ON IRREGULAR FIXED POINTS

S.K. Kaul

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Throughout this paper (X, d) will be a metric space with metric d, and h a homeomorphism of X onto itself. For any real number $r > 0$, and $p \in X$, $U(p, r)$ will denote the open r-sphere about p. Any point $p \in X$ is called <u>regular</u> [3] if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(p, y) < \delta$ implies $d(h^n(p), h^n(y)) < \epsilon$ for all integers n, where h^n denotes the iterates of h for $n > 0$, of h^{-1} for $n < 0$, and h^o is the identity. Any point of X which is not a regular point is called an irregular point. Let $I(h)$ denote the set of all the irregular points of X and $R(h) = X - I(h)$. Lim inf and Lim sup are defined as in [4].

We shall prove the following:

THEOREM 1. Let X be locally compact and connected. If $p \in I(h)$, $h(p) = p$ and $I(h)$ is zero dimensional at p, then there exists a $q \in R(h)$ such that $p \in \operatorname{Lim\,sup}_{n \to \pm\infty} h^n(q)$.

1. LEMMA 1. Let $p \in X$, $h(p) = p$ and U, V be open sets containing p such that $\operatorname{cl} V \subset U$. Let $N \subset V$ be a connected set containing p. If there exists a $y \in N$ such that $h^n(y) \notin U$ for some integer n, then there exists an $x \in N$ such that $h^n(x) \in \operatorname{cl} U - V$.

<u>Proof.</u> Suppose there does not exist any such point in N. Set $A = h^n(N) \cap (X - \operatorname{cl} U)$ and $B = h^n(N) \cap V$. Then A, B are

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non-empty, $\mathrm{cl}\, A \cap B = \phi = A \cap \mathrm{cl}\, B$, and $A \cup B = h^n(N)$. Hence, A, B define a separation of $h^n(N)$ contradicting the fact that $h^n(N)$ is connected. This proves the Lemma.

LEMMA 2. Let X be locally compact and connected. If $p \in I(h)$, $h(p) = p$ and X is 0-dimensional at p , then for sufficiently small $\varepsilon > 0$, and U, V open sets containing p such that $\mathrm{cl}\, V \subset U \subset U(p, \varepsilon)$ and any $r > 0$ such that $U(p, r) \subset V$, there exists a $y \in U(p, r)$ and an integer m such that $h^m(y) \in \mathrm{cl}\, U - V$.

<u>Proof.</u> Since X is locally compact and $p \in I(h)$ there exists an $\bar{\varepsilon}$ such that for any $\varepsilon \leq \bar{\varepsilon}$, $\mathrm{cl}\, U(p, \varepsilon)$ is compact and for any $\delta > 0$ there exists a pair (x, n), where $d(p, x) < \delta$ and n is an integer, such that $d(h^n(p), h^n(x)) > \varepsilon$. Since $h(p) = p$, $d(p, h^n(x)) > \varepsilon$.

Let U, V and r be as in the Lemma. If X is locally connected at p then the result follows from Lemma 1. Let us suppose then that X is not locally connected at p . Assuming that the Lemma is not true for some r we shall prove a contradiction.

Let $\{r_n\}$ be a monotone sequence of real numbers converging to zero and $r_1 = r$. For all pairs (x, n) such that $d(p, h^n(x)) > \varepsilon$, where $d(p, x) < r_1$, and n is an integer, let (x_1, n_1) denote one for which $|n_1|$ is least.

For any $y \in U(p, r)$ let $c(y)$ denote the component of $U(p, r)$ containing y . Note then that $x_1 \notin c(p)$, since, from Lemma 1, this leads to a contradiction because $h^{n_1}(x_1) \notin U$. Also $h^{n_1}[c(x_1)] \cap \mathrm{cl}\, U = \phi$. For if not then from the above assumption $h^{n_1}[c(x_1)] \cap (\mathrm{cl}\, U - V) = \phi$ and, therefore, $h^{n_1}[c(x_1)] \cap V \neq \phi$. But then a separation of $h^{n_1}[c(x_1)]$ can be defined contradicting that it is connected. It is clear from the same reasoning that $h^{n_1 - 1}[c(x_1)]$ or $h^{n_1 + 1}[c(x_1)]$, depending upon whether n_1 is positive or negative respectively, is contained in V .

