Neutrino Gas in Equilibrium with Self-Interaction

M. Sirera[1] and A. Perez[2]

[1] Departamento de Astronomia y Astrofísica, Universidad de Valencia, 46100 Burjassot (Valencia) Spain;
Miguel.Sirera@uv.es

[2] Departamento de Física Teórica, Universidad de Valencia, 46100 Burjassot (Valencia) Spain;
Armando.Perez@uv.es

Summary. We study a neutrino gas in equilibrium both at $T = 0$ and at finite temperature. The neutrinos are assumed to be massive Dirac particles with two generations interacting with each other via neutral currents and with a background of matter. We analyze the main properties of the neutrino eigenmodes in the medium, such as effective masses and mixing angle.

1 Introduction

The neutrino behavior in matter is important in several astrophysical scenarios, especially is the case of supernova explosions, where neutrino interactions and oscillations can change the shock dynamics [4, 11, 14, 15, 16, 17, 18]. A crucial ingredient in this scenario is the neutrino self- interaction [1, 3, 12, 20]. As it has been remarked, such interactions are non-diagonal in flavor space, and give raise to new phenomena in the oscillatory behavior [5, 9, 10]. Here we analyze in detail the equilibrium of a neutrino gas both at $T = 0$ and at finite temperature. The neutrinos are supposed to be massive Dirac particles with two generations, namely electron-neutrinos ν_e and muon-neutrinos ν_μ, which are assumed to interact among them via neutral currents and with a background of matter. This might be the situation for neutrinos in a supernova inside the neutrino sphere, if self-interactions are effective enough to reach the equilibrium, where we assume that the equilibrium is not only thermal but also chemical between the two generations [2, 5]. To this end, we use a method based on Wigner functions, which has been shown to be appropriate to describe both the equilibrium and kinematics of many-particle neutrino systems with mixing [19].

In this work neutrinos are treated as extremely relativistic particles of positive helicity (antineutrinos of negative helicity), whose states of defined momentum $\overrightarrow{p}$ are also of defined energy (or effective masses different of their vacuum masses) i.e. what is called the "quasi-particle approximation." We make the assumption that the background of matter is electrically neutral and consisting only of nucleons and electrons ($n_e = n_p$).

https://doi.org/10.1017/S0252921100009350 Published online by Cambridge University Press

2 Hartree Approximation

First we treat the gas in the so-called "Hartree approximation," i.e., without taking into account the effects of the statistical correlations. At first, let us suppose that we have only neutrinos which are interacting among them (without a background of matter). In this case, starting from the general equation of motion and imposing proper conditions, we obtain that the equation of motion for the Wigner Function of neutrinos in equilibrium is [19]

$$(\gamma p - M)F(p) = \frac{G_F}{2\sqrt{2}} \int d^4k Tr[\gamma_\mu(1-\gamma^5)F(k)]\gamma^\mu(1-\gamma^5)F(p), \quad (1)$$

where G_F is the Fermi constant, M^{ab} is the mass matrix and $F^{ab}(p)$ the Wigner Function for the neutrinos. Both of them have flavor indices in the generation space, such that $a, b = e, \mu$.

From this equation, and assuming that neutrinos basically consist on their left-chirality projections, we can obtain a specific equation for neutrinos (with negative helicity), where $n_\nu = n_1 + n_2 = n_{\tilde{\nu}_1} + n_{\tilde{\nu}_2} - n_{\tilde{\overline{\nu}}_1} - n_{\tilde{\overline{\nu}}_2}$ is the total density of neutrinos (with $p^0 \simeq |\overrightarrow{p}|$) minus the corresponding antineutrino density (with $p^0 \simeq -|\overrightarrow{p}|$)

Let us now assume that, in addition to the self-interaction among the neutrinos, we have an electrically neutral background of matter composed by electrons and nucleons. In this case, the effective masses of neutrinos are [6, 7, 19]

$$M_{1,2}^2 = 1/2(A_c + \Sigma) \mp 1/2(A_c^2 + \Delta^2)^{1/2} + A_n, \quad (2)$$

where

$$\begin{aligned} \Sigma &= m_1^2 + m_2^2 \\ \Delta &= m_1^2 - m_2^2 \\ A_c &= 2\sqrt{2}G_F|\overrightarrow{p}|n_e \\ A_n &= 2\sqrt{2}G_F|\overrightarrow{p}|n_\nu - \sqrt{2}G_F|\overrightarrow{p}|n_n \end{aligned} \quad (3)$$

being n_e the number density of electrons (minus antielectrons) and n_n the number density of neutrons (minus antineutrons).

