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ON COMPLEX PROJECTIVE HYPERSURFACES

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In this paper we prove various results concerning monodromy groups associated with nonsingular complex projective hypersurfaces. Most of these results are already known but proofs are either unavailable or are algebraic and require a lot of machinery. The groups in question are those obtained from the second Lefschetz theorem (see (**1**)) applied to (a) the general Veronese variety, (b) a nonsingular projective hypersurface. By embedding the monodromy group of an extraordinary local isolated singularity (discovered by Libgober (**8**)) in these global monodromy groups we obtain necessary and sufficient conditions for the global groups to be finite. For case (a) we also obtain information on the structure of the dual to the Veronese variety which is of use when considering the monodromy group. The author gratefully acknowledges the financial support of the Stiftung Volkswagenwerk for a vist to the IHES during which this paper was written.

1.

In what follows $H_*(A; R)$ denotes the reduced singular homology groups of a space A with coefficients in some ring R; if R is omitted integer coefficients are understood. The fundamental group of A is denoted by $\pi_1(A)$, the omission of a base point is for notational convenience and will cause no problems. Finally P^m denotes complex projective space of dimension m, $\check{P}^m$ the dual space of hyperplanes in P^m.

We first consider the Veronese variety V. If d is an integer greater than zero the space of all projective hypersurfaces in P^n of degree d forms a projective space P^N of dimension $N=\binom{n+d}{n}-1$. Let $F_0, \ldots, F_N$ be the set of basis monomials of degree d in $n+1$ variables: we have a well defined embedding $F=(F_0, \ldots, F_N)\colon P^n \to P^N$, and $V=F(P^n)$ is the Veronese variety of type (n, d).

Given any smooth variety $U\subset P^m$ let $\check{U}$ denote the dual variety of tangent hyperplanes $H\in \check{P}^m$. If we now set $X=\{(y, H)\in P^N\times \check{P}^N : H\notin \check{V},\ y\in H\cap V\}$ and denote the obvious projection $X\to \check{P}^N-\check{V}$ by Φ, this is a locally trivial fibre bundle and if W is a typical fibre we wish to study the representation of $\pi_1(\check{P}^N-\check{V})$ as a group of automorphisms of the R module $H_*(W; R)$.

We now consider the structure of $\check{V}$, indeed we give another interpretation of Φ. If P^N is the space of hypersurfaces as above let D denote the subset of singular such. Setting $X_1=\{(x, F)\in P^n\times P^N-D\colon F(x)=0\}$ and denoting projection $X_1\to P^N-D$ by

Φ_1 we have a commuting diagram

$$\begin{tikzcd} X_1 \arrow[r, "\theta_1"] \arrow[d, "\Phi_1"'] & X \arrow[d, "\Phi"] \\ P^N - D \arrow[r, "\theta_2"'] & \check{P}^N \check{V} \end{tikzcd}$$

where $\theta_1(x, F) = (F(x), F^*)$, $\theta_2(F) = F^*$ and $F^* = (a_0, \ldots, a_N)^*$ is the hyperplane $\sum_0^N a_i x_i = 0$. Moreover θ_1 and θ_2 are diffeomorphisms and the natural extension of θ_2 takes D to V.

A theorem of Zariski (see (**6**)) asserts that for generic lines L (resp. planes P) in $\check{P}^N$ the homomorphism induced by the inclusion $\pi_1(L - \check{V}) \to \pi_1(\check{P}^N - \check{V})$ (resp. $\pi_1(P - \check{V}) \to \pi_1(\check{P}^N - \check{V})$) is a surjection (resp. an isomorphism). Consequently it would be advantageous to describe some stratification of $\check{V}$ (or equivalently D) down to codimension 1 at least. This we now do, working with D and assuming $d \geqq 3$; for $d = 2$ see the last part of Section 2.

Consider the polynomials $F \in P^N$ whose hypersurfaces $\{F = 0\} \subset P^n$ are of the following types:

(i) $F = 0$ has at least 3 singularities,
(ii) $F = 0$ has 2 singularities one of which is not of type A_1,
(iii) $F = 0$ has 1 singularity which is not of type A_1 or A_2.

