Consider a decreasing sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ of closed sets. If $V_i = \mathbb{R}^d$ for all i or $V_n = \varnothing$ for a sufficiently large n, the conclusion is obviously satisfied, so we may assume that there exists k such that X_i is a cubical set for all $i \geq k$. In particular, X_k can be written as $X_k = \bigcup_{j=1}^{m} Q_j$, where $Q_j \in \mathcal{K}^d$. By Proposition 2.1, there exists at most $m3^d$ elementary cubes included in X_k and at most $2^{m3^d} - 2$ proper cubical subsets of X_2. Thus $V_{i+1} = V_i$ for all but finitely many i, and the conclusion follows. ■

Definition 3.7 Let $(X, \mathcal{T})$ be a topological space. A closed set $V \subset X$ is *irreducible* if, given any decomposition $V = V_1 \cup V_2$ with V_1, V_2 closed, we must have $V = V_1$ or $V = V_2$.

Proposition 3.8 *Let $V \in \mathcal{V}^d$. Then V is irreducible if and only if $V = \mathbb{R}^d$ or V is an elementary cube.*

Proof First, observe that $\mathbb{R}^d$ is irreducible, because all other elements of $\mathcal{V}^d$ are cubical sets. Cubical sets are bounded and $\mathbb{R}^d$ is not, so it cannot be a union of two cubical sets. Let $V \subset \mathbb{R}^d$ be an irreducible closed set. If $V \neq \mathbb{R}^d$, then V is a cubical set, so it may be written as a union of n elementary cubes, $V = \bigcup_{i=1}^{n} Q_i$, $Q_i \in \mathcal{K}^d$. We argue by induction on n that V is an elementary cube. If $n = 1$, $V = Q_1$ is an elementary cube. If $n > 1$, consider the decomposition $V = V_1 \cup V_2$ with $V_1 = Q_1$ and $V_2 = \bigcup_{i=2}^{n} Q_i$. Since V is irreducible, either $V = V_1$ or $V = V_2$ and the induction hypothesis applies to both cases.

Suppose that $V = Q$ is an elementary cube and consider its decomposition $Q = V_1 \cup V_2$ to two closed, hence cubical, subsets. Then the cell $\mathring{Q}$ intersects either V_1 or V_2, and the conclusion follows from Proposition 2.2(iv). ■

Proposition 3.9 *For any $V \in \mathcal{V}^d$ there exists a unique family of irreducible sets $\{V_k\}_{k=1,2,\ldots,n}$ such that $V_j \not\subset V_k$ for $j \neq k$ and $V = \bigcup_{k=1}^{n} V_k$.*

Proof The set $V = \mathbb{R}^d$ is irreducible, so we may assume that V is a cubical set, so it can be written as a finite union of elements of $\mathcal{K}^d$. By the definition of a maximal face, it can be written as $V = \bigcup\{Q \in \mathcal{K}^d \mid Q \in \mathcal{K}_{\max}(X)\}$. This union extends over a finite set, and it remains to show that it is unique. Suppose that $V = \bigcup_{k=1}^{n} V_k$, where V_k are irreducible and $V_j \not\subset V_k$. By Proposition 3.8, V_k is an elementary cube for each k. We need to show that $\{V_k \mid k = 1, \ldots, n\} = \mathcal{K}_{\max}(V)$. Suppose that $Q \in \mathcal{K}_{\max}(V)$. Since $\bigcup_{k=1}^{n} V_k = V$, there exists k such that $\mathcal{Q} \cap V_k \neq \varnothing$. By Proposition 2.2(vi), $Q \subset V_k$. Since Q is maximal, $Q = V_k$. Thus

$$\mathcal{K}_{\max}(V) \subset \{V_k \mid k = 1, \ldots, n\}. \tag{8}$$

The reverse inclusion is shown by contradiction. Suppose that $V_j \notin \mathcal{K}_{\max}(V)$ for some j. Then there exists $Q \in \mathcal{K}_{\max}(V)$ such that V_j is a proper face of Q. By (8), $Q = V_k$ for some k. But $V_j \not\subset V_k$, a contradiction. ■

We end this section with a remark that the statements of all propositions in this section hold true for relative cubical topology in a given cubical set.

4 Cubical Maps and Their Homology

4.1 Cubical Maps

Having recalled the definition of homology of a cubical set in Section 2, we now want to extend this definition to maps $f\colon X \to Y$ where $X, Y \subset \mathbb{R}^d$ are cubical sets. Following [6], we will define the homomorphism $H_*(f)$ induced in homology for a class of maps satisfying the following two conditions:

(i) $f(Q) \in \mathcal{K}(Y)$ for every $Q \in \mathcal{K}(X)$.
(ii) The restriction $f_{|Q}$ to every $Q \in \mathcal{K}(X)$ is affine linear.

These conditions somewhat mimic the definition of simplicial maps in the simplicial homology theory. The difference between these two classes of maps is that vertices of a simplex are affine independent, whereas vertices of an elementary cube are not. Thus, any simplicial vertex map admits a unique linear extension to each simplex, and the passage from a combinatorial concept of a simplicial vertex map to a topological concept of a piecewise continuous map is very natural. This is not true for maps defined on vertices of elementary cubes, which cause condition (ii) to be restrictive and not natural. However, the only purpose of this condition is to obtain a continuous map. When the cubical topology introduced in Section 3 is considered, condition (ii) is not necessary and the definition of a cubical map can be stated as follows.

Definition 4.1 Let X, Y be cubical sets. A map $f\colon X \to Y$ is called a *cubical map* if it is a continuous and closed map with respect to the relative cubical topology in X and Y.

Here is a more explicit equivalent formulation.

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Proposition 4.2 *Let X, Y be cubical sets. A map $f\colon X \to Y$ is a cubical map if and only if*

(i) $f^{-1}(Q)$ *is a cubical set for every* $Q \in \mathcal{K}(Y)$*,*
(ii) $f(Q) \in \mathcal{K}(Y)$ *for every* $Q \in \mathcal{K}(X)$*.*

Proof By definition of the relative cubical topology, f is continuous if and only if $f^{-1}(A)$ is a cubical set in X for every cubical set A in Y. Since every cubical set is a finite union of elementary cubes and a finite union of cubical sets is a cubical set, this is equivalent to (i).