3 Corrections to the Hartree Approximation

The collective effects in the neutrino gas are brought in by means the statistical correlations, which modify the equation of motion of the Wigner Function (or the left-quirality part $F_L(x)$), which now becomes

https://doi.org/10.1017/S0252921100009350 Published online by Cambridge University Press

$$[\gamma p - \sqrt{2}G_F \gamma a + \sqrt{2}G_F \int d^4 q \gamma F_L(q)\gamma - M]F_L(p) = 0. \tag{4}$$

where the four-vector a is defined as

$$a_\mu = \int d^4 k Tr \gamma_\mu F_L(k). \tag{5}$$

Obviously, the third term of this equation provides us with an additional correction to the corresponding equation in the Hartree approximation. After some algebra, we arrive to the following expression for the effective masses in the medium:

$$M^{*2}_{1,2}\left(p^0\right) = \frac{1}{2}\left[\Sigma + 2\sqrt{2}G_F\left(3n_\nu + n_e - n_n\right)p^0\right] \mp \frac{1}{2}\Delta^*, \tag{6}$$

where

$$\Delta^* = \left[\left(2\sqrt{2}G_F p^0\left(n_e + \delta\right) - \Delta\cos 2\theta_0\right)^2 + \left(4\sqrt{2}G_F p^0 n_{12} - \Delta\sin 2\theta_0\right)^2\right]^{1/2} \tag{7}$$

is the effective mass difference. In Eq. (6) the upper (lower) sign corresponds to M_1^* (M_2^*) ,where Σ and Δ have been defined in Eq. (3). The mixing angle is now given by

$$\sin 2\theta = \frac{\Delta\sin 2\theta_0 - 4\sqrt{2}G_F p^0 n_{12}}{\Delta^*}. \tag{8}$$

The quantities that appear in these equations are defined in the following manner:

$$\begin{aligned}
n_{\nu_e} &= \tfrac{1}{2\pi^2}\int_0^\infty d|\boldsymbol{q}||\boldsymbol{q}|^2[c^2 f_1(q) + s^2 f_2(q)],\\
n_{\nu\mu} &= \tfrac{1}{2\pi^2}\int_0^\infty d|\boldsymbol{q}||\boldsymbol{q}|^2[s^2 f_1(q) + c^2 f_2(q)],\\
n_{\nu_{12}} &= \tfrac{1}{2\pi^2}\int_0^\infty d|\boldsymbol{q}||\boldsymbol{q}|^2 cs[f_1(q) - f_2(q)],
\end{aligned} \tag{9}$$

where $f_1(q)$ and $f_2(q)$ are the Fermi statistical distribution functions for each generation, corresponding to quasi-particles with well-defined effective masses, and s and c are the *sin* and *cos* of the rotation angle θ which relates the eigenstates of effective masses to flavor eigenstates. In this way, n_{ν_e} (n_{ν_μ}) is the number density of electron (muon) neutrinos and $n_{\nu_{12}}$ contains interference effects. Analogously, we can define the number densities for antineutrinos, so we have that $n(e) = n_{\nu_e} - \bar{n}_{\nu_e}$ is the net electron neutrino number density, $n(\mu) = n_{\nu_\mu} - \bar{n}_{\nu_\mu}$ is the net muon neutrino number density, $n_\nu = n(e) + n(\mu)$ is the total number neutrino density, $n_{12} = n_{\nu_{12}} - \bar{n}_{\nu_{12}}$, and $\delta = n(e) - n(\mu)$ is a statistical parameter of asymmetry between the two flavors.

