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Roitman's Theorem for Singular Projective Varieties

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Abstract. We construct an Abel–Jacobi mapping on the Chow group of 0-cycles of degree 0, and prove a Roitman theorem, for projective varieties over $\mathbb{C}$ with arbitrary singularities. Along the way, we obtain a new version of the Lefschetz Hyperplane theorem for singular varieties.

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1. Introduction

If X is a smooth complete variety of dimension n over the complex field $\mathbb{C}$, there is a natural map $CH^n(X)_{\deg 0} \to \mathrm{Alb}\,(X)$ from the Chow group of 0-cycles of degree 0 (modulo rational equivalence) to the Albanese variety of X. It is known that this is surjective, and has a 'large' kernel if $H^0(X, \Omega^i_{X/\mathbb{C}}) \neq 0$ for some $i \geqslant 2$. A famous result of Roitman [R] asserts that $CH^n(X)_{\deg 0} \to \mathrm{Alb}\,(X)$ is always an isomorphism on torsion subgroups. Our aim in this paper is to generalize this theorem to reduced projective varieties with arbitrary singularities.

We begin with some background. Let X be any reduced projective variety of dimension n over $\mathbb{C}$. Let X_{sing} (the singular locus) denote the set of points $x \in X$ such that the module of Kähler differentials $\Omega^1_{\mathcal{O}_{x,X}/\mathbb{C}}$ is not a free module of rank n; thus X_{sing} includes points x through which X has a component of dimension $< n$. For $Y \subset X$ a closed subvariety containing X_{sing}, Levine and Weibel [LW] have defined the relative Chow (cohomology) group of 0-cycles $CH^n(X, Y)$ to be the quotient of the free Abelian group on points of $X - Y$ modulo a suitable notion of rational equivalence (given by the subgroup of cycles of the form $(f)_C$ where C is a 'Cartier curve' on (X, Y) and f a rational function on C which is a unit at points of $C \cap Y$). If Y has codimension $\geqslant 2$, the relations may also be given by sums of cycles $(f)_C$, where C is any curve which does not meet Y. If X is integral and Y has codimension $\geqslant 2$, Levine [L3] and Collino [Co] have shown that Roitman's theorem holds for $CH^n(X, Y)$, if we take $\mathrm{Alb}\,X$ to be the Albanese variety of any resolution of singularities of X.

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In this paper, we will take $CH^n(X)$ to be the group $CH^n(X, X_{\mathrm{sing}})$ as defined in [LW] (see also [L2]).

In a recent paper [BPW], the authors show that if X is a projective surface, then the Albanese variety of X, now *defined* to be the group

$$J^2(X) := \frac{H^3(X, \mathbb{C})}{F^2H^3(X, \mathbb{C}) + \mathrm{im}\, H^3(X, \mathbb{Z})},$$

is in fact a semi-Abelian variety (here F^2 denotes the subspace for the Hodge filtration of the mixed Hodge structure). They construct a natural homomorphism $CH^2(X) \twoheadrightarrow \mathbb{Z}^{\oplus t}$, where X has t irreducible components of dimension 2; the kernel is denoted by $CH^2(X)_{\deg 0}$. They then construct a homomorphism $CH^2(X)_{\deg 0} \to J^2(X)$, using simplicial techniques and Chern classes with values in Deligne cohomology. The main theorem of [BPW] is that this map is an isomorphism on torsion subgroups. Their proof is technically quite complicated, and uses particular features of the two-dimensional case, like the relationship with K_2, which implies (via excision, double-relative K-groups, etc.) a formula for the Chow group in terms of that of its normalization, and a quotient of SK_1 groups of possibly nonreduced curves.

This motivates our main result.

THEOREM 1.1. *Let X be a reduced projective variety over $\mathbb{C}$ of dimension n.*

(a) *Define*

$$J^n(X) := \frac{H^{2n-1}(X, \mathbb{C})}{F^nH^{2n-1}(X, \mathbb{C}) + \mathrm{im} H^{2n-1}(X, \mathbb{Z})}.$$

Then $J^n(X)$ is the group of closed points of a semi-Abelian variety over $\mathbb{C}$ (which we again denote by $J^n(X)$). The association $X \mapsto J^n(X)$ is a contravariant functor from n-dimensional projective varieties over $\mathbb{C}$ to semi-Abelian varieties over $\mathbb{C}$.

(b) *There is a degree map $\deg_X: CH^n(X) \to \mathbb{Z}^{\oplus t}$, where X has t irreducible components of dimension n. Let $CH^n(X)_{\deg 0}$ denote the kernel of $\deg_X$. Then there is a surjective Abel–Jacobi map $AJ^n_X: CH^n(X)_{\deg 0} \to J^n(X)$. If $U = X_0 - X_{\mathrm{sing}}$, where X_0 is any n-dimensional component of X, and $x_0 \in U$ is a base point, then the map $x \mapsto AJ^n_X([x] - [x_0])$, defined on closed points of U, is induced by a morphism $U \to J^n(X)$.*

(c) *If $Y \subset X$ is a reduced subscheme of dimension m which fits into a chain of subschemes $Y = Y_m \subset Y_{m+1} \subset \cdots \subset Y_n = X$, such that Y_j is a reduced Cartier divisor on Y_{j+1} for each $j < n$, then there is a commutative diagram*

$$\begin{array}{ccc} CH^m(Y)_{\deg 0} & \longrightarrow & CH^n(X)_{\deg 0} \\ \downarrow{\scriptstyle AJ^m_Y} & & \downarrow{\scriptstyle AJ^n_X} \\ J^m(Y) & \longrightarrow & J^n(X) \end{array}$$

(where $CH^m(Y) \to CH^n(X)$ *is induced by the obvious map on* 0*-cycles determined by the inclusion* $Y \subset X$*). The map* $J^m(Y) \to J^n(X)$ *is independent of the particular chain* $\{Y_m, Y_{m+1}, \dots\}$ *chosen.*

(d) *(Lefschetz theorem) If in (c) above,* Y *is a general complete intersection of very ample divisors in* X*, then* $J^m(Y) \to J^n(X)$ *is an isomorphism if* $\dim Y = m \geqslant 2$*, and is surjective on* N*-torsion for each* N*, if* $m = 1$ *(in particular,* $J^1(Y) \twoheadrightarrow J^n(X)$ *in this case).*

(e) *(Roitman theorem) The map* AJ_X^n *is an isomorphism on torsion subgroups.*

Here part (d) is obtained as a corollary of the following version of the Lefschetz Hyperplane Theorem, valid for singular projective varieties, which may be of independent interest (see Section 4.1 below). Part (a) ('without base points'), formulated for cohomology with $\mathbb{C}$-coefficients, has been obtained earlier [GNPP]; we thank the referee for providing us this reference. The rather technical statement made in (b) is needed in the proof of the Roitman Theorem (part (e) of the Main Theorem). Following the referee's suggestions, the two separate 'Lefschetz Theorems' stated in an earlier version of our paper are now combined.