Again by definition, f is a closed map if and only if $f(A)$ is a cubical set in Y for every cubical set A in X. By the previous arguments, this is equivalent to the condition that $f(Q)$ is a cubical set for every $Q \in \mathcal{K}(X)$. We show, by contradiction, that $f(Q)$ must be an elementary cube. Suppose that $f(Q)$ is a cubical set which is not an elementary cube. By Proposition 3.8, there are two cubical sets R and S, neither equal to $f(Q)$, such that $f(Q) = R \cup S$. Then $Q = f^{-1}(R) \cup f^{-1}(S)$. Since Q is irreducible and $f^{-1}(R)$ and $f^{-1}(S)$ are cubical, we must have $Q = f^{-1}(R)$ or $Q = f^{-1}(S)$, so $f(Q) = R$ or $f(Q) = S$, a contradiction. ■

The following property of cubical maps will be used later.

Proposition 4.3 *Let X, Y be cubical sets and $f\colon X \to Y$ a cubical map. For any $Q \in \mathcal{K}(X)$ $\dim f(Q) \leq \dim Q$.*

Proof We argue by induction on the dimension $k = \dim Q$. If $k = 0$, Q is a singleton and so is $f(Q)$, hence $\dim f(Q) = \dim Q = 0$. Suppose that the conclusion holds for a given $k \geq 0$. Let $Q \in \mathcal{K}_{k+1}(X)$ and $m = \dim f(Q)$. If $m = 0$, we are done. If $m > 0$, there are two opposite faces P_+ and P_- of $f(Q)$ of dimension $m-1$. Since P_+ and P_- are disjoint elementary cubes, $f^{-1}(P_+)$ and $f^{-1}(P_-)$ are two disjoint proper cubical subsets of Q. Therefore, $\dim f^{-1}(P_+) \leq k$ and $\dim f^{-1}(P_-) \leq k$. By induction hypothesis, $k \leq m-1$, so $\dim Q = k+1 \leq m = \dim f(Q)$. ■

The identity map id_X obviously is a cubical map, and it is easy to check that the composition $g \circ f$ of two cubical maps is a cubical map. Thus we may form a category Cub whose objects are cubical sets and morphisms are cubical maps.

Note that cubical maps are not necessarily continuous with respect to the Euclidean topology. For example, any surjective function $f\colon [0,1] \to [0,1]$ such that $f^{-1}(\{0,1\}) = \{0,1\}$ is a cubical map. We can modify values of a cubical function inside elementary cells freely, as long as images of elementary cubes remain the same. Therefore it makes sense to define an equivalence relation for cubical maps $f, g\colon X \to Y$ by setting $f \sim g$ if and only if $f(Q) = g(Q)$ for all $Q \in \mathcal{K}(X)$. The equivalence class of f is called the *cubical class* of f. We will soon see that any cubical map contains in its cubical class a representative which is continuous and whose restriction to any elementary cube is affine linear, that is, a linear map possibly composed with a translation. Before proceeding further, it is helpful to have some examples of cubical maps.

Example 4.4 An inclusion of cubical sets $i\colon A \hookrightarrow X$ is a cubical map. The following maps of the Euclidean space, when restriced to a cubical set and its image, become cubical maps:

(i) Projection, $p\colon \mathbb{R}^d \to R^{d-1}$, $p(\mathbf{x}) = (x_2, x_3, \dots, x_d)$;
(ii) Coordinate immersion, $j\colon \mathbb{R}^d \to \mathbb{R}^{d+1}$; $j(\mathbf{x}) = (m, x_1, x_2, \dots, x_d)$, $m \in \mathbb{Z}$;
(iii) Translation, $\mathbf{x} \mapsto \mathbf{m} + \mathbf{x}$, where $\mathbf{m} \in \mathbb{Z}^d$;
(iv) Transpose, $(x_i, x_{i+1}) \mapsto (x_{i+1}, x_i)$;
(v) Inversion, $x_i \mapsto -x_i$.

A composition of cubical maps is a cubical map, hence more maps can be generated from the above examples. We proceed towards an explicit formula which implies, in particular, that any cubical map can be obtained by composing the maps listed in Example 4.4, up to the cubical equivalence class.

In the sequel, the following notation is be helpful. We first put $\mathbb{N}_d = \{1, 2, \dots, d\}$. Next, given an elementary cube $Q = I_1 \times I_2 \times \dots \times I_d \in \mathcal{K}^d$, we put

$$\operatorname{ess}(Q) := \{i \in \mathbb{N}_d \mid I_i \text{ is non-degenerate}\}.$$

Theorem 4.5 *Let $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d'}$ be cubical sets. The cubical class of any cubical map $g\colon X \to Y$ contains a map f with the following property. For all $Q \in \mathcal{K}(X)$, the restriction to Q of $f = (f_1, f_2, \dots, f_{d'})$ can be expressed coordinate-wise by the formula*

$$f_i(\mathbf{x}) = m_i + \epsilon_i x_{\mu(i)}, \tag{9}$$

where $i \in \mathbb{N}_{d'}$, $m_i \in \mathbb{Z}$, $\epsilon_i \in \{-1, 0, 1\}$ and μ is a function from $\mathbb{N}_{d'}$ to $\mathbb{N}_d$. Moreover, ϵ_i and m_i are uniquely determined by i, $\mu(i)$ is uniquely determined by i unless $\epsilon_i = 0$, and the function $\nu_{f,Q}\colon \operatorname{ess}(f(Q)) \to \operatorname{ess}(Q)$ such that $\nu_{f,Q}(k) = \mu(k)$ is injective and uniquely determined by f and Q. Conversely, any map defined on elementary cubes in X by (9) is a cubical map.

Proof The construction of f on each elementary cube Q proceeds by induction on $k = \dim Q$.