(Here A_1 and A_2 are the first two simple singularities, see for example (**2**)). In each case F satisfies at least 3 conditions and the set of such F form a constructible subset of P^N of codimension 3. (See (**12**) p. 37 for a definition of constructible set). The complement of this set in D is the disjoint union of the following sets:

(i) those F with $F = 0$ having 1 A_1 singularity,
(ii) those F with $F = 0$ having 2 A_1 singularities,
(iii) those F with $F = 0$ having 1 A_2 singularity.

We shall refer to these sets respectively as A_1, $2A_1$, A_2.

Theorem 1.1. (a) *D is an irreducible hypersurface in P^N.*

(b) *$A_1, 2A_1, A_2$ are constructible smooth subsets of P^N.*

(c) *Locally at $F \in 2A_1$ the triple $(P^N, D, 2A_1)$ is analytically equivalent to* $(\mathbb{C}^N$, $\{y \in \mathbb{C}^N : y_0 \cdot y_1 = 0\}$, $\{y \in \mathbb{C}^N : y_0 = y_1 = 0\})$ *at* $0 \in C^N$.

(d) *Locally at $F \in A_2$ the triple (P^N, D, A_2) is analytically equivalent to* $(\mathbb{C}^N$, $\{y \in \mathbb{C}^N : y_0^3 = y_1^2\}$, $\{y \in \mathbb{C}^N : y_0 = y_1 = 0\}$ *at* $0 \in \mathbb{C}^N$.

Proof. Let G denote the projective general linear group $PGL(n+1, \mathbb{C})$. There is a natural action of G on P^N by "change of co-ordinates", $\alpha : G \times P^N \to P^N$.

(a) If $L = \{F \in P^N : F = 0$ has a singularity at $(1 : 0 \ldots : 0)\}$ clearly L is a linear subspace of P^N, and $D = \alpha(G \times L)$ will be irreducible because G and L are.

Parts (b), (c) and (d) are proved by the same method as that employed in (**4**) where the case $n = d = 3$ is considered. For example when considering A_2 we take as our normal form

$$F = x_0^{d-2}(x_1^2 + \ldots + x_{n-1}^2) + x_0^{d-3}(x_n^3 + f_3(x_1, \ldots, x_n)) + f_d(x_0, \ldots, x_n)$$

where f_j is homogeneous of degree j and the degree of x_0 in any monomial of f_d is $\leqq d-4$. We also exclude all f_3 and f_d which give singularities on $F=0$ other than at $(1:0:\ldots:0)$ and the singularity here is clearly an A_2. For any fixed F_0 in the above form one easily verifies that the space of forms $F+t_1x_nx_0^{d-1}+t_2x_0^d$ is transverse to the G orbit of F_0. The methods described in (**4**) now show that the A_2 stratum meets the transversal precisely when $t_1=t_2=0$ and so is smooth. It is constructible by Chevalley's theorem (**12**, p. 37) since we have a constructible normal form. Assertion (d) is proved as in (**4**) by showing that our transversal is essentially a versal unfolding of an A_2 singularity, the parameters in the normal form being redundant. The set of singular hypersurfaces in the transversal is thus identified with the discriminant set of a versal unfolding of $x_1^2+\ldots+x_{n-1}^2+x_n^3$, and thus is locally the product of a cusp with some affine space.

Corollary 1.2. *A generic line in P^N (resp. $\check{P}^N$) meets D (resp. $\check{V}$) in a finite set of points. The corresponding hypersurfaces (resp. hyperplane sections) have a single A_1 singularity.*

Corollary 1.3. *A generic plane section of D (resp. $\check{V}$) is an irreducible curve with only double points and cusps for its singularities.*