LEFSCHETZ THEOREM

(a) *Let* X *be a reduced projective variety over* $\mathbb{C}$ *of dimension* n *and let* Y *be a general hyperplane section. Then the the Gysin map* $H^i(Y, \mathbb{Z}(j)) \to H^{i+2}(X, \mathbb{Z}(j+1))$ *is an isomorphism for* $i < \dim Y$ *and surjective for* $i = \dim Y$.

(b) *Let* X *be as above,* $\pi: \widetilde{X} \to X$ *the normalization, and* $A \subset X$ *a closed subvariety such that if* $U = X - A$ *and* $V = \widetilde{X} - \pi^{-1}(A)$*, the following are satisfied:*

   (i) V *is nonsingular of dimension* n,
   (ii) $V \to U$ *is the normalization of* U,
   (iii) *if* $W = \pi^{-1}(U_{\text{sing}})$*, then* U_{sing} *and* W *are nonsingular of dimension* $n-1$ *and* $W \to U_{\text{sing}}$ *is an analytic covering space.*

   Let $C \subset X$ *be a reduced curve such that*

   (i′) C *is a local complete intersection in* X.
   (ii′) $C \cap X_{\text{sing}}$ *is reduced, and supported at smooth points of* X_{sing}.
   (iii′) $C \cap (X - X_{\text{sing}})$ *has only plane curve singularities.*
   (iv′) $C \cap A = \emptyset$.

   Let Y *be a general hypersurface section of* X *of sufficiently large degree, and which contains* C*. Then the Gysin map* $H^i(Y, \mathbb{Z}(j)) \to H^{i+2}(X, \mathbb{Z}(j+1))$ *is an isomorphism for* $i < \dim Y$ *and surjective for* $i = \dim Y$.

Thus, there are 3 main results in the paper: the construction of the Abel–Jacobi map on rational equivalence classes for arbitrary dimensional varieties, and the proofs of the Lefschetz theorem (with $\mathbb{Z}$-coefficients, and with base conditions) and the Roitman theorem. We comment further on the proofs.

The Abel–Jacobi map is constructed on the level of 0-cycles using Deligne's 1-motives [D] (or equivalently, extension classes for mixed Hodge structures [C]). To show that rational equivalence is preserved, we use a moving lemma for Cartier curves, together with functoriality properties of mixed Hodge structures with respect to point blow-ups, pullbacks and Gysin maps. The discussion of the functoriality for the Gysin maps uses M. Saito's theory [MS] of Mixed Hodge Modules, to give a clean exposition, though ad hoc (messy) arguments of a more elementary type are possible in the cases at hand.

The proof of the Roitman Theorem is motivated by Bloch's proof of Roitman's theorem for smooth varieties, as outlined in his book [B]. The proof is in 2 stages. First, one proves the result for surfaces; this is the main result of [BPW] (we have another proof of this case, which does not use Deligne–Beilinson cohomology; this has been omitted for reasons of space). The second step is a reduction of the general case to that of surfaces, using the moving lemma for Cartier curves, point blow-ups and the Lefschetz Theorem (the version with base conditions).

The Lefschetz Theorem is proved using a result of Verdier [V], that any morphism of varieties is locally trivial (in the complex topology) over some Zariski open subset of the target. We remark that this differs from the technique of [GNPP] (that of *hyperrésolutions cubiques*).

We believe that our proof should generalize to yield Roitman's theorem over fields of positive characteristic, for torsion relatively prime to the characteristic, at least if we assume resolution of singularities (this was one reason for giving a new proof for the surface case as well). This is because, though we have extensively made use of the complex topology in our presentation, we have mostly only used either 'topological' or 'motivic' arguments while doing so. We hope to return to this later.

In (b) of the Main Theorem above, we can understand the algebricity of the map $U \to J^n(X)$ from two points of view. The first way is to deduce it from a general, purely algebraic description of the 1-motive associated (following Deligne [D]) to $H^{2n-1}(Y, \mathbb{Z}(n))$, for any n-dimensional (possibly noncomplete) variety Y over $\mathbb{C}$, and in particular, an algebraic construction of the semi-Abelian variety $J^n(X)$. This will be proved elsewhere [BS]; we mention this here, however, to further bring out the analogy with the Albanese variety of a smooth complete variety. This approach is also related to Serre's construction [Se] (see [Me] for an exposition) of a generalized Albanese variety for a noncomplete smooth variety, applied to components of $X - X_{\text{sing}}$.

We will sketch below a second proof of algebricity which reduces it to the case of curves, where it is known.

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2. Cartier Curves

We first recall the definition of $CH^n(X) = CH^n(X, X_{\text{sing}})$ from [LW] (see also [L2]). A *Cartier curve* on X is a subscheme $C \subset X$ such that

(i) C is pure of dimension 1 (*i.e.*, C has no 0-dimensional irreducible components);
(ii) no component of C is contained in X_{sing};
(iii) if $x \in C \cap X_{\text{sing}}$, then the ideal of C in $\mathcal{O}_{x,X}$ is generated by a regular sequence (of $n-1$ elements).
(iv) $C \cap (X - X_{\text{sing}})$ has no embedded points (*i.e.*, is Cohen–Macaulay), and any component C' of C which is disjoint from X_{sing} appears in C with multiplicity 1 (*i.e.*, if $\eta \in C'$ is the generic point, then $\mathcal{O}_{\eta,C}$ is reduced).

In particular, if $n = 2$, then C is an effective Cartier divisor on X (but not conversely). Note that in (iii), if $x \in C \cap X_{\text{sing}}$, then $\dim \mathcal{O}_{x,X} = n$, since no irreducible component of C is contained in any component of X of dimension $< n$.