Let $k = 0$. Then $g(Q) \in \mathcal{K}_0^{d'}(Y)$ by Proposition 4.3, so we may write

$$g(Q) = \prod_{i=1}^{d'} [l_i], \ l_i \in \mathbb{Z}.$$

Hence $f_i(Q) = l_i + 0$ is a unique function of the form (9) except that μ is arbitrary because $\epsilon_i = 0$. The function $\nu_{g,Q}$ is not defined in this case, because $\operatorname{ess}(g(Q)) = \varnothing = \operatorname{ess}(Q)$. Thus we may put $f_{|Q} := g_{|Q}$.

Suppose that the construction is done for all elementary cubes of dimension $k \geq 0$ so that the restriction of g and f to the k-th skeleton of X,

$$X^{(k)} = \bigcup \{Q \in \mathcal{K}_i(X) \mid i \leq k\},$$

satisfies the conclusion of the theorem. Consider $Q \in \mathcal{K}_{k+1}(X)$.

If $\mathrm{ess}(g(Q)) = \varnothing$, we get as previously, $g_i(Q) = l_i{+}0$ and $f_{|Q} := g_{|Q}$. If $\mathrm{ess}(g(Q)) \neq \varnothing$, choose $n \in \mathrm{ess}(g(Q))$ and let $g_n(Q) = [r_n, r_n + 1]$. Put

$$\begin{aligned} P &= g(Q), \\ P_0 &= g_1(Q) \times g_2(Q) \times \cdots \times g_{n-1}(Q) \times [r_n] \times g_{n+1}(Q) \times \cdots \times g_{d'}(Q), \\ Q_0 &= Q \cap g^{-1}(P_0). \end{aligned}$$

Since P_0 is a proper face of P, $Q_0 \subsetneq Q$. It follows from Proposition 3.8 that Q_0 is an elementary cube. Indeed, suppose that R_1, R_2 are cubical sets such that $Q_0 = R_1 \cup R_2$. Then $g(R_1) \cup g(R_2) = P_0$, but g is a cubical map and P_0 is an elementary cube, hence $g(R_1) = P_0$ or $g(R_2) = P_0$. Consequently, $R_1 = Q_0$ or $R_2 = Q_0$. Hence, Q_0 is an elementary cube and a proper face of Q. We show that $\dim Q_0 = k$. Indeed, if $\dim Q_0 < k$, there exists $R_0\mathcal{K}(X)$ such that $Q_0 \subsetneq R_0 \subsetneq Q$, and then $g(Q_0) = P_0 \subsetneq g(R_0) \subsetneq g(Q) = P$. This is impossible, because the three sets are elementary cubes and $\dim g(Q) = \dim P_0 + 1$. Thus $\dim Q_0 = \dim P_0 = k$.

By the induction hypothesis, f_{Q_0} is defined coordinate-wise by formulas $f_{0i}(\mathbf{x}) = m_{0i} + \epsilon_{0i} x_{\mu_0(i)}$ for $i \in \mathbb{N}_{d'}$, and ϵ_{0i}, m_{0i}, and $\mu_0(i)$ are uniquely determined by i. Since $\dim_{Q_0} = k$, there is a unique $j \in \mathrm{ess}(Q)$ such that $j \notin \mathrm{ess}(Q_0)$. The j-th component I_j of Q can be written as $I_j = [l_j, l_j + 1]$, and the j-th component I_{0j} of Q_0 is either $[l_j]$ or $[l_j + 1]$.

In the case $I_{0j} = [l_j]$, we put $f_i(\mathbf{x}) = f_{0i}(\mathbf{x})$ for all $i \neq n$ and $f_n(\mathbf{x}) = r_n - l_j + x_j$. This uniquely determines $m_n = r_n - l_j$, $\epsilon_n = 1$ and $\mu(n) = j$.

In the case $I_{0j} = [l_j + 1]$, we put $f_i = f_{0i}$ for all $i \neq n$ and $f_n = r_n + 1 + l_j - x_j$. This uniquely determines $m_n = r_n + 1 + l_j$, $\epsilon_n = -1$ and $\mu(n) = j$.

We show that $\nu_{f,Q} \colon \mathrm{ess}(f(Q)) \to \mathrm{ess}(Q)$ such that $\nu_{f,Q}(k) = \mu(k)$ is an injective function. Consider $a, b \in \mathrm{ess}(f(Q))$, $a \neq b$. If $a \neq n$ and $b \neq n$, then $a, b \in \mathrm{ess}(f(Q_0))$ and, by induction hypothesis, $a \neq b$ implies $\nu(a) \neq \nu(b)$. If $a = n \neq b$, then $\nu(a) = \nu(n) = j$ and $b \in \mathrm{ess}(f(Q_0))$. However, j is not in the image of ν_{f,Q_0}, hence $\nu(a) \neq \nu(b)$.

The converse statement is obvious. ■

Note that the coordinate function f_i in the formula (9) for a given elementary cube Q depends only on one coordinate of $\mathbf{x}$, namely $x_{\mu(i)}$. Thus, we may introduce cubical functions $f^i \colon I_{\mu(i)} \to J_i$ defined on elementary intervals appearing in $Q = \prod_{j=1}^{d} I_j$, $f(Q) = \prod_{i=1}^{d'} J_i$, given by

$$f^i(t) = m_i + \epsilon_i t. \tag{10}$$

With the help of these one-dimensional functions, the formulas (9) for $i \in N_{d'}$ can be replaced by the formula

$$f(\mathbf{x}) = (f^1(x_{\mu(1)}), f^2(x_{\mu(2)}), \ldots, f^{d'}(x_{\mu(d')})). \tag{11}$$

It is clear that the maps defined by (9) and (11) are affine linear on each elementary cube, and since the formulas coincide on common faces of elementary cubes, they extend to a map $f \colon X \to Y$ which is continuous in Euclidean topology. Thus every cubical class contains a representative which is continuous in the traditional sense.

4.2 Induced Chain Maps

We shall now proceed towards the definition of $H_*(f)$, the homomorphism induced by a cubical map f. First, let us introduce the following notation. Given $A, B \subset \mathbb{N}$ and a function $\alpha\colon A \to B$, we define its sign by

$$\operatorname{sgn}\alpha := \begin{cases} (-1)^{\operatorname{card}\{(i,j)\in A^2 \mid \alpha_j<\alpha_i\}} & \text{if } \alpha \text{ is bijective,} \\ 0 & \text{otherwise,} \end{cases}$$

where card stands for the number of elements of a set.