The dispersion relation for neutrinos and antineutrino mass eigenstates, can be written as:

$$p^2 - M_{1,2}^{*2}\left(p^0\right) = 0 \tag{10}$$

and provides (as an implicit equation) the energy p^0 as a function of the momentum $|\boldsymbol{p}|$. As a first approximation, one can use the fact that, under most situations of interest, neutrinos are extremely relativistic particles. Thus, for neutrinos one can replace p^0 by $|\boldsymbol{p}|$. In this way, the above dispersion equation can be approximately solved as:

$$p_0 = \sqrt{|\boldsymbol{p}|^2 + M_{1,2}^{*2}\left(|\boldsymbol{p}|\right)} \tag{11}$$

To obtain the corresponding formulae for antineutrinos we only have to change $|\boldsymbol{p}|$ to $-|\boldsymbol{p}|$ in the two previous equations.

An interesting consequence of Eq. (8) is that the condition for the MSW resonance is modified with respect to the situation where there is not a neutrino background. In fact, the condition for the resonance is now:

$$p^0 = \frac{\Delta\left[(\delta + n_e)\cos 2\theta_0 + 2n_{12}\sin 2\theta_0\right]}{2\sqrt{2}G_F\left[(\delta + n_e)^2 + 4n_{12}^2\right]} \tag{12}$$

This new condition can be of interest if $\sin 2\theta_0 \simeq 1$, as suggested for both solar and atmospheric neutrino oscillation values (for a review, see [8]). In this case, the MSW resonance might be dominated by the neutrino background.

4 Low-Temperature Limit

Since the gas is in total equilibrium, the state is characterized by a single chemical potential μ of the neutrinos, and the temperature T of the gas. Here we treat, for the sake of simplicity, the low-temperature case ($T/\mu \ll 1$). In this case, we can apply the "Sommerfeld expansion" in order to carry out the Fermi integrals and calculate the thermodynamical variables of the system. In this type of calculation, we have a series expansion in T/μ. Using this, we can calculate the total neutrino density to the lowest order in T. In this calculation we can neglect the antiparticles. The total density turns out to be

$$n_\nu = n_{\nu_e} + n_{\nu_\mu} = \frac{1}{2\pi^2}\int_0^\infty d|\boldsymbol{p}||\boldsymbol{p}|^2[(f_1(p) + f_2(p)] \simeq \\ \frac{2\mu^3 - 3\sqrt{2}G_F\mu^2(n_e - n_n) + 2\pi^2\mu T^2 - \sqrt{2}\pi^2 G_F(n_e - n_n)T^2}{6\pi^2 + 9\sqrt{2}G_F\mu^2}, \tag{13}$$

where we are using the high density approximation, in which the neutrino vacuum masses are negligible as compared to the self-energies of the interaction.

https://doi.org/10.1017/S0252921100009350 Published online by Cambridge University Press

We can now calculate the variable δ of asymmetry starting from its definition. We restrict ourselves to the value $\theta_0 = \pi/4$ and since we are in the high-density limit, then one finds $\theta \simeq 90^o$ for the in-medium mixing angle, and thus $\cos 2\theta \simeq -1$. This implies that ν_e can be approximately identified with ν_2 (the heaviest eigenstate, according to our convention) and, similarly, ν_μ corresponds to ν_1. This identification can also be seen from the corresponding distribution functions, Eq. (9). Since the mass difference is usually very small (as compared to the chemical potential and temperature), we have

$$f_1(p) \simeq f_2(p) \tag{14}$$

which implies that

$$n_{\nu_{12}} \ll n_{\nu_e} \simeq n_{\nu_\mu} \tag{15}$$

To illustrate this assertion, let us calculate the asymmetry parameter δ for a low-temperature neutrino gas

$$\begin{aligned} \delta = n_{\nu_e} - n_{\nu_\mu} &\simeq -\tfrac{1}{2\pi^2}\textstyle\int_0^\infty d|\boldsymbol{p}||\boldsymbol{p}|^2[(f_1(p) - f_2(p)] \\ &= -\tfrac{3\sqrt{2}\mu^2 G_F n_e + \sqrt{2}\pi^2 G_F n_e T^2}{6\pi^2} \end{aligned} \tag{16}$$