If C is a Cartier curve on X, with generic points $\eta_1, \ldots, \eta_r$, $S = (C \cap X_{\text{sing}}) \cup \{\eta_1, \ldots, \eta_r\}$, and $\mathcal{O}_{S,C}$ is the semilocal ring on C of the points S, then there is a natural map on unit groups $\theta_C\colon \mathcal{O}^*_{S,C} \to \oplus_{i=1}^r \mathcal{O}^*_{\eta_i,C}$. Define $R(C, X) = \operatorname{image} \theta_C$. Note that this depends on the pair (C, X).

Remark. In [LW], the discussion of rational functions on page 108 seems to suggest that θ_C is injective; this will not be true, unless $\mathcal{O}_{S,C}$ is Cohen–Macaulay. However this point does not affect the rest of the discussion there. In the Definition 1.2 of a Cartier curve, they do not include the condition (iv) above, but their Lemma 1.3 shows that for the purposes of the Chow group, it suffices to restrict ourselves to such curves (i.e., no additional elements in $R^n(X)$ are obtained by allowing the more general Cartier curves which do not satisfy (iv)).

We now define the divisor $(f)_C$ of an element $f \in R(C, X)$, for any Cartier curve C. Let C_i denote the maximal Cohen–Macaulay subscheme of C supported on the irreducible component with generic point η_i. Then $\mathcal{O}_{\eta_i,C} = \mathcal{O}_{\eta_i,C_i}$, and for any $x \in C_i$, the map $\mathcal{O}_{x,C_i} \to \mathcal{O}_{\eta_i,C_i}$ is the inclusion of a one-dimensional Cohen–Macaulay local ring into its total quotient ring. If f_i is the component in $\mathcal{O}_{\eta_i,C_i}$ of $f \in R(C, X)$, then for any closed point $x \in C_i$, we can write $f_i = a_x/b_x$ where $a_x, b_x \in \mathcal{O}_{x,C_i}$ are nonzero divisors; now let

$$(f_i)_{C_i} = \sum_{x \in C_i} \big(\ell(\mathcal{O}_{x,C_i}/a_x\mathcal{O}_{x,C_i}) - \ell(\mathcal{O}_{x,C_i}/b_x\mathcal{O}_{x,C_i})\big) \cdot [x],$$

which is a 0-cycle supported on C_i (since f_i is a unit for all but finitely many x). Finally, set

$$(f)_C = \sum_i (f_i)_{C_i}.$$

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(This definition of $(f)_C$ is easily seen to be equivalent to Definition 2 of [LW].)

DEFINITION. The *Chow group of 0-cycles* $CH^n(X) = CH^n(X, X_{\text{sing}})$ is defined to be the quotient $CH^n(X) = Z^n(X)/R^n(X)$, where $Z^n(X)$ is the free Abelian group on (closed) points of $X - X_{\text{sing}}$, and $R^n(X)$ is the subgroup generated by cycles $(f)_C$ for all Cartier curves C in X, and all $f \in R(C, X)$.

Levine ([L2], Lemma 1.4) has shown that if X is integral, then $R^n(X)$ is also the group generated by cycles $(f)_C$ where C ranges over *integral* Cartier curves in X, and $f \in R(C, X)$. His proof shows that if X is reduced, then $R^n(X)$ is generated by $(f)_C$ for reduced Cartier curves C. The following is a useful technical lemma refining Levine's assertion. If $\delta = \sum_{i=1}^m n_i P_i$, let $\operatorname{supp}(\delta) = \{P_1, \ldots, P_m\} \subset X$. We will use versions of Bertini's theorem in the proof below, and later in the paper; a general Bertini theorem which implies whatever we need is Satz 5.2 of [F].

LEMMA 2.1. *Let X be a reduced, quasi-projective variety, and let $\delta \in R^n(X)$. Let $A \subset X_{\text{sing}}$ be a closed subset of codimension $\geqslant 2$ in X, and $D \subset X$ be a reduced effective divisor. Let D_1 be the union of components of D which are contained in X_{sing}. Then we can find a reduced (possibly reducible) Cartier curve C, and $f \in R(C, X)$, such that $(f)_C = \delta$, and such that* (i) *$C \cap D$ is 0-dimensional* (ii) *$C \cap D_1$ is a reduced 0-dimensional scheme, and* (iii) *$C \cap A = \emptyset$.*

Proof. We first sketch Levine's argument, and then indicate how it needs to be modified; we use his notation to help the reader make the comparison with his proof. We find it most convenient to do this as a sublemma.

SUBLEMMA 1. *Let X be a reduced, quasi-projective variety, $A \subset X_{\text{sing}}$ a closed subset of codimension $\geqslant 2$ in X, and D a reduced divisor in X. Let D_1 be the union of the components of D which are contained in X_{sing}. Let $\gamma = (f)_Z$ for some Cartier curve Z and $f \in R(Z, X)$. Then there exists a reduced Cartier curve C on X, and an element $g \in R(C, X)$, such that*

(a) $\gamma = (g)_C$
(b) $C \cap A = \phi$, and $C \cap D$ is finite, and $C \cap D_1$ is finite and reduced.

Proof. First note that since Z is a Cartier curve on X, the general complete intersection surface $T \subset X$ containing Z is reduced. We will remove from T any 0-dimensional irreducible components; this does not change T in a neighbourhood of Z; also, the modified T is still a local complete intersection in X. Clearly $T \cap X_{\text{sing}}$ has dimension $\leqslant 1$, $T \cap D$ is a reduced divisor in T, and $T \cap A$ is a finite set (it may be empty).

For any such general surface T containing Z, following Levine, take L to be a high power of a given very ample line bundle. Then choose s_∞ and s_0 to be general sections of the line bundles L and $L^{\otimes N}$ respectively, for some large enough N, such that if $C_0 = (s_0)$, $C_\infty = (s_\infty)$ are the respective divisors of zeroes, then the following conditions are satisfied:

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(i) C_∞ contains a given finite set S of points of $T - X_{\text{sing}}$. The finite set S has the following structure: first choose a general effective divisor Z' such that $Z+Z'$ is a very ample Cartier divisor on T, then choose Z_∞ a general effective Cartier divisor linearly equivalent to $Z+Z'$; now take S to be the union of the nonregular locus of f on Z with $(Z\cap Z')\cup\big((Z\cup Z')\cap Z_\infty\big)$. We only need to use below that S is the union of the nonregular locus of f (on Z), the set where Z is not a Cartier divisor in T (*i.e.*, where Z is not a local complete intersection in X), and a set 'in general position' in a suitable sense. (Note that at any point $x \in Z$ where Z is not a Cartier divisor in T, we have $x \in Z\cap Z'$, since $Z+Z'$ is a Cartier divisor by choice.)