Definition 4.6 Let $f\colon X \to Y$ be a cubical map, $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d'}$ cubical sets. The *homomorphism induced by f on k-chains* $f_{\#k} : C_k(X) \to C_k(Y)$ is defined on the generators $\widehat{Q} \in \mathcal{K}_k^d(X)$ by induction on d as follows.

(i) Let $k = 0$ and $d = 1$. Then $Q = [l]$ for some $l \in \mathbb{Z}$ and we put

$$f_{\#0}(\widehat{[l]}) = \widehat{[f(l)]}.$$

(ii) Let $k = 1$ and $d = 1$. Then $Q = [l, l+1]$ for some $l \in \mathbb{Z}$ and we put

$$f_{\#1}(\widehat{[l, l+1]}) = \begin{cases} \widehat{[f(l), f(l+1)]} & \text{if } f(l) < f(l+1), \\ -\widehat{[f(l+1), f(l)]} & \text{if } f(l) > f(l+1), \\ 0 & \text{if } f(l) = f(l+1). \end{cases}$$

(iii) Let $d > 1$, $Q = \prod_{i=1}^d I_i$, $\dim f(Q) = n$, and let $l_1 < l_2 < \cdots < l_n$ be the indices in $\operatorname{ess}(f(Q))$. We define

$$f_{\#k}(\widehat{Q}) = \operatorname{sgn}(f, Q) \mathop{\diamond}_{i=1}^{n} f_{\#1}^i(\widehat{I}_{\nu_{f,Q}(l_i)}) \tag{12}$$

where $\operatorname{sgn}(f, Q) := \operatorname{sgn}(\nu_{f,Q})$ is defined in Theorem 4.5, and f^i is defined in (10).

Note that the image of a k-chain is always a k-chain because if $\dim Q \neq \dim f(Q)$, then $\operatorname{sgn}(f, Q) = 0$.

Theorem 4.7 *The family of homomorphisms $f_\# := \{f_{\#k}\}\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ is a* chain map*, that is, it commutes with the boundary operator. More explicitly, for any $k \in \mathbb{N}, k \neq 0$ we have*

$$\partial_k \circ f_{\#k} = f_{\#k-1} \circ \partial_k. \tag{13}$$

Proof Obviously it is enough to verify (13) on elements of the canonical basis $\widehat{Q} \in \mathcal{C}(X)$. If $\operatorname{emb} Q = 1$, the verification is straightforward. Thus assume $Q = \prod_{i=1}^d I_i$ with $d > 1$. Let $I_i = [a_i, b_i]$ for some $a_i \in \mathbb{Z}$ and $b_i \in \{a_i, a_i+1\}$. Let $\mu\colon \mathbb{N}_{d'} \to \mathbb{N}_d$ be as in Theorem 4.5. Let $A := \operatorname{ess}(Q)$, $B := \operatorname{ess}(f(Q))$ and $\nu := \nu(f, Q) = \mu_{|B} \colon B \to A$. If ν is not a bijection, then one easily verifies that both sides of (13) are zero. Thus assume that ν is a bijection. For $i \in B$ put

$$
\begin{aligned}
s_i &:= \operatorname{card}\{\nu(j) \in A \mid j < i\} = \operatorname{card}\{j \in B \mid j < i\}, \\
p_i &:= \operatorname{card}\{j \in B \mid j < i \text{ and } \nu(j) > \nu(i)\}, \\
n_i &:= \operatorname{card}\{j \in B \mid j > i \text{ and } \nu(j) < \nu(i)\}, \\
t_i &:= \operatorname{card}\{j \in B \mid \nu(j) < \nu(i)\} = \operatorname{card}\{\nu(j) \in A \mid \nu(j) < \nu(i)\}, \\
B_i &:= B \setminus \{i\}, \\
r_i &:= \operatorname{card}\left\{(l, m) \in B_i^2 \mid l < m \text{ and } \nu(l) > \nu(m)\right\}, \\
\gamma_i &:= (-1)^{\sum_{j=1}^{i-1} \dim f_\#^j(\widehat{I}_{\nu(j)})}, \\
\epsilon_i &:= \operatorname{sgn}(f, Q)\gamma_i .
\end{aligned}
$$

Since ν is bijective, $\dim f_\#^i(\widehat{I}_{\nu(i)}) = \dim \widehat{I}_{\nu(i)}$, therefore

$$\gamma_i = (-1)^{\sum_{j=1}^{i-1} \dim \widehat{I}_{\nu(j)}} = (-1)^{s_i}.$$

Let

$$Q_a^i := \prod_{j=1}^{i-1} I_j \times [a_i] \times \prod_{k=i+1}^{d} I_k, \quad Q_b^i := \prod_{j=1}^{i-1} I_j \times [b_i] \times \prod_{k=i+1}^{d} I_k.$$

Note that $\operatorname{ess}(Q_a^i) = B_i = \operatorname{ess}(Q_b^i)$, therefore $\operatorname{sgn}(f, Q_a^i) = (-1)^{r_i} = \operatorname{sgn}(f, Q_b^i)$. Denote this common value by δ_i. We have

$$\operatorname{sgn}(f, Q) = (-1)^{\operatorname{card}\left\{(l,m)\in B^2 \mid l<m \text{ and } \nu(l)>\nu(m)\right\}} = (-1)^{r_i+p_i+n_i}.$$

Therefore $\delta_i = (-1)^{r_i} = \operatorname{sgn}(f, Q)(-1)^{p_i+n_i}$. Consequently,

$$\epsilon_i\delta_i = \operatorname{sgn}(f, Q)^2(-1)^{s_i+p_i+n_i} = (-1)^{t_i}.$$

