Proceedings of the Edinburgh Mathematical Society (1990) **33**, 321–326 ©

DIFFERENTIAL ALGEBRA OF THE "EVEN ORDER KORTEWEG–DE VRIES EQUATIONS"

by JOHN M. VEROSKY

(Received 11th May 1989)

In the quotient ring of differential polynomials modulo cubic terms the usual odd order hierarchy of Korteweg–de Vries equations can be supplemented by an even order hierarchy. The full (even and odd) sequence is generated by an Olver recursion operator of order one and any pair has zero bracket in the quotient ring. The even order equations do not possess a Hamiltonian structure and thus their related Rosencrans densities are considered.

1980 *Mathematics subject classification* (1985 Revision): 35G20.

1. Introduction

Consider the algebra P of differential polynomials in one dependent variable u and one independent variable x. It consists of polynomials in the variable $u_0, u_1, u_2, \ldots$ where u_n is the nth derivative of u and it has a Lie algebra structure with bracket $[p,q]=v(p)q-v(q)p$ where the derivation $v(k)$ is defined by (sum on repeated indices)

$$v(k)=D^j(k)\frac{\partial}{\partial u^j}.$$

D^j is just the derivative operator D to the jth power. If $u_t=p$ and $u_s=q$ are evolution equations then the bracket measures the commutativity of the flows: $[p,q]=q_t-p_s=u_{st}-u_{ts}$. It is well known that the usual sequence of higher (odd) order Kortweg–de Vries (KdV) equations k_{2n+1} all have mutually zero brackets and thus have mutually commuting evolutionary flows (see Olver's book [2] for a thorough exposition of these and other standard facts from the formal variational calculus). The first three members of the KdV sequence are

$$\bar{k}_1=u_1$$

$$\bar{k}_3=u_3+u_0u_1$$

$$\bar{k}_5=u_5+\tfrac{5}{3}u_0u_3+\tfrac{10}{3}u_1u_2+\tfrac{5}{6}u_0^2u_1$$

and the whole sequence can be generated by a recursion operator $R=D^2+\tfrac{2}{3}u_0+\tfrac{1}{3}u_1D^{-1}$

where the inverse D^{-1} always acts on polynomials in the image of D in the case of the KdV sequence and thus poses no problem of definition [3]. Thus $k_{2n+3}=Rk_{2n+1}$ for the KdV case. If R had a square root S then there would be a complete (both even and odd) set of KdV equations k_n for each n related by $k_{n+1}=Sk_n$, but S does not exist as an operator expressible without an infinite series of negative powers of D. Thus there are no even KdV equations in the algebra P.

In this paper we make the observation that if we restrict ourselves to the quotient ring P/P^3 where P^3 is the ideal of polynomials of degree three or more (generated by cubic monomials), then the square root S and the resulting even KdV equations exist in that quotient ring. The ring P/P^3 consists of linear and quadratic differential polynomials, and the complete KdV sequence is

$$k_1=u_1$$

$$k_2=u_2+\tfrac{1}{3}u_0^2$$

$$k_3=u_3+u_0u_1$$

$$k_4=u_4+\tfrac{4}{3}u_0u_2+u_1^2$$

$$k_5=u_5+\tfrac{5}{3}u_0u_3+\tfrac{10}{3}u_1u_2 \text{ etc.}$$

The odd order ones have the same linear and quadratic terms as the usual KdV equations. The operator generating this sequence is

$$S=D+\tfrac{1}{3}u_0D^{-1}$$

which has square

$$S^2=(D+\tfrac{1}{3}u_0D^{-1})^2=R+\tfrac{1}{9}u_0D^{-1}u_0D^{-1}$$

and differs from R by a quadratic term which leads to at least cubic terms when applied to any polynomial. Thus S^2 and R are equivalent on P/P^3. Strictly speaking we should be using $P(D^{-1})/P^3(D^{-1})$ which is the formal extension of P/P^3 by D^{-1} but the D^{-1} terms never arise in the generation of the full P/P^3 KdV sequence just as they never do in the ordinary sequence.

2. The recursion operator

It can be shown using Olver's criteria ([2] and [3]) that S is a recursion operator for k_2. In fact we have the following.