Consider, for example, the core of a protoneutron star (see, for example, [13]). For a density equal to the saturation density and an electron-to-baryon fraction $Y_e = 0.3$ one has $n_e = 1.1 \times 10^8$ MeV^3. If the neutrino chemical potential is $\mu = 100$ MeV then the asymmetry density over the neutrino number density is

$$\delta/n_\nu \sim -\frac{G_F n_e}{\mu} \sim -2.6 \times 10^{-5} \tag{17}$$

in accordance with the above discussion.

5 Conclusions

In conclusion, we can say that to obtain more accurate results when treating the neutrino gas, we have to consider the collective effects by including the correlations, which, under the conditions assumed above, give a non-diagonal term in the effective mass matrix. Therefore, in addition to a modification in the effective masses of eigenstates, there is a change in the in-medium mixing angle, as compared to the Hartree result. Also, the condition for the MSW resonance (when a matter background is included) differs from the usual MSW condition.

On the other hand, and considering the high-density case, the effective mass difference is much larger than the vacuum difference, and one finds $\theta \simeq$

90^o for the in-medium mixing angle. Under these circumstances, the flavor states can be approximately identified with mass eigenstates. Since both $f_1(p)$ and $f_2(p)$ share the same chemical potential and the vacuum masses are negligible one finds, in practice, that in equilibrium the two neutrino flavors are mixed in the same proportion. In fact, the estimations we have presented for typical values inside a supernova show that the flavor asymmetry δ is very small.

References

1. K.N. Abazajian, J.F. Beacom, N.F. Bell: Phys. Rev. D **66**, 013008 (2002)
2. N.F. Bell, A.A. Rawlinson, R.F. Sawyer: hep-ph 0304082 (2003)
3. A.D. Dolgov, S.H. Hansen, S. Pastor, S.T. Petcov, G.G. Raffelt, D.V. Semikoz: Nuc. Phys. **632**, 363 (2002)
4. G.L. Fogli, E. Lisi, D. Montanino, A. Palazzo: Phys. Rev. D **65**, 073008 (2002)
5. A. Friedland, C. Lunardini: hep-ph 0304055 (2003)
6. C.W. Kim, A. Pevsner: In: *Neutrinos in Physics and Astrophysics,* ed. Harwood (Academic Pub., 1993)
7. D. Nötzold, G. Raffelt: Nuc. Phys. **307**, 924 (1988)
8. S. Pakvasa, J.W.F. Valle: hep-ph 0301061 (2003)
9. J. Pantaleone: Phys. Lett. B **287**, 128 (1992)
10. J. Pantaleone: astro-ph 9405008 (1994)
11. S. Pastor, G. Raffelt: Phys. Rev. Lett. **89**, 191101 (2002)
12. S. Pastor, G. Raffelt, D.V. Semikoz: Phys. Rev. D **65**, 053011 (2002)
13. Mad. Prakash, I. Bombaci, Man. Prakash, P.J. Ellis, J.M. Latimer, R. Knorren: Phys. Rep. **280**, 1 (1997)
14. Y.-Z. Qian, G.M. Fuller, G.J. Mathews, R.W. Mayle, J.R. Wilson, S.E. Woosley: Phys. Rev. Lett. **71**, 1965 (1993)
15. Y.-Z. Qian, G.M. Fuller: Phys. Rev. D **51**, 1479 (1995)
16. G. Raffelt: Nuc. Phys. **95**, 183 (2001)
17. G. Raffelt: New Astron. Rev. **46**, 699 (2002)
18. G. Sigl: Phys. Rev. D **51**, 4035 (1995)
19. M. Sirera, A. Pérez: Phys. Rev. D **59**, 125011 (1999)
20. Y.Y.Y. Wong: Phys. Rev. D **66**, 025015 (2002)