(ii) Let h be the rational function on $Z\cup Z'\cup Z_\infty$ which is f on Z, and 1 on $Z'\cup Z_\infty$. By construction, the nonregular locus of h on $Z\cup Z'\cup Z_\infty$ is contained in S. Hence there exists a sufficiently large $N>0$ such that $s' = s_\infty^{\otimes N}\mid_{Z\cup Z'\cup Z_\infty}\cdot h$ is a regular section of $L^{\otimes N}\mid_{Z\cup Z'\cup Z_\infty}$; let s_0 be a general regular section of $L^{\otimes N}$ which lifts s'.

Let C_0', C_∞' be the curves obtained by removing the 0-dimensional components of C_0, C_∞. Levine shows that for suitable functions $g_0 \in R(C_0', X)$, $g_\infty \in R(C_\infty', X)$, we have an equation between 0-cycles $\gamma = (f)_Z = (g_0)_{C_0'} + (g_\infty^{-N})_{C_\infty'}$.

Levine notes that C_0', C_∞' are reduced, and without common components, from Bertini's theorem (since T is a reduced surface, L is sufficiently ample on T, and s_0, s_∞ are sufficiently general sections). Actually, Levine works in the situation where X is irreducible, so that C_0, C_∞ are also irreducible, and so $C_0 = C_0'$, $C_\infty = C_\infty'$. However, his argument gives the stated conclusions.

Since s_∞ and s_0 are general (subject to the conditions mentioned above), it is easy to see that that s_∞ and s_0 may be chosen so that

(1) $C_0\cap C_\infty\cap X_{\text{sing}} = \phi$;
(2) $(C_0\cup C_\infty)\cap A = \phi$, and $(C_0\cup C_\infty)\cap(T\cap D)$ is a 0-dimensional scheme
(3) $(C_0\cup C_\infty)\cap D_1$ is a reduced 0-dimensional scheme.

(recall $T\cap A$ is a finite subset of T_{sing}, while $T\cap D$ is a reduced curve in T). The only subtlety here is in (3): the point is that Z is a local complete intersection in T at all points of $T\cap X_{\text{sing}}$, and in particular at points of $T\cap D_1$, and also f is a unit at these points. Hence the finite set S of 'base points' considered earlier can be taken to be disjoint from D_1. Now (3) follows from Bertini's theorem.

Since by construction, C_0', C_∞' are reduced Cartier curves on X with no common irreducible component, and since $C_0'\cap C_\infty'\cap X_{\text{sing}} = \phi$, their union $C = C_0'\cup C_\infty'$ is also a Cartier curve, and $R(C, X) = R(C_0', X)\oplus R(C_\infty', X)$. Hence if g is the function defined by $g\mid_{C_0'} = g_0$, $g\mid_{C_\infty'} = g_\infty^{-N}$, then $g\in R(C, X)$ and $(g)_C = \gamma$. □

Now given *any* $\delta \in R^n(X)$ as in the statement of the lemma, we claim that we may write $\delta = (f)_Z$ for some reduced Cartier curve Z and some $f \in R(Z, X)$. Indeed, first write $\delta = \sum_{i=1}^r (f_i)_{Z_i}$ for suitable Cartier curves Z_i and elements $f_i \in R(Z_i, X)$. We argue by induction on r; the case $r = 1$ is covered by the

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sublemma. Applying the sublemma to $\gamma_1 = (f_1)_{Z_1}$ (and $A = D = \phi$), we may assume without loss of generality that Z_1 is reduced; now apply the sublemma to $\gamma_2 = (f_2)_{Z_2}$ and $A = (Z_1 \cap X_{\text{sing}})$, and any suitable $D \supset Z_1$. Hence we may assume that Z_2 is also reduced, Z_1 and Z_2 have no common components, and $Z_1 \cap Z_2 \cap X_{\text{sing}} = \phi$. But then $Z_1 \cup Z_2$ is also a reduced Cartier curve on X, and $R(Z_1 \cup Z_2, X) = R(Z_1, X) \oplus R(Z_2, X)$. Hence we have an expression for δ as a sum of divisors of functions on $r - 1$ Cartier curves.

Now finally apply the sublemma once more, to replace the given reduced Cartier curve Z and function f by another such pair, where in addition $Z \cap A = \emptyset$, $Z \cap D$ is finite, and $Z \cap D_1$ is a reduced 0-dimensional scheme, for the original choices of A and D. □

3. The Semi-Abelian Variety $J^n(X)$

In this section, our goal is to study the properties of $J^n(X)$, and in particular, to prove parts (a), (c) of the Main Theorem, and the algebricity assertion in (b). The surjectivity assertion in (b) will be proved in the next section.

We begin with (a), which is implicit in Deligne's paper [D]. Let X be a reduced projective scheme of dimension n over $\mathbb{C}$, and

$$J^n(X) := \frac{H^{2n-1}(X, \mathbb{C})}{F^n H^{2n-1}(X, \mathbb{C}) + \operatorname{im} H^{2n-1}(X, \mathbb{Z})}.$$

We find it more convenient to work with the (equivalent) definition

$$J^n(X) := \frac{H^{2n-1}(X, \mathbb{C}(n))}{F^0 H^{2n-1}(X, \mathbb{C}(n)) + \operatorname{im} H^{2n-1}(X, \mathbb{Z}(n))}.$$

By [D] (8.2.4)(ii), $H^{2n-1}(X, \mathbb{Z}(n))/(\text{torsion})$ is a torsion-free mixed $\mathbb{Z}$-Hodge structure whose nonzero Hodge numbers $h^{p,q}$ have (p, q) in the set

$$H = \{(-1, -1), (-1, 0), (0, -1)\}.$$

Further, the pure $\mathbb{Q}$-Hodge structure $\operatorname{Gr}_W^{-1} H^{2n-1}(X, \mathbb{Q}(n))$ is polarizable, since (by [D], (8.2.5)) it is a Tate twist of a mixed $\mathbb{Q}$-Hodge substructure of $H^{2n-1}(\widetilde{X}, \mathbb{Q})$, where $\widetilde{X}$ is a resolution of singularities of X. By [D], (10.1.3), there is an equivalence of categories between torsion-free mixed $\mathbb{Z}$-Hodge structures M with Hodge numbers in H such that $\operatorname{Gr}_W^{-1}(M \otimes \mathbb{Q})$ is polarizable, and the category of semi-abelian varieties over $\mathbb{C}$ (and algebraic homomorphisms between them). Since $X \mapsto H^{2n-1}(X, \mathbb{Z}(n))/(\text{torsion})$ is a contravariant functor on the category of n-dimensional projective varieties, (a) is proved. Thus $J^n(X)$ is the group of closed points of a semi-abelian variety over $\mathbb{C}$, which we also denote by $J^n(X)$.