Theorem 1. *Let $k_{n+1}=S^nu_1$. For any $i,j>0$ the bracket $[k_i,k_j]\equiv 0$ where $\equiv$ is* mod P^3.

Proof. Olver's criteria says that S is a recursion operator for k_2 if $d(-u_t+k_2)S-Sd(-u_t+k_2)=0$ where the differential of the evolution equation $u_t=k_2$

$$d(-u_t+k_2)=-D_t+\frac{\partial k_2}{\partial u_j}D^j$$

is used. D_t is the total time derivative. A simple operator computation shows that

$$d(-u_t+k_2)S-Sd(-u_t+k_2)=-\tfrac{2}{9}Du_0D^{-1}u_0$$

which is zero on P/P^3 because a quadratic (in u) operator $Du_0D^{-1}u_0$ acting on any element in P always results in an element in P^3 (really $P^3(D^{-1})$) which is equivalent to zero. Hence we have

$$\begin{aligned}[k_2,k_i]&=v(k_2)k_i-v(k_i)k_2\\&=v(k_2)k_i-d(k_2)k_i=(v(k_2)-d(k_2))k_i\\&=-d(-u_t+k_2)k_i=-d(-u_t+k_2)S^{i-2}k_2\\&\equiv -S^{i-2}d(-u_t+k_2)k_2=-S^{i-2}[k_2,k_2]=0,\end{aligned}$$

using the algebraic relations between v,d, and the bracket (thus any bracket $[k_2,k_i]$ will contain at least cubic terms). For example $[k_2,k_3]=\frac{1}{3}u_0^2u_1$. Next, observe that by the Jacobi identity for the bracket, for any $i,j>0$ we have $[[k_i,k_j],k_2]\equiv 0$. Obviously $[k_i,k_j]\equiv q$ which is a pure quadratic polynomial but $[q,k_2]\equiv[q,u_2]$ is yet another quadratic. The only quadratic $\equiv 0$ is zero. Hence $[q,u_2]=0$ in P, not just in P^3. But by a standard theorem (Theorem 5.22 in [**2**]) the only pure quadratic polynomial q commuting with u_2 must be zero. Thus $q=0$ and consequently $[k_i,k_j]\equiv 0$. This concludes the proof.

An argument similar to the second part of the above proof also based on the Jacobi identity and quadratic terms will prove the Tu commutativity theorem for P/P^3. (See [**1**], [**6**] for the original theorem, p. 311 in [**2**] for a simplified proof and [**7**] for its occurrence in certain systems.)

3. Rosencrans densities

The odd order $\bmod P^3$ KdV equations still have Hamiltonian structure $u_t=DE[H]$ (see [**2**, **4**]) where the conserved densities H are taken $\bmod P^4$ (P^4 is generated by quartic monomials) since the Euler operator $E=(-D)^j(\partial/\partial u_j)$ takes u_j derivatives reducing the degree of H by one. For example the Hamiltonian form of the 5th order $\bmod P^3$ KdV equation is

$$u_t=DE[\tfrac{1}{2}u_2^2-\tfrac{5}{6}u_0u_1^2]$$

$$=u_5+\tfrac{10}{6}u_0u_3+\tfrac{10}{3}u_1u_2.$$

Rosencrans [5] has observed that if $u_r=p$ and $u_s=q$ are two flows commuting with a Hamiltonian system $u_t=DE[H]$ then $T=pD^{-1}q$ is a conserved density of the Hamiltonian system. That is, the integral of T over space is independent of time. We can modify his proof to obtain:

Theorem 2. *Let $u_r=p$ and $u_s=q$ commute with $u_t=DE[H]$ in P/P^3. Then $T=pD^{-1}q$ is a conserved density* mod P^4 *for $u_t=DE[H]$.*

Proof. Consider the following computation of the integrand (where we may thus use $\equiv$ mod image of D).