From equation (3) and Definition 4.6 we get

$$
\begin{aligned}
(\partial \circ f)_\#(\widehat{Q}) &= \sum_{i=1}^{d} \epsilon_i \Big(\mathop{\Diamond}_{h=1}^{i-1} f_\#^h(\widehat{I}_{\mu(h)}) \Big) \diamond \partial f_\#^i(\widehat{I}_{\mu(i)}) \diamond \Big(\mathop{\Diamond}_{h=i+1}^{d} f_\#^h(\widehat{I}_{\mu(h)}) \Big) \\
&= \sum_{i\in B} \epsilon_i \Big(\mathop{\Diamond}_{h=1}^{i-1} f_\#^h(\widehat{I}_{\nu(h)}) \Big) \diamond f_\#^i(\widehat{[b_{\nu(i)}]}) \diamond \Big(\mathop{\Diamond}_{h=i+1}^{d} f_\#^h(\widehat{I}_{\nu(h)}) \Big) \\
&\quad - \sum_{i\in B} \epsilon_i \Big(\mathop{\Diamond}_{h=1}^{i-1} f_\#^h(\widehat{I}_{\nu(h)}) \Big) \diamond f_\#^i(\widehat{[a_{\nu(i)}]}) \diamond \Big(\mathop{\Diamond}_{h=i+1}^{d} f_\#^h(\widehat{I}_{\nu(h)}) \Big)
\end{aligned}
$$

$$= \sum_{i\in B} \epsilon_i \delta_i f_\# \Big(\big(\mathop{\Diamond}_{h=1}^{i-1} \widehat{I}_{\nu(h)} \big) \diamond \widehat{[b_{\nu(i)}]} \diamond \big(\mathop{\Diamond}_{h=i+1}^{d} \widehat{I}_{\nu(h)} \big) \Big)$$

$$- \sum_{i\in B} \epsilon_i \delta_i f_\# \Big(\big(\mathop{\Diamond}_{h=1}^{i-1} \widehat{I}_{\nu(h)} \big) \diamond \widehat{[a_{\nu(i)}]} \diamond \big(\mathop{\Diamond}_{h=i+1}^{d} \widehat{I}_{\nu(h)} \big) \Big)$$

$$= f_\# \Big(\sum_{i\in B} (-1)^{t_i} \big(\mathop{\Diamond}_{h=1}^{i-1} \widehat{I}_{\nu(h)} \big) \diamond \partial \widehat{I}_{\nu(i)} \diamond \big(\mathop{\Diamond}_{h=i+1}^{d} \widehat{I}_{\nu(h)} \big) \Big)$$

$$= f_\# \Big(\sum_{l\in A} (-1)^{\operatorname{card}\{m\in A \mid m<l\}} \big(\mathop{\Diamond}_{h=1}^{l-1} \widehat{I}_h \big) \diamond \partial \widehat{I}_l \diamond \big(\mathop{\Diamond}_{h=l+1}^{d} \widehat{I}_h \big) \Big)$$

$$= f_\# \Big(\sum_{l=1}^{d} (-1)^{\sum_{j=1}^{l-1} \dim \widehat{I}_j} \big(\mathop{\Diamond}_{h=1}^{l-1} \widehat{I}_h \big) \diamond \partial \widehat{I}_l \diamond \big(\mathop{\Diamond}_{h=l+1}^{i-1} \widehat{I}_h \big) \Big)$$

$$= f_\# \circ \partial(\widehat{Q}).$$

∎

The correctness of the following definition is a standard consequence of the property (13) of any chain map on chain complexes.

Definition 4.8 Let $f\colon X \to Y$ be a cubical map and $f_\#\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ the induced chain map. The homomorphism $H_k(f)\colon H_k(X) \to H_k(Y)$ induced by $f_\#$ on quotient groups is called the the *k-th homology of f*. The family of maps $H_*(f) = \{H_k(f)\}\colon H_*(X) \to H_*(Y)$ is called the *homology map* of f.

Lemma 4.9 *The definition of a chain map induced by a cubical map is functorial, in the following sense.*

(i) *Given a cubical set X,* $(\mathrm{id}_X)_{\#k} = \mathrm{id}_{C_k(X)}$ *for all k.*
(ii) *Given two cubical maps* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *on cubical sets,* $g_{\#k} \circ f_{\#k} = (g \circ f)_{\#k}$ *for all k.*

Proof Let $X \subset \mathbb{R}^d, Y \subset \mathbb{R}^{d'}$ and $Z \subset \mathbb{R}^{d''}$. It is enough to verify (i) and (ii) on elements of the canonical basis $\widehat{Q} \in \mathcal{K}_k^d(X)$. Put $Q = \prod_{i=1}^d I_i$.

(i) Since $\nu_{\mathrm{id}_{X,Q}}\colon \mathbb{N}_d \to \mathbb{N}_d$ is the identity, we have

$$(\mathrm{id}_X)_{\#k}(\widehat{Q}) = \operatorname{sgn}(\mathrm{id}_X, Q) \mathop{\Diamond}_{i=1}^{d} (\mathrm{id}_X)_{\#1} \widehat{I}_{\nu(i)} = \mathop{\Diamond}_{i=1}^{d} \widehat{I}_i = \widehat{Q}.$$

(ii) First let $d = d' = d'' = 1$. If $\dim Q = 0$ then $Q = [l]$ for some $l \in \mathbb{Z}$ and

$$\big(g_{\#0} \circ f_{\#0}\big)\,[l] = g_{\#0}\widehat{[f(l)]} = [\widehat{(g \circ f)(l)}] = (g \circ f)_{\#0}\widehat{[l]}.$$

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If $\dim Q = 1$, then $Q = [l, l+1]$ for some $l \in \mathbb{Z}$. Put $a = f(l)$, $b = f(l+1)$, $c = g(a)$, and $d = g(b)$. We have

$$
\begin{aligned}
(g_{\#1} \circ f_{\#1})(\widehat{[l, l+1]}) &= \begin{cases} g_{\#1}\widehat{[a,b]} & \text{if } a < b, \\ -g_{\#1}\widehat{[b,a]} & \text{if } b < a, \\ g_{\#1}0 & \text{if } a = b, \end{cases} \\
&= \begin{cases} \widehat{[c,d]} & \text{if } a < b \text{ and } c < d, \\ -\widehat{[d,c]} & \text{if } a < b \text{ and } d < c, \\ -\widehat{[d,c]} & \text{if } b < a \text{ and } d < c \\ \widehat{[c,d]} & \text{if } b < a \text{ and } c < d, \\ 0 & \text{otherwise}, \end{cases} \\
&= \begin{cases} \widehat{[c,d]} & \text{if } c < d, \\ -\widehat{[d,c]} & \text{if } d < c, \\ 0 & \text{otherwise}, \end{cases} \\
&= (g \circ f)_{\#1}(\widehat{[l, l+1]}).
\end{aligned}
$$