Next, we construct the 'Abel–Jacobi map' $AJ_X^n\colon CH^n(X)_{\deg 0} \to J^n(X)$. Let $X_1, \ldots, X_t$ be the n-dimensional irreducible components of X, and let X_0 be the

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union of the irreducible components of dimension $< n$ (note that $X_0 \subset X_{\rm sing}$). If $\delta \in Z^n(X)$ is a 0-cycle supported on $X - X_{\rm sing}$, then we can uniquely write $\delta = \sum_{i=1}^t \delta_i$ with δ_i supported on $X_i - \cup_{j\neq i} X_j$. Define $\deg_{X_i}(\delta) = \deg \delta_i$, where 'deg' denotes the usual degree of a 0-cycle (*i.e.*, $\deg(\sum_j n_j[x_j] = \sum_j n_j)$. Then we have a (surjective) degree homomorphism $\deg_X\colon Z^n(X) \twoheadrightarrow \mathbb{Z}^{\oplus t}$, defined by $[\delta] \mapsto (\deg_{X_1}(\delta), \ldots, \deg_{X_t}(\delta))$.

As is easily seen, another description of the degree homomorphism is as follows: if $S = \operatorname{supp} \delta$, then there is a map

$$H^{2n}_S(X, \mathbb{Z}(n)) \to H^{2n}(X, \mathbb{Z}(n)) \cong \oplus_{i=0}^t H^{2n}(X_i, \mathbb{Z}(n)) \cong \mathbb{Z}^{\oplus t},$$

since $H^{2n}(X_0, \mathbb{Z}(n)) = 0$, and $H^{2n}(X_i, \mathbb{Z}(n)) \cong \mathbb{Z}$ for $1 \leqslant i \leqslant t$. Then the image of the cohomology class of δ in $H^{2n}_S(X, \mathbb{Z}(n))$ (which is the free Abelian group on the points in $\operatorname{supp}(\delta)$) is just $\deg_X(\delta) \in \mathbb{Z}^{\oplus t}$.

We claim that the degree map induces a well-defined homomorphism $\deg_X\colon CH^n(X) \to \mathbb{Z}^{\oplus t}$. For this, we must show that if C is a Cartier curve, and $f \in R(C, X)$, then $\deg_{X_i}((f)_C) = 0$ for each i. If $C_1, \ldots, C_r$ are the irreducible components of C, let f_j be the restriction of f to C_j, so that $(f)_C = \sum (f_j)_{C_j}$. Notice that at any point of $C_j \cap X_i \cap X_k$ with $k \neq i$, which is necessarily a singular point of X, we have $f_j \in \operatorname{image}(\mathcal{O}^*_{x,C_j} \to \mathcal{O}^*_{\eta_j,C_j})$ (where the generic point of C_j is η_j); hence for all j, we see that $(f_j)_{C_j}$ *has support disjoint from* $X_i \cap X_k$ *for all* $i \neq k$.

Now fix any component X_i of X. Suppose $C_j \subset X_i$ for $j \leqslant s$, and $C_j \not\subset X_i$ for $j>s$; then for $\delta = (f)_C$, we see from the preceeding paragraph that $\delta_i = \sum_{j\leqslant s} (f_j)_{C_j}$. Hence $\deg(\delta_i) = \sum_{j\leqslant s} \deg(f_j)_{C_j} = 0$.

We define $CH^n(X)_{\deg 0} = \ker(\deg_X\colon CH^n(X) \to \mathbb{Z}^{\oplus t})$.

Let δ be a zero cycle on X of degree 0 (*i.e.*, $\deg_{X_i}(\delta) = 0$ for all irreducible components X_i of X), and let $\operatorname{supp}(\delta) = S \subset X - X_{\rm sing}$. Consider the long exact sequence of cohomology groups

$$\begin{aligned} \cdots \to\ & H^i_S(X, \mathbb{Z}(n)) \to H^i(X, \mathbb{Z}(n)) \\ \to\ & H^i(X - S, \mathbb{Z}(n)) \to H^{i+1}_S(X, \mathbb{Z}(n)) \to \cdots. \end{aligned}$$

The groups occurring in this sequence carry natural mixed $\mathbb{Z}$-Hodge structures, such that the maps in the sequence are morphisms of mixed Hodge structures ([D], (6.3), and (8.3.9) applied to the inclusion $X - S \hookrightarrow X$).

For any $x \in X - X_{\rm sing}$, we have an isomorphism $H^{2n}_x(X, \mathbb{Z}(n)) \cong \mathbb{Z}$ as mixed Hodge structures, and $H^{2n-1}_S(X, \mathbb{Z}(n)) = 0$. Hence $H^{2n}_S(X, \mathbb{Z}(n)) \cong \mathbb{Z}[S]$, the free abelian group on S, regarded as a pure Hodge structure (of type (0,0)), and we have an exact sequence of mixed Hodge structures

$$\begin{aligned} 0 \to\ & H^{2n-1}(X, \mathbb{Z}(n)) \to H^{2n-1}(X - S, \mathbb{Z}(n)) \\ \to\ & H^{2n}_S(X, \mathbb{Z}(n)) \to H^{2n}(X, \mathbb{Z}(n)) \to 0. \end{aligned} \tag{1}$$

In particular, δ yields a map of mixed Hodge structures $\mathbb{Z} \cdot [\delta] \to H^{2n}_S(X, \mathbb{Z}(n))$. Now $H^{2n}(X, \mathbb{Z}(n)) \cong \oplus_{i=1}^t H^{2n}(X_i, \mathbb{Z}(n)) \cong \mathbb{Z}^{\oplus t}$. Since δ has degree 0, we see

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that the composite $\mathbb{Z}\cdot[\delta] \to H^{2n}_S(X,\mathbb{Z}(n)) \to H^{2n}(X,\mathbb{Z}(n)) = \mathbb{Z}^{\oplus t}$ is trivial. Hence from the exact sequence (1) we obtain a pull-back exact sequence of mixed Hodge structures

$$0 \to H^{2n-1}(X,\mathbb{Z}(n)) \to M \to \mathbb{Z}\cdot[\delta] \to 0. \tag{2}$$

Hence $M/(\text{torsion})$ is a mixed $\mathbb{Z}$-Hodge structure with Hodge numbers from the set $\{(-1,-1),(-1,0),(0,-1),(0,0)\}$, such that $\mathrm{Gr}^{-1}(M\otimes\mathbb{Q}) = \mathrm{Gr}^{-1}(H^{2n-1}(X,\mathbb{Q}(n)))$ is polarizable. By [D], (10.1.3), we see that $M/(\text{torsion})$ determines a 1-motive over $\mathbb{C}$, yielding a homomorphism $[M]_*\colon\mathbb{Z}\to J^n(X)$. We define $AJ^n_X(\delta) = [M]_*(1)$; this is often called the *extension class* of the mixed Hodge structure M in (2); see [C], for example.