$$D_t pD^{-1}q=p_tD^{-1}q+pD^{-1}q_t$$

$$=v(DE[H])p\cdot D^{-1}q+pD^{-1}v(DE[H])q$$

$$\equiv v(p)DE[H]D^{-1}q+pD^{-1}v(q)DE[H] \qquad \operatorname{mod} P^4$$

$$\equiv -qD^{-1}v(p)DE[H]+pD^{-1}v(q)DE[H] \qquad \operatorname{mod}\operatorname{im} D$$

$$=-qv(p)E[H]+pv(q)E[H]$$

because $v(p)$ and $v(q)$ commute with D

$$=(pv(q)-qv(p))E[H]$$

$$=v(q)(pE[H])-v(p)(qE[H])-(v(q)p-v(p)q)E[H]$$

since $v(p)$ and $v(q)$ are derivations

$$\equiv v(q)(pE[H])-v(p)(qE[H]) \qquad \operatorname{mod} P^4$$

$$\equiv v(q)(v(p)H)-v(p)(v(q)H) \qquad \operatorname{mod}\operatorname{im} D$$

by definition of E and v and integration by parts

$$=v([q,p])H\in P^3\cdot P=P^4\equiv 0 \qquad \operatorname{mod} P^4$$

by a standard identity for v.

In other words $D_tpD^{-1}q=a+b$ where $a\in P^4$ and $b\in\operatorname{im} D$. Hence

$$\frac{d}{dt}\int pD^{-1}q\,dx=\int a\,dx \quad \text{where} \quad a\in P^4.$$

For example

$$k_2D^{-1}k_3=(u_2+\tfrac{1}{3}u_0^2)(u_2+\tfrac{1}{2}u_0^2)$$

$$=u_2^2+\tfrac{5}{6}u_0^2u_2+\tfrac{1}{6}u_0^4$$

$$=u_0^2+\tfrac{5}{6}u_0^2u_2+\tfrac{5}{36}u_0^4+\tfrac{1}{36}u_0^4$$

$$=\text{usual 2nd order KdV conserved density}+\tfrac{1}{36}u_0^4.$$

In fact all of the higher order KdV densities H_n can be generated recursively mod P^4 by

$$k_2D^{-1}k_{2n+1}=k_2D^{-1}DE[H_n]=k_2E[H_n]\equiv\pm H_{n+1} \quad \bmod P^4.$$

The orders are correct because $E[H_n]$ has order $2n$ with highest order term u_{2n}. Thus $k_2E[H_n]$ has highest order term $u_2u_{2n}\equiv\pm u_{n+1}^2 \quad \bmod \operatorname{im} D$.

In general we have

$$k_{2m}D^{-1}k_{2n+1}\equiv H_{n+m} \qquad \bmod P^4$$

$$k_{2m}D^{-1}k_{2n}\equiv 0 \qquad \bmod P^4, \bmod \operatorname{im} D$$

$$k_{2m+1}D^{-1}k_{2n+1}\equiv 0 \qquad \bmod P^4, \bmod \operatorname{im} D.$$

The first formula shows how an even KdV equation can be used to produce higher order conserved densities when the Rosencrans product with an odd KdV is taken. The second says that two evens only produce a trivial (in the image of D) density and the third really says that the Poisson bracket of H_m and H_n is zero, reflecting the commutativity of k_{2m+1} and k_{2n+1} because for Hamiltonian equations the Rosencrans density and the Poisson bracket coincide (see [5]).

Acknowledgement. I would like to thank SERC for financial suppoort and am grateful to Peter Olver and Steven Rosencrans for many helpful discussions on the formal variational calculus.

REFERENCES

1. N. Kh. Ibragimov and A. B. Shabat, Evolution equations with non-trivial Lie–Bäcklund group, *Functional Anal. Appl.* **14** (1980), 19–28.

2. P. J. Olver, *Applications of Lie Groups to Differential Equations* (GTM **107**, Springer-Verlag, New York, 1986).

3. P. J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* **18** (1977), 1212–1215.

4. P. J. Olver, On the Hamiltonian structure of evolution equations, *Math. Proc. Cambridge Philos. Soc.* **88** (1980), 71–88.

5. S. I. Rosencrans, Conservation laws generated by pairs of non-Hamiltonian symmetries, *J. Differential Equations* **43** (1982), 305–322.

6. G. Z. Tu, A commutativity theorem of partial differential operators, *Comm. Math. Phys.* **77** (1980), 289–297.

7. J. M. Verosky, Tu commutativity for quasilinear evolution equations, *Phys. Lett. A* **119** (1986), 25–27.

Department of Mathematics
Heriot-Watt University
Riccarton, Edinburgh EH14 4AS
Scotland