Now let $d, d', d'' \geq 1$, not all equal to 1. We use abbreviations $\nu_f = \nu_{f,Q}$ and $\nu_g = \nu_{f(Q)}g$. We let $k_1 < k_2 < \cdots < k_m$ be the essential indices of $f(Q)$ and $l_1 < l_2 < \cdots < l_n$ the essential indices of $g(f(Q))$.

By the linearity of $g_\#$ and by (12),

$$
\begin{aligned}
(g_{\#k} \circ f_{\#k})(\widehat{Q}) &= g_{\#k}\big(\operatorname{sgn}(f, Q) \mathop{\Diamond}_{i=1}^{m} f_{\#k}^{k_i} \widehat{I}_{\nu_f(k_i)}\big) \\
&= \operatorname{sgn}(\nu_f)\, g_{\#k}\big(\mathop{\Diamond}_{i=1}^{m} f_{\#k}^{k_i} \widehat{I}_{\nu_f(k_i)}\big) \\
&= \operatorname{sgn}(\nu_f)\ \operatorname{sgn}(\nu_g) \mathop{\Diamond}_{j=1}^{n} g_{\#k}^{l_j}\big(f_{\#k}^{\nu_f(l_j)}(\widehat{I}_{\nu_f(\nu_g(l_j))})\big).
\end{aligned}
$$

Since $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$ for any permutations σ, τ, using the result proved in the case $d = d' = d'' = 1$, we get

$$
\begin{aligned}
(g_{\#k} \circ f_{\#k})(\widehat{Q}) &= \operatorname{sgn}(\nu_f \circ \nu_g)\big(\mathop{\Diamond}_{j=1}^{n} (g^{l_j} \circ f^{\nu(l_j)})_{\#k} \widehat{I}_{\nu_f \circ \nu_g(l_j)}\big) \\
&= \operatorname{sgn}(\nu_f \circ \nu_g)\big(\mathop{\Diamond}_{j=1}^{n} (g \circ f)_{\#k}^{l_j} \widehat{I}_{\nu_f \circ \nu_g(l_j)}\big) \\
&= \operatorname{sgn}(\nu_{g \circ f})\big(\mathop{\Diamond}_{j=1}^{n} (g \circ f)_{\#k}^{l_j} \widehat{I}_{\nu_{f \circ g}(l_j)}\big) = (g \circ f)_{\#k}. \qquad \blacksquare
\end{aligned}
$$

By standard homological algebra arguments, Lemma 4.9 implies the following.

Theorem 4.10 *H_* is a functor from* Cub *to the category of graded groups. More explicitly:*

(i) *Given a cubical set X, $H_*(\mathrm{id}_X) = \mathrm{id}_{H_*(X)}$.*
(ii) *Given cubical maps $f\colon X \to Y$ and $g\colon Y \to Z$ on cubical sets, $H_*(g) \circ H_*(f) = H_*(g \circ f)$.*

The following examples are related to first three maps in Example 4.4.

Example 4.11 Assume $A \subset X$ are cubical sets. If $i\colon A \to X$ is the inclusion map, then $i_\#\colon \mathcal{C}(A) \to \mathcal{C}(X)$ is also an inclusion map.

Example 4.12 Consider the elementary cubes $Q = [0, 1]^d$, $Q' = [0, 1]^{d-1}$ and the projection map $p\colon Q \to Q'$ given by $p(x_1, x_2, x_3, \ldots, x_d) := (x_2, x_3, \ldots, x_d)$. Any face P of Q can be written as $P = I_1 \times P'$, where I_1 can be $[0, 1]$, $[0]$, or $[1]$, and $P' = p(P)$ is a complementary face of P. The induced chain map $p_\#\colon \mathcal{C}(Q) \to \mathcal{C}(Q')$ is given by

$$\pi_{\#k}(\widehat{P}) := \begin{cases} \widehat{P'} & \text{if } I_1 = [0] \text{ or } I_1 = [1], \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Example 4.13 Let Q and Q' be as in Example 4.12. The map $j\colon Q' \to Q$ given by

$$j(x_1, x_2, x_3, \ldots, x_{d-1}) := (0, x_1, x_2, \ldots, x_{d-1})$$

is a cubical map, and the induced chain map $p_\#\colon \mathcal{C}(Q') \to \mathcal{C}(Q)$ is given by

$$j_{\#k}(c) := \widehat{[0]} \diamond c.$$

Note that $pj = \mathrm{id}_{Q'}$, therefore $p_\# j_\# = (pj)_\# = \mathrm{id}_{\mathcal{C}(Q')}$. Next, $j_\# p_\# = (jp)_\#$ is chain homotopic to $\mathrm{id}_{\mathcal{C}(Q)}$, with the chain homotopy $D_k\colon C_k(Q) \to C_{k+1}(Q)$ given by

$$D_k(\widehat{P}) := \begin{cases} \widehat{[0,1]} \diamond \widehat{P'} & \text{if } I_1 = [1], \\ 0 & \text{if } I_1 = [0], \\ 0 & \text{if } I_1 = [0,1], \end{cases}$$

where $P = I_1 \times P' \in \mathcal{K}(Q)$ is as in Example 4.12. It follows that $H_*(pj)\colon H_*(Q') \to H_*(Q)$ is the inverse of $H_*(pj)\colon H_*(Q) \to H_*(Q')$. Consequently, $H_*(Q') \cong H_*(Q)$.

From the result presented in Example 4.13, one can conclude, by induction on d, that $Q = [0, 1]^d$ is *acyclic*, that is, its homology is isomorphic to homology of a point:

$$H_k(Q) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is probably the simplest way of proving it without using the homotopy invariance theorem, whose proof is more involved.