From the continuity of h^{n_1} and h^{-n_1} there exists an $s_1 > 0$ such that if $d(p, x) < s_1$ then $h^{n_1}(x) \in V$ and $h^{-n_1}(x) \in V$. Set $\delta_1 = r_1$ and $\delta_2 = \min(s_1, r_1)$. Again there exists, as above, a pair (x_2, n_2), $d(p, x_2) < \delta_2$ and n_2 an integer, such that $|n_2|$ is the least integer for which $h^{n_2}(x_2) \notin U$. From the choice of δ_2 it is clear that $|n_2| > |n_1|$. Iterating this process we get pairs (x_i, n_i) and numbers δ_i, $i=1, \ldots$ such that (1) $\lim_{i\to\infty} \delta_i = 0$ and (2) $h^{n_i}[c(x_i)] \cap \mathrm{cl}\, U = \phi$. Assuming without loss of generality that all n_i are positive, we have furthermore, (3) $h^{n_i - 1}[c(x_i)] \subset V$ and (4) $n_i > n_j$ if $i > j$.

All the elements $c(x_i)$ are distinct for $i = 1, 2, \ldots$ and from (1) all except a finite number of them intersect any open set containing p. Therefore $\liminf_{i\to\infty} c(x_i)$ contains p and is non-empty. Hence $N = \limsup_{i\to\infty} c(x_i)$ is a connected set [4, (9.1), p. 14] and contains p.

Clearly $N \subset c(p)$. Furthermore, since, $\mathrm{cl}[c(x_i)] \cap$ boundary $U(p, r) \neq \phi$ [4, (10.1), p. 16] and boundary $U(p, r)$ is compact, $N \cap$ boundary $U(p, r) \neq \phi$. Hence N is non-degenerate.

Since $I(h)$ is zero dimensional at p and N is connected and non-degenerate there exists a $y \in N \cap R(h)$ but $y \notin$ boundary $U(p, r)$. Let $d(V, X-U) = \varepsilon_o$; then $\varepsilon_o > 0$. From the regularity of y there exists an $\eta > 0$ such that $d(x, y) < \eta$ implies that $d(h^n(x), h^n(y)) < \varepsilon_o$ for all integers n. Since $y \in N = \limsup_{i\to\infty} c(x_i)$, $U(y, \eta) \cap c(x_i) \neq \phi$ for infinitely many values of i. Let $x \in c(x_i) \cap U(y, \eta)$. Then $d(h^{n_i}(x), h^{n_i}(y)) < \varepsilon_o$. But since $h^{n_i}[c(x_i)] \cap \mathrm{cl}\, U = \phi$, from the choice of ε_o, $h^{n_i}(y) \notin V$. Again, by our assumption that the Lemma is not true $h^{n_i}(y) \notin \mathrm{cl}\, U - V$, hence $h^{n_i}(y) \notin \text{cl } U$. But, since $y \in c(p)$, Lemma 1 leads again to a contradiction of the assumption. This completes the proof of Lemma 2.

Proof of Theorem 1. Since p is irregular and $h(p) = p$ there exists an $\varepsilon > 0$ such that $U(p, \varepsilon)$ is compact and for any $\delta > 0$ there exists a pair (x, n), where $d(x, p) < \delta$, n is an integer and $d(p, h^n(x)) > \varepsilon$. Since $I(h)$ is zero dimensional at p, and $p \in I(h)$, there exists an open set V containing p such that $\text{cl } V \subset U(p, \varepsilon)$, boundary $V \cap I(h) = \phi$, and $a = d(\text{cl } V, X - U(p, \varepsilon)) > 0$. Since X is connected, boundary $V \cap R(h) \neq \phi$. Let $V_i = \{x: d(x, V) < a/2^i\}$, $i = 1, 2, \ldots$.

Let $\delta_1 > 0$ and $U(p, \delta_1) \subset V$. From Lemma 2 there exists a $y_1 \in U(p, \delta_1)$ and an integer n_1 such that $h^{n_1}(y_1) \in \text{cl } V_1 - V$. It is easy to see that this process can be iterated to get a sequence of positive real numbers $\{\delta_i\}$ converging to zero, such that, for each i, $i = 1, \ldots,$ there exists a $y_i \in U(p, \delta_i)$ and an integer n_i such that $h^{n_i}(y_i) \in \text{cl } V_i - V$ and $|n_i| > |n_j|$ if $i > j$. Since $\text{cl } V_1 - V$ is compact, the sequence $\{h^{n_i}(y_i)\}$ contained in $\text{cl } V_1 - V$ has a convergent subsequence converging to a point q of $\text{cl } V_1 - V$. We may assume without loss of generality that the above sequence itself converges to q. Since $h^{n_i}(y_i) \in \text{cl } V_i - V$ and $\lim_{i \to \infty} a/2^i = 0$ $q \in$ boundary $V \subset R(h)$.