One can give a more concrete description of $AJ^n_X(\delta)$ as follows: considering the Hodge numbers involved, we see that

$$\frac{H^{2n-1}(X,\mathbb{C}(n))}{F^0H^{2n-1}(X,\mathbb{C}(n))} \cong \frac{M\otimes\mathbb{C}}{F^0M\otimes\mathbb{C}};$$

if $\alpha\colon\mathbb{Z}\to M$ is a splitting of the exact sequence (2), then

$$\operatorname{im}\alpha(1) \in \frac{M\otimes\mathbb{C}}{F^0M\otimes\mathbb{C}} \cong \frac{H^{2n-1}(X,\mathbb{C}(n))}{F^0H^{2n1-}(X,\mathbb{C}(n))}$$

depends on the choice of the splitting α, but the image of $\alpha(1)$ in the quotient

$$J^n(X) = \frac{H^{2n-1}(X,\mathbb{C}(n))}{F^0H^{2n-1}(X,\mathbb{C}(n)) + \operatorname{im} H^{2n-1}(X,\mathbb{Z}(n))}$$

is independent of the splitting α. One checks easily (see [D], diagram on page 56) that $AJ^n_X(\delta) = \operatorname{im}(\alpha(1)) \in J^n(X)$. This description of $AJ^n(X)$ amounts to the 'classical' definition of the Abel–Jacobi map via integrals.

It is easy to see that the above construction yields a homomorphism $AJ^n_X\colon Z^n(X)_{\deg 0} \to J^n(X)$ (this is obvious from the second description of AJ^n_X, for example). We claim that $AJ^n_X(R^n(X)) = 0$, so that we have a well-defined homomorphism $AJ^n_X\colon CH^n(X)_{\deg 0} \to J^n(X)$. To see this, we first show that if C is a reduced Cartier curve in X, then there is a commutative diagram

$$\begin{array}{ccc} Z^1(C)_{\deg 0} & \xrightarrow{AJ^1_C} & J^1(C) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Z^n(X)_{\deg 0} & \xrightarrow{AJ^n_X} & J^n(X) \end{array}$$

This will involve the construction of a certain *Gysin map*. Let X be a (reduced) variety, $Y\subset X$ an effective Cartier divisor, and $Z\subset Y$ a subvariety. Then we will

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construct Gysin maps $H^i_Z(Y, \mathbb{Z}(r)) \to H^{i+2}_Z(X, \mathbb{Z}(r+1))$ for any $i \geqslant 0$, and any integer r, which will be morphisms of mixed Hodge structures (the mixed Hodge structures exist as a result of [D], (8.3.9)).

Since Y is an effective Cartier divisor on X, it has a cohomology class $[Y] \in H^2_Y(X, \mathbb{Z}(1))$ (one way to define this is to notice that Y has a class in $H^1_Y(X, \mathcal{O}^*_X)$, and to use the boundary map $H^1_Y(X, \mathcal{O}^*_X) \to H^2_Y(X, \mathbb{Z}(1))$ in the exponential sequence). Define the Gysin map above to be the cup product with the class of Y,

$$H^i_Z(Y, \mathbb{Z}(r)) \xrightarrow{\cup[Y]} H^{i+2}_Z(X, \mathbb{Z}(r+1)).$$

This cup product map may be defined by replacing Y by an open tubular neighbourhood U in X, which has a retraction $\rho: U \to Y$, such that the pair (Y, Z) is a strong deformation retract of $(U, \rho^{-1}(Z))$ (such a tubular neighbourhood exists since X has a triangulation with Y as a subcomplex, for example); this yields an isomorphism $H^i_Z(Y, \mathbb{Z}(r)) \cong H^i_{\rho^{-1}(Z)}(U, \mathbb{Z}(r))$. We then use the excision isomorphisms $H^*_Z(X, \mathbb{Z}(n)) \cong H^*_Z(U, \mathbb{Z}(n))$. Thus the cup product takes values in $H^{i+2}_{Y\cap\rho^{-1}(Z)}(U, \mathbb{Z}(r+1)) = H^{i+2}_Z(X, \mathbb{Z}(r+1))$.

LEMMA 3.1. *Under the above conditions, the Gysin map*

$$H^i_Z(Y, \mathbb{Z}(r)) \xrightarrow{\cup[Y]} H^{i+2}_Z(X, \mathbb{Z}(r+1)).$$

is a morphism of mixed Hodge structures. Equivalently,

$$H^i_Z(Y, \mathbb{Q}(r)) \xrightarrow{\cup[Y]} H^{i+2}_Z(X, \mathbb{Q}(r+1))$$

is a morphism of $\mathbb{Q}$-mixed Hodge structures.

Proof. This can be quickly deduced as a formal consequence of M. Saito's theory of Mixed Hodge Modules [MS], by giving a sheaf-theoretic construction of the cup-product. We outline the argument. We make use of some (more or less standard) notation. For any $\mathbb{C}$-scheme T, let $\pi^T: T \to \operatorname{Spec}\mathbb{C}$ be the structure morphism, and let $\mathbb{Q}_T = (\pi^T)^*\mathbb{Q}$ for any $\mathbb{C}$-scheme T; we write Rf_*, $Rf_!$ for the total derived images of f_*, $f_!$ (and their liftings to Mixed Hodge Modules), and $Rf^!$ for the right adjoint to $Rf_!$. We use f^* for the left adjoint to Rf_*, since f^* is exact. We use $\mathcal{D}(MHM(T))$ to denote the derived category of Mixed Hodge Modules on T in the sense of [MS].