We now return to the study of the monodromy group of $\Phi: X\rightarrow \check{P}^N-\check{V}$; let W denote a fibre of Φ. The first Lefschetz hyperplane theorem asserts that the inclusion $W\hookrightarrow V$ induces an isomorphism in H_p, $p\leqq n-1$. But V is diffeomorphic to P^n so using Poincare duality $H_p(W)=Z$ for p even and $p\neq n-1$, and it is 0 for p odd and $p\neq n-1$. It is also easy to see that $H_{n-1}(W)$ is free, and we calculate its rank as follows. If W_1 is a fibre of Φ_1 (i.e. a hypersurface in P^n) and b is a generator of $H^2(P^n)$, W_1 has dual Chern class $(1+db)$, and so its total Chern class $c(W_1)=(1+b)^{n+1}(1+db)^{-1}\mid W_1$. If μ resp. μ' are the canonical generators of $H_{2n}(P^n)$ resp. $H_{2n-2}(W_1)$ then $\langle b^{n-1}\mid W_1, \mu'\rangle=\langle db^n, \mu\rangle$, where $\langle , \rangle$ is the Kronecker product. So the Euler characteristic $\chi(W)=\chi(W_1)=\langle c_{n-1}(W_1), \mu'\rangle=d^{-1}\{(1-d)^{n+1}+(n+1)d-1\}$, and hence rank $H_{n-1}(W)=\{\frac{1}{2}(1+(-1)^{n+1})+d^{-1}((d-1)^{n+1}+(-1)^{n+1}(d-1))\}$. (Compare Milnor and Stasheff **11**, §11) but beware of the mistake in problem 16.D). Now rank $H_{n-1}(P^n)=\frac{1}{2}(1+(-1)^{n+1})$ so the space of vanishing cycles for Φ, Van $\Phi\subset H_{n-1}(W)$ has dimension $d^{-1}\{(d-1)^{n+1}+(-1)^{n+1}(d-1)\}$ and we have $\frac{1}{2}(1+(-1)^{n+1})$ invariant cycles.

It is amusing to use the Zeuthen Segre formula (**1**) to compute the number of singular members of a generic pencil of hyperplane sections of V, i.e. the degree of $\check{V}$ or D. All members of the pencil have a codimension two linear subspace of P^N in common, the axis of the pencil, which intersects V in a variety which is the complete intersection of two nonsingular hypersurfaces U and W say. The dual class of $U\cap W$ is d^2b^2 while $c(U\cap W)=(1+b)^{n+2}(1+db)^{-2}\mid U\cap W$. It follows that

$$\begin{aligned}\chi(U\cap W)&=d^2\left\{\sum_{j=0}^{n-2}\frac{(n-1-j)!\,(n+1)!}{j!\,(n+1-j)!}(-1)^{n-j}d^{n-2-j}\right\}\\&=\{((n-1)d+2)(1-d)^n+d(n+1)-2\}\cdot d^{-1}.\end{aligned}$$

If μ_2 is the number of singular members of the (generic) pencil the Zeuthen Segre formula states that

$$\chi(P^n)=\chi(V)=2\cdot\chi(W)-\chi(U\cap W)+(-1)^n\mu_2$$

from which it follows that $\mu_2=(d-1)^n(n+1)$.

It is also of interest to compute the number of double points δ and cusps κ in the plane section of Corollary 1.3. This can be done for $n=1,2$ as follows.

For $n=1$ we claim that this section is the dual of the generic rational plane curve of degree d, C_d. For if C_d has the parametric equations $(x_0\colon x_1\colon x_2)=(f_0(y_0,y_1)\colon f_1(y_0,y_1)\colon f_2(y_0,y_1))$ the line $\sum_0^2\lambda_i x_i=0$ is tangent to C_d if and only if the discriminant of $\sum_0^2\lambda_i f_i$ is zero. The curve C_d has degree d, class $2(d-1)$, $3(d-2)$ inflexions and $\frac{1}{2}(d-1)(d-2)$ double points which are its only singularities. By Plucker's formulae $\check{C}_d$ has degree $2(d-1)$, $2(d-2)(d-3)=\delta$ double points and $3(d-2)=\kappa$ cusps.

For $n=2$ let V_1 be a generic projection of the Veronese surface $V\subset P^N$, $N=\frac{1}{2}n(n+3)$, into P^3. The dual $\check{V}_1$ can be identified with a plane section of the dual $\check{V}$. Since V_1 is the image of a generic projection its numerical characters are those of V, and provided one believes the formulae of Cayley these are easily computed. Indeed in the notation of Baker (**3**) p. 162 the degree of V is $d^2=\mu_0$, the genus of a general section $p=\frac{1}{2}(d-1)(d-2)$, so the rank of a general section $\mu_1=3d(d-1)$. The class μ_2 of V we computed above is $3(d-1)^2$, while the type ν_2 of V is $6(d-1)^2$ (this follows, for example, because the arithmetic genus p_a of V vanishes, since V is rational). The numbers we now require are T (resp. i) the number of tangent planes through a generic point which are bitangential (resp. inflexional). A short computation shows that $T=3/2(d-1)(d-2)(3d^2-3d-11)$ (our δ) while $i=12(d-1)(d-2)$ (our κ).

2.