4.3 Combinatorial Vietoris Theorem

Here is a combinatorial version of the Vietoris theorem [7].

Theorem 4.14 *Let X, Y be cubical sets and $f\colon X \to Y$ a cubical map. If f is surjective and $f^{-1}(Q)$ is acyclic for each $Q \in \mathcal{K}(Y)$, then $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism.*

Proof We construct by induction a chain map $\psi = \{\psi_k : C_k(X) \to C_k(Y)\}$ and we prove that it is a homological inverse of f_*.

For $k = 0$, let $\widehat{Q} \in C_0(X)$ be an elementary 0-chain. Since f is surjective and cubical, there exists a $P \in \mathcal{K}_0(X)$ such that $f(P) = Q$. Then $f_\#(\widehat{P}) = \widehat{Q}$ and we put $\psi_0(\widehat{Q}) := \widehat{P}$.

Suppose now that $k \geq 1$ and that $\psi_i\colon C_i(X) \to C_i(Y)$ has been constructed for $i = 1, 2, \ldots, k-1$ so that

$$|\psi_i(\widehat{Q})| \subset f^{-1}(Q) \text{ for all } Q \in \mathcal{K}_i(Y), \tag{15}$$

$$\psi_{i-1}\partial_i = \partial_i\psi_i. \tag{16}$$

Note that then for any $Q \in \mathcal{K}_k(Y)$,

$$|\psi_{k-1}(\partial\widehat{Q})| \subset f^{-1}\big(|\partial\widehat{Q}|\big) \subset f^{-1}\big(|\widehat{Q}|\big) \subset f^{-1}(Q).$$

By the induction hypothesis, $\psi_{k-1}\partial\widehat{Q} \in \mathbb{Z}_{k-1}(f^{-1}(Q))$. Since $f^{-1}(Q)$ is acyclic, its reduced homology $\tilde{H}_*(f^{-1}(Q))$ is zero. Therefore, there exists a $c \in C_k(f^{-1}(Q))$ such that $\partial c = \psi_{k-1}\partial\widehat{Q}$. In the case $k > 1$, this is straightforward. In the case $k = 1$, it follows from the fact that Q is an interval so, by the definition of Ψ_0, $\psi_0\partial\widehat{Q}$ is a difference of two vertices, thus it is a reduced cycle. We put $\psi_k\widehat{Q} := c$.

Thus the map ψ is constructed. We will show now that

$$f_\# \circ \psi = \mathrm{id}_{C(Y)}\,. \tag{17}$$

The proof is again by induction. For $k = 0$, the assertion follows immediately from the definition of ψ_0. Suppose that $\psi_{\#i} \circ \psi_i = \mathrm{id}_{C_i(Y)}$ for $0 \leq i \leq k-1$. Given any $Q \in \mathcal{K}_k(Y)$, we have

$$\partial\widehat{Q} = f_{\#k-1} \circ \psi_{k-1}(\partial\widehat{Q}) = \partial_k \circ f_{\#k} \circ \psi_k(\widehat{Q})$$

and, by the definition of ψ,

$$|f_{\#k} \circ \psi(\widehat{Q})| \subset f(|\psi(\widehat{Q})|) \subset f(f^{-1}(Q)) \subset Q.$$

Therefore $f_{\#k} \circ \psi_k(\widehat{Q})$ is a k-chain which has the same boundary as $\widehat{Q}$. It follows that $f_{\#k} \circ \psi_k(\widehat{Q}) - \widehat{Q}$ is a cycle in Q. However, $\tilde{H}_*(Q) = 0$, hence, every k-cycle in Q is a boundary. Since $\dim Q = k$, the only $(k+1)$-dimensional boundary in Q is zero. Thus $f_{\#k} \circ \psi(\widehat{Q}) = \widehat{Q}$.

In the last step, we will show that $\psi \circ f_\#$ is chain homotopic to $\mathrm{id}_{C(X)}$. To do this we construct by induction a chain homotopy $D = \{D_i \colon C_i(X) \to C_{i+1}(X)\}$ such that

$$\partial_{i+1} \circ D_i + D_{i-1} \circ \partial_i = \psi_i \circ f_{\# i} - \mathrm{id}_{C_i(X)}, \tag{18}$$

$$|D_i(c)| \subset f^{-1}(Q) \text{ for any } c \in C_i(f^{-1}(Q)) \text{ and } Q \in \mathcal{K}_i(Y). \tag{19}$$

Let $k = 0$ and take any $P \in \mathcal{K}_0(X)$. Let $Q := f(P)$ and let $c := \psi_0(\widehat{Q})$. Then $\psi_0 \circ f_\#(\widehat{P}) = c$. Since $|c| \cup |\widehat{P}| \subset f^{-1}(Q)$, we have $|c - \widehat{P}| \subset f^{-1}(Q)$. Since $\tilde{H}_*(f^{-1}(Q)) = 0$, there exists a $c' \in C_1(f^{-1}(Q))$ such that

$$\partial c' = c - \widehat{P} = \left(\psi_0 \circ f_{\#0} - \mathrm{id}_{C_0(X)}\right)(\widehat{P}).$$

We put $D_0(\widehat{P}) := c'$.