We claim that $\lim_{i \to \infty} h^{-n_i}(q) = p$. Let $\varepsilon_o > 0$ be arbitrary. Since $q \in R(h)$ there exists an $\eta > 0$ such that $d(x, q) < \eta$ implies that $d(h^n(x), h^n(q)) < \varepsilon_o/2$ for all integers n. Since $q = \lim_{i \to \infty} h^{n_i}(y_i)$, $d(q, h^{n_i}(y_i)) < \eta$ for $i \geq N_1$ for some integer N_1. Hence $d(h^{-n_i}(q), y_i) < \varepsilon_o/2$ for $i \geq N_1$. Again since $\lim_{i \to \infty} \delta_i = 0$ there exists an integer N_2 such that $\delta_i < \varepsilon_o/2$ for $i \geq N_2$, that is, $d(p, y_i) < \varepsilon_o/2$ for $i \geq N_2$. Hence for $i \geq \max(N_1, N_2)$,

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$$d(h^{-n_i}(q), p) \leq d(h^{-n_i}(q), y_i) + d(y_i, p) < \varepsilon_o .$$

This proves the above claim and hence the theorem.

2. EXAMPLE. Let $\{p_i : i \text{ is an integer}\}$ be any set of real numbers such that $p_i < p_{i+1}$ for each i, $\lim_{i\to\infty} p_i = 1$ and $\lim_{i\to -\infty} p_i = -1$. For each i let L_i denote the line segment $\{(x, y): -1 \leq x \leq 1 \text{ and } y = p_i\}$ in the Euclidean 2-space with the usual topology and M_i be its reflection in the line $y = x$. Let $L = \bigcup \{L_i : i \text{ is an integer}\}$ and $M = \bigcup\{M_i : i \text{ is an integer}\}$. Let

$$X = M \cup L \cup \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$$

have the relative topology. Let

$$h : X \to X$$

be defined as follows:

$$\begin{aligned} h(p_i, p_j) &= (p_i, p_j) \text{ if } p_i = \pm 1 \text{ and } p_j = \pm 1 \\ &= (p_i, p_{j+1}) \text{ if } p_i = \pm 1 \text{ and } p_j \neq \pm 1 \\ &= (p_{i+1}, p_j) \text{ if } p_i \neq \pm 1 \text{ and } p_j = \pm 1 \\ &= (p_{i+1}, p_{j+1}) \text{ if } p_i \neq \pm 1 \neq p_j . \end{aligned}$$

In the last case (p_i, p_j) is the initial end point of two line segments whose terminal end points are (p_{i+1}, p_j) and (p_i, p_{j+1}) such that no coordinate is 1 or -1. For points of these line segments h is defined by linear extension onto the line segments

$$[(p_{i+1}, p_{j+1}), (p_{i+2}, p_{j+1})] \quad \text{and} \quad [(p_{i+1}, p_{j+1}), (p_{i+1}, p_{j+2})] .$$

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It is not difficult to see that h is a homeomorphism of X onto itself. The set of points $\{(\pm 1, p_i)\} \cup \{(p_i, \pm 1)\}$, $i = 0, \pm 1, \ldots$ is the set of irregular non-fixed points of h and $\{(1, 1), (1, -1), (-1, -1), (1, -1)\}$ is the set of fixed irregular points of X under h. I(h) is zero dimensional and compact; R(h) is connected and so is X. X is locally connected but not locally compact at any point of I(h). The points (-1, 1) and (1, -1) are fixed irregular points, but for no $y \in R(h)$ does $\operatorname{Lim\,sup}_{n \to \pm\infty} h^n(y)$ contain either of them (cf. Theorem 1).

REMARK. It is interesting to compare Theorem 1 above with similar results - Lemma 10 of [1] and Lemma 1 of [2]. Also one may ask the question whether the main theorem of [2] can be obtained with fewer assumptions - in particular without assuming X locally connected. The above example indicates, however, that local compactness is essential.

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University of Calgary

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