Libgober in his paper (**8**) produced a very degenerate isolated singularity which occurs on hypersurfaces of degree d in P^n. If we set

$$f_c=x_1^d+x_2^{d-1}+x_2x_3^{d-1}+\dots+x_{n-1}x_n^{d-1}+c,\quad F_c=x_0^d f_c(x_1/x_0,\dots,x_n/x_0)$$

the singularity in question is $f_0=0$, which appears on the hypersurface $F_0=0$, and is weighted homogeneous with weights

$$\left(1/d,\,1/d-1,\dots,\frac{(d-1)^k+(-1)^{k+1}}{d(d-1)^k},\dots,\frac{(d-1)^{n-1}+(-1)^n}{d(d-1)^{n-1}}\right).$$

A short computation shows that this singularity has Milnor number $\xi=d^{-1}\{(d-1)^{n+1}+(-1)^{n+1}(d-1)\}$ (see (**8**) but beware of the misprint on p. 199, there is a sign missing in (ii)). Choose a sufficiently small ball neighbourhood B for f_0 at $(0,\dots,0)\in\mathbb{C}^n$ (corresponding to $(1\colon 0\colon\dots\colon 0)\in P^n$); for small c the variety $f_c=0$ intersects B in a smooth manifold homotopy equivalent to a wedge of $\xi(n-1)$-spheres. We may choose arbitrarily small constants $\lambda_1,\dots,\lambda_n$ such that $f_0+\sum_1^n\lambda_i x_i=0$ has nondegenerate critical points all inside B (see (**10**) Appendix B). In fact using Mather's results on generic projections (**9**) one can ensure that the resulting critical values are distinct. For if graph $f_0=\{(x,f_0(x))\colon x\in\mathbb{C}^n\}$ consider the map graph $f_0\to\mathbb{C}$ induced by the projection $(x,t)\mapsto t$. By (**9**) we may choose a linear map $\mathbb{C}^{n+1}\to\mathbb{C}$

arbitrarily close to the above projection inducing a map graph $f_0 \to \mathbb{C}$ with generic singularities i.e. only A_1 critical points with distinct critical values. Such a linear map is of the form $(x, t) \mapsto \sum \beta_i x_i + s \cdot t$ with the β_i small and s close to 1; choose $\lambda_i = s^{-1} \cdot \beta_i$.

Working with P^N, D and Φ_1 we consider the pencil of hypersurfaces $a(F_0 + \sum_1^n \lambda_i x_i x_0^{d-1}) + b x_0^d = 0$. There are ξ critical points of this pencil corresponding to $a = 1$, $b = -\left(\text{critical values of } f_0 + \sum_1^n \lambda_i x_i\right)$. Fixing c sufficiently small the usual procedure (see (**7**)) associates to each critical point a vanishing cycle in $f_c = 0$. They (or rather their inclusions in $H_{n-1}(F_c = 0)$) are also vanishing cycles for Φ_1; we now show that these vanishing cycles remain independent after inclusion.

Set $\{F_c = 0\} = W$, $\{f_c = 0\} \cap B = N$, W-int. $N = M$, $S = \partial B$, $\partial M = \partial N = K$, $h =$ the monodromy transformation of f_0 and Δ_n the characteristic polynomial of h. Fixing d we have a family of polynomials Δ_n; we claim that $\Delta_{2m}(1) = 1$, $\Delta_{2m-1}(1) = d$ for $m \geqq 1$. Clearly $\Delta_1 = (t^d - 1)(t-1)^{-1}$, so $\Delta_1(1) = d$ (compare (**10**)). Libgober shows that $\Delta_{n+1}\Delta_n = (t^{d(d-1)^n} - 1)(t^{(d-1)^n} - 1)^{-1}$ so $\Delta_{n+1}(1)\ \Delta_n(1) = d$ and the result is clear. The Wang sequence of the Milnor fibration $N \hookrightarrow S - K \to S^1$ is

$$0 \to H_n(S-K; \mathbb{Q}) \to H_{n-1}(N; \mathbb{Q}) \xrightarrow{h-1} H_{n-1}(N; \mathbb{Q}) \to H_{n-1}(S-K; \mathbb{Q}) \to 0$$