Now let $i_1: Y \to X$, $i_2: Z \to Y$ be the inclusions, and let $i = i_1 \circ i_2: Z \to X$ be the composite. The (rational) cohomology class of Y in $H^2_Y(X, \mathbb{Q}(1))$ corresponds to a morphism of mixed Hodge structures $\mathbb{Q} \to H^2_Y(X, \mathbb{Q}(1))$. The Gysin map in the lemma is obtained from the cup-product pairings of Mixed Hodge structures (actually, from the special case $j = 2$, $s = 1$)

$$H^i_Z(Y, \mathbb{Q}(r)) \otimes H^j_Y(X, \mathbb{Q}(s)) \to H^{i+j}_Z(X, \mathbb{Q}(r+s)).$$

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These in turn (since $Ri_{j*} = Ri_{j!}$ for $j = 1, 2$) result from a pairing in $\mathcal{D}(MHM(X))$

$$Ri_! Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_{1!} Ri_1^! \mathbb{Q}_X \to Ri_! Ri^! \mathbb{Q}_X,$$

which we now construct.

There is a natural isomorphism (projection formula)

$$Ri_! Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_{1!} Ri_1^! \mathbb{Q}_X \cong Ri_! \left(Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i^* Ri_{1!} Ri_1^! \mathbb{Q}_X \right);$$

this isomorphism is valid on applying Saito's functor Rat, since it holds in the derived category of sheaves of Abelian groups (see [KS], Prop. 2.6.6), and hence also holds in $\mathcal{D}(MHM(X))$, because Rat is faithful. Hence by adjunction, it suffices to construct a pairing in $\mathcal{D}(MHM(Z))$

$$Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i^* Ri_{1!} Ri_1^! \mathbb{Q}_X \to Ri^! \mathbb{Q}_X.$$

Now $i^* Ri_{1!} = i_2^* i_1^* Ri_{1!} = i_2^* i_1^* Ri_{1*}$, and so, using the natural transformation $i_1^* Ri_{1*} \to (\text{identity})_Y$, there is a canonical morphism

$$Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i^* Ri_{1!} Ri_1^! \mathbb{Q}_X \to Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i_2^* Ri_1^! \mathbb{Q}_X,$$

and we are reduced to constructing a pairing

$$Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i_2^* Ri_1^! \mathbb{Q}_X \to Ri^! \mathbb{Q}_X = Ri_2^! Ri_1^! \mathbb{Q}_X.$$

By adjunction, this is equivalent to a pairing in $\mathcal{D}(MHM(Y))$

$$Ri_{2!} \left(Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i_2^* Ri_1^! \mathbb{Q}_X \right) \to Ri_1^! \mathbb{Q}_X.$$

Again we have a projection formula

$$Ri_{2!} \left(Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} i_2^* Ri_1^! \mathbb{Q}_X \right) \cong Ri_{2!} Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_1^! \mathbb{Q}_X,$$

and, using the natural tranformation $Ri_{2!} Ri_2^! \to (\text{identity})_Y$, a canonical morphism

$$Ri_{2!} Ri_2^! i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_1^! \mathbb{Q}_X \to i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_1^! \mathbb{Q}_X.$$

So we are further reduced to constructing a pairing in $\mathcal{D}(MHM(Y))$

$$i_1^* \mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_1^! \mathbb{Q}_X \to Ri_1^! \mathbb{Q}_X.$$

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Since $i_1^*\mathbb{Q}_X = \mathbb{Q}_Y$, there is in fact a canonical isomorphism

$$i_1^*\mathbb{Q}_X \overset{\mathbb{L}}{\otimes} Ri_1^!\mathbb{Q}_X \cong Ri_1^!\mathbb{Q}_X.$$

This gives the desired pairing. □

In particular, taking $Z = Y$ yields a map $H^i(Y, \mathbb{Z}(r)) \to H_Y^{i+2}(X, \mathbb{Z}(r+1))$. Composing using the map ('forget the supports') $H_Y^{i+2}(X, \mathbb{Z}(r+1)) \to H^{i+2}(X, \mathbb{Z}(r+1))$, we obtain a morphism of mixed Hodge structures $H^i(Y, \mathbb{Z}(r)) \to H^{i+2}(X, \mathbb{Z}(r+1))$ which we also call a Gysin map.

LEMMA 3.2. *Let X be a reduced projective variety of dimension n over $\mathbb{C}$, and let $Y \subset X$ be an m-dimensional reduced subscheme, which fits into a chain $Y = Y_m \subset Y_{m+1} \subset \cdots \subset Y_n = X$ such that each Y_j is a reduced Cartier divisor on Y_{j+1}. Then $Y \cap X_{\rm sing} \subset Y_{\rm sing}$, and there is a commutative diagram*

$$\begin{array}{ccc} Z^m(Y)_{\deg 0} & \xrightarrow{AJ_Y^m} & J^m(Y) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Z^n(X)_{\deg 0} & \xrightarrow{AJ_X^n} & J^n(X), \end{array}$$

where i is the map on 0-cycles induced by the inclusion $Y - Y_{\rm sing} \subset X - X_{\rm sing}$, and j is a homomorphism of semi-Abelian varieties induced by a morphism of mixed Hodge structures $H^{2m-1}(Y, \mathbb{Z}(m)) \to H^{2n-1}(X, \mathbb{Z}(n))$.

Proof. It clearly suffices to prove the lemma when Y is a reduced Cartier divisor in X. Using the Gysin map $H^{2n-3}(Y, \mathbb{Z}(n-1)) \to H^{2n-1}(X, \mathbb{Z}(n))$, we obtain a homomorphism of semi-Abelian varieties $J^{n-1}(Y) \to J^n(X)$.

Let $\delta \in Z^{n-1}(Y)_{\deg 0}$. It suffices to show that the 2 maps $Z^{n-1}(Y)_{\deg 0} \to J^n(X)$, occurring in the diagram above, have the same value on δ.