Now suppose that for $i = 0, 1, 2, \dots, k-1$ the maps $D_i \colon C_i(X) \to C_{i+1}(X)$ are constructed so that properties (18) and (19) are satisfied. Take any $P \in \mathcal{K}_k(X)$. Let $Q := f(P)$ and let $c := \psi_k(\widehat{Q}) = \psi_k(f_\#(\widehat{P}))$. Since both $|\widehat{P}|$ and $|c|$ are in $f^{-1}(Q)$, the induction hypothesis (19) and the subadditivity of support in [9, Ch. 2, Proposition 2.19(iv)] imply that

$$|c - \widehat{P} - D_{k-1}\partial_k \widehat{P}| \subset |c| \cup |\widehat{P}| \cup |D_{k-1}\partial_k \widehat{P}| \subset f^{-1}(Q).$$

Since $\tilde{H}_*(f^{-1}(|Q|) = 0$, there exists a $c' \in C_{k+1}(f^{-1}(Q))$ such that

$$\partial c' = c - \widehat{P} - D_{k-1}\partial_k \widehat{P} = \left(\psi_k \circ f_{\# k} - \mathrm{id}_{C_k(X)} - D_{k-1}\partial_k\right)(\widehat{P}).$$

It remains to define $D_k(\widehat{P}) := c'$. Then (18) is obviously satisfied, and (19) follows when the construction is completed for all cubes P in $f^{-1}(Q)$. ■

5 Application to Computing Homology of Maps

From now on, by a continuous map we mean a map which is continuous with respect to the Euclidean topology. As we pointed out in Section 4.1, the class of cubical maps is small. In particular, it is too small to obtain a counterpart of the theorem stating that every continuous map may be approximated by simplicial maps. Since this approximation theorem is crucial in the definition of simplicial homology of continuous maps, one can see that there is no way to carry over the definition of the homology of a continuous map from the simplicial case to the cubical case by means of approximation. One way to overcome this difficulty is by considering cubical multivalued maps and their homology. This approach is presented in [9]. The main difficulty of this approach does not lie in the construction of the multivalued map itself, but in the construction of the so-called chain selector of the multivalued map. In particular it requires solving a large linear equation for each elementary cube in the domain of the multivalued map.

However, it turns out that approximation, which is convenient in the case of simplicial homology, may be replaced by a Cartesian approach, which is natural for cubical homology. This approach is used in [11] to present a new algorithm for computing homology of continuous maps. Since the presentation there is technical and oriented on efficiency, in this section we will describe the Cartesian approach without the technicalities that hide the main idea.

The construction is based on the definition of the homology of a multivalued map via projections from the graph given in [7] together with an idea from [1]. Let X, Y be two cubical sets, and let $f\colon X \to Y$ be continuous in Euclidean topology. Our goal is to define the homology of f in terms of homology of some cubical maps. Recall that the graph of f is the set $\operatorname{graph}(f) := \{(x, y) \in X \times Y \mid y = f(x)\}$. Obviously, $\operatorname{graph}(f)$ is not a cubical set unless f is locally constant. However, we may consider a cubical set $Z \subset X \times Y$ such that $\operatorname{graph}(f) \subset Z$. Let $p_Z\colon Z \to X$ and $q_Z\colon Z \to Y$ denote projections to X and Y, respectively. Then we have the following commutative diagram of continuous maps.

(20)
$$\begin{tikzcd} & Z \arrow[dl, "p_Z"'] \arrow[dr, "q_Z"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

We know from Example 4.4 that p_Z and q_Z are cubical maps and therefore their homology is well defined. Since homology is functorial, the homology of f must satisfy $H_*(f) \circ H_*(p_Z) = H_*(q_Z)$. This may be solved for $H_*(f)$ if $H_*(p_Z)$ is an isomorphism. Since obviously p_Z is surjective, by Theorem 4.14 $H_*(p_Z)$ is an isomorphism if

(21)
$$\forall x \in X \text{ the set } p_Z^{-1}(x) = \{x\} \times Y \ \cap \ Z \text{ is acyclic.}$$

The simplest candidate for Z is $\operatorname{ch}(f)$, the closed hull of f in $X \times Y$. In practice, it often fulfills the acyclicity condition of (21). In the case when the acyclicity condition of (21) fails, one has to go through the process of subdivision or, similarly to the multivalued approach presented in [9], so-called rescaling (changing units). One can prove that with a sufficiently fine rescaling the closed hull of the graph satisfies the acyclicity condition of (21).

We now turn our attention to the algorithm computing $H_*(f)$.

First observe that computing the homology of a cubical map is very simple: via formulas in Definition 4.6 one obtains the associated chain map, which reduces the problem to elementary linear algebra.

In order to find an algorithm computing the homology of a continuous map one has to answer first the fundamental question what does it mean to have a continuous map on input of an algorithm. Algorithms can process only finite amounts of data, which suggests that the continuous maps must be somehow coded with some finite code. A possibility is to restrict the class of considered maps to polynomials with rational coefficients and to feed the algorithm with the coefficients. However, such an approach is very restrictive, because in many problems the continuous maps of interest are not polynomials. Even worse, the necessity of coding may lead to the incorrect conclusion that it is necessary to restrict the algorithm to some countable class of continuous maps.

Fortunately, the homology of maps is preserved under homotopy, which suggests searching for a coding which could be shared by many maps having the same homology. Exactly this type of coding is provided by the closed hull of the graph of the map. The construction of the closed hull may be done in many ways and we do not want to go into detail here. The typical approach is based on interval arithmetic [12]. Some ways to achieve this task for certain classes of continuous functions are discussed in [9, 14, 15].

Therefore the outline of the algorithm is as follows. Given a continuous map $f\colon X \to Y$:

(1.) Fix a grid scale in $X \times Y$.
(2.) Construct the closed hull Z of the graph of the map f.
(3.) Compute the projections $p_X\colon Z \to X$ and $p_Y\colon Z \to Y$.
(4.) Verify the condition (21) and if it fails, take a smaller grid scale and go back to 2.
(5.) Compute the homology maps $p_X^*\colon Z \to X$ and $p_Y^*\colon Z \to Y$.
(6.) Compute the inverse p_X^{*-1} (the inverse exists by Vietoris–Begle theorem).
(7.) Compute and return the composition $p_Y^* p_X^{*-1}$.

As we already stated, the strength of this algorithm lies in the fact that one avoids solving the large number of large linear equations needed in the algorithm presented in [9]. However, a direct application of this algorithm would not be efficient for another reason. The problem is that with introduction of the graph one raises the dimension of the problem from the maximum of dimensions of the cubical sets X and Y to the sum of these dimensions. The solution is to perform some preprocessing, which allows one to replace the graph by another set in $X \times Y$ whose dimension is the same as the dimension of X. The preprocessing is quite complicated but leads to an algorithm which has been implemented and performs well in concrete applications. The details of the algorithm outlined in this section together with all the technicalities which make it efficient are presented in [11].

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