Since $\Delta(1) \neq 0$ the homomorphism $h-1$ is an isomorphism so $H_n(S-K; \mathbb{Q}) = 0$. By Alexander and Poincare duality $H_n(S-K; \mathbb{Q}) \cong H^{n-2}(K; \mathbb{Q}) \cong H_{n-1}(K; \mathbb{Q}) = 0$. It follows from the universal coefficient theorem that $H_{n-1}(K)$ is finite. We now consider the Mayer Veitoris sequence of the decomposition $W = M \underset{K}{\cup} N$,

$$\ldots \to H_n(W) \to H_{n-1}(K) \to H_{n-1}(N) \oplus H_{n-1}(M) \to H_{n-1}(W) \to \ldots$$

Since $H_{n-1}(K)$ is finite its image in $H_{n-1}(N)$, which is free, is zero, so the homomorphism $H_{n-1}(N) \oplus \{0\} \to H_{n-1}(W)$ is injective. Reverting to rational coefficients and counting dimensions it is clear that the inclusions of the vanishing cycles in N are a basis for the space of vanishing cycles in $H_{n-1}(W; \mathbb{Q})$. By the Picard-Lefschetz formula the monodromy operators for f_0 associated with the vanishing cycles $e_1, \ldots, e_\xi$ say are determined by their intersection matrix as are the corresponding operators in the global monodromy group. We have thus shown:

Theorem 2.1. *The local monodromy group of* $f_0 = 0$ *is isomorphic to a subgroup of the global monodromy group of* Φ_1.

One easily checks that f_0 is simple (as in (**2**)) if and only if $(n, d) = (1, d)$, $(2, 3)$, $(2, 4)$ or (3.3). If f_0 is not simple its monodromy group is infinite for by (**2**) any such singularity specializes to a simple elliptic singularity $\tilde{E}_k$ $k = 6, 7$ or 8. Consequently the monodromy group of f_0 will have a subgroup with some quotient isomorphic to the monodromy group of some $\tilde{E}_k$ which by (**5**) are all infinite. On the other hand if n is even the monodromy group will not be finite since the operators are not reflexions. One can now deduce one half of

Corollary 2.2. *The monodromy group of* Φ_1 (or Φ) *is finite if and only if* $(n, d) = (1, d)$, $(3, 3)$ *and it is then respectively* $S(d-1)$ (*symmetric group*), $W(E_6)$ (*the Weyl group of type* E_6).

The actual assertion that the group is finite is trivial in the first case, for the second compare Todd (**13**); $W(E_6)$ is indeed the group of symmetries of the 27 lines on the cubic surface.

Let us now clear up the excluded case $d=2$. Any quadric can be written as $\sum_0^n q_{ij}x_ix_j=0$, with $q_{ij}=q_{ji}$ and we stratify the space of quadrics by the rank of the matrix (q_{ij}). These strata are orbits under the natural G action with representatives $\sum_0^r x_i^2$, $0\leqq r\leqq n$. If $r\leqq n-1$ one easily checks that the orbit has codimension $\geqq 3$. Thus a generic plane section of the variety D gives a nonsingular plane curve of degree $n+1$, and $\pi_1(P^N-D)$ is cyclic of order $n+1$.

For the monodromy we note that if we replace f_c above by $x_1^2+x_2^2+\ldots+x_n^2+c$ the same procedure works, when n is odd, for $\Delta=(t+(-1)^{n+1})$. It follows that the monodromy representation is the canonical epimorphism $\mathbb{Z}_{n+1}\to\mathbb{Z}_2$ for n odd, while for n even the monodromy group is obviously trivial.

Note that in all cases we can also locate the invariant cycle when n is odd (there are no invariant cycles when n is even). For choosing a $\frac{1}{2}(n+1)$ plane cutting W transversally and missing B we obtain an $(n-1)$ cycle whose intersection number with each of the e_i is zero.

3.

The methods of Section 2 can be used to obtain results on the monodromy group of a nonsingular complex projective hypersurface. One simply notes that $x_1^d+x_2^{d-1}x_0+x_2x_3^{d-1}+\ldots+x_{n-1}x_n^{d-1}+x_{n+1}^d=x_0^d$ is a nonsingular projective hypersurface, and the plane $x_0=x_{n+1}$ cuts this in a hypersurface with a singularity of the type described in Section 2. It is not difficult now to choose a generic pencil of hyperplane sections of the above hypersurface one of whose elements is close to $x_0=x_{n+1}$, and embed the local monodromy group of the singularity in the global monodromy group of the hypersurface. The details are left to the reader.

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