Let $S = \mathrm{supp}\,(\delta)$. There is a commutative diagram with exact rows

$$\begin{array}{ccccc} H^{2n-3}(Y, \mathbb{Z}) & \longrightarrow & H^{2n-3}(Y - S, \mathbb{Z}) & \longrightarrow & H_S^{2n-2}(Y, \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2n-1}(X, \mathbb{Z}) & \longrightarrow & H^{2n-1}(X - S, \mathbb{Z}) & \longrightarrow & H_S^{2n}(X, \mathbb{Z}), \end{array}$$

where the vertical maps are Gysin maps. The commutativity of the diagram follows from the functoriality of the cup product, and the compatibility of cup products with the boundary map in the exact sequences of pairs and triples (see, for example, Spanier [S], Ch. 5, Sect. 6, No. 12). This implies that, if M_Y, M_X are the mixed Hodge structures associated to δ which are used to define $AJ_Y^{n-1}(\delta)$ and $AJ_X^n(\delta)$

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respectively, then we have a commutative diagram of mixed Hodge structures with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2n-3}(Y, \mathbb{Z}(n-1)) & \longrightarrow & M_Y & \longrightarrow & \mathbb{Z}[\delta] & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \text{Gysin}} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & H^{2n-1}(X, \mathbb{Z}(n)) & \longrightarrow & M_X & \longrightarrow & \mathbb{Z}[\delta] & \longrightarrow & 0.
\end{array}$$

Hence we have a morphism of 1-motives $M_Y \to M_X$, i.e., a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{(M_Y)_*} & J^{n-1}(Y) \\
\Big\| & & \Big\downarrow \\
\mathbb{Z} & \xrightarrow{(M_X)_*} & J^n(X)
\end{array}$$

This implies that the two maps $AJ_X^n \circ i$ and $j \circ AJ_Y^{n-1}$ have the same value on δ. □

Remark. In the above lemma, the map $J^m(Y) \to J^n(X)$ is clearly unique, once we know that $AJ_Y^m \colon Z^m(Y)_{\deg 0} \to J^m(Y)$ is surjective. Thus the map $J^m(Y) \to J^n(X)$ will not depend on the chain $\{Y_m, Y_{m+1}, \ldots\}$. We prove the surjectivity of AJ in the next section, as a consequence of Lemma 3.7.

We also need another functoriality property of AJ.

LEMMA 3.3. *Let $f \colon X \to Y$ be a morphism between n-dimensional varieties, and $\delta \in Z^n(Y)$ a 0-cycle of degree 0 on Y such that $f^*(\delta) \in Z^n(X)$ (here $f^*(\delta)$ is the inverse image of δ as a cycle). Then $\deg_X f^*(\delta) = 0$. If $f^* \colon J^n(Y) \to J^n(X)$ is the morphism induced by $f^* \colon H^{2n-1}(Y, \mathbb{Z}(n)) \to H^{2n-1}(X, \mathbb{Z}(n))$, then $AJ_X^n(f^*(\delta)) = f^* AJ_Y^n(\delta)$.*

Proof. If $S = \operatorname{supp}(\delta)$, then we have a diagram of mixed Hodge structures with exact rows

$$\begin{array}{ccccccccc}
0 & \to & H^{2n-1}(Y, \mathbb{Z}(n)) & \to & H^{2n-1}(Y - S, \mathbb{Z}(n)) & \to & H_S^{2n}(Y, \mathbb{Z}(n)) & \to & H^{2n}(Y, \mathbb{Z}(n)) \\
 & & \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f^*} \\
0 & \to & H^{2n-1}(X, \mathbb{Z}(n)) & \to & H^{2n-1}(X - f^{-1}(S), \mathbb{Z}(n)) & \to & H_{f^{-1}(S)}^{2n}(X, \mathbb{Z}(n)) & \to & H^{2n}(X, \mathbb{Z}(n))
\end{array}$$

This immediately implies the lemma. □

LEMMA 3.4. *Let X be projective of dimension n, and let $f \colon Y \to X$ be the blow up of a smooth point $x \in X$.*

(i) *The cycle-theoretic direct image map $f_* \colon Z^n(Y) \to Z^n(X)$ induces an isomorphism $f_* \colon CH^n(Y) \to CH^n(X)$.*

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(ii) *There is a commutative diagram*

$$\begin{array}{ccc} Z^n(Y)_{\deg 0} & \xrightarrow{AJ_Y^n} & J^n(Y) \\ {\scriptstyle f_*}\big\downarrow & & \big\uparrow{\scriptstyle \cong\, f^*} \\ Z^n(X)_{\deg 0} & \xrightarrow{AJ_X^n} & J^n(X) \end{array}$$

where f_* is induced by the cycle-theoretic direct image, and is surjective, and f^* is an isomorphism.

Proof. That $f^*: J^n(X) \to J^n(Y)$ is an isomorphism follows from the corresponding isomorphism on integral cohomology $H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(Y, \mathbb{Z})$, which in turn follows from the formula for the cohomology of a blow up of a smooth point. The cycle theoretic direct image, defined on generators by $f_*[P] = [f(P)]$ for any point $P \in Y - Y_{\text{sing}}$, is clearly a surjective homomorphism. Since f is the blow up of a smooth point, f induces a one-to-one correspondence between the sets of n-dimensional components of X and Y, so that we may view $\deg_X$ and $\deg_Y$ as homomorphism taking values in the same Abelian group $\mathbb{Z}^{\oplus t}$. With this convention, we verify at once that $\deg_X \circ f_* = \deg_Y$. Hence there is an induced surjection $f_*: Z^n(Y)_{\deg 0} \to Z^n(X)_{\deg 0}$.

If C is any Cartier curve on X, its strict transform C' is a Cartier curve on Y, such that $f_*: R(C') \to R(C)$ is an isomorphism. Hence there is a well-defined, surjective homomorphism $f_*: CH^n(Y) \to CH^n(X)$, and $\ker(CH^n(Y) \to CH^n(X))$ is generated by 0-cycles (necessarily of degree 0, since the degree maps for $Z^n(X)$ and $Z^n(Y)$ are compatible) which are supported on the exceptional divisor of f. But any 0-cycle of degree 0 supported on the exceptional divisor ($\cong \mathbb{P}^{n-1}$) is clearly in $R^n(Y)$. Hence $f_*: CH^n(Y) \to CH^n(X)$ is an isomorphism.

So it remains to prove the commutativity of the square in (ii). For this, it suffices to prove that if $P, Q \in Y$ are smooth points, then $f^* \circ AJ_X^n([f(P)] - [f(Q)]) = AJ_Y^n([P] - [Q])$. If $f(P) = f(Q)$, then we claim both sides are 0. For the right side, this is because any two points in the exceptional divisor lie on a line $L \cong \mathbb{P}^1$ contained in $Y - Y_{\text{sing}}$; now apply Lemma 3.2 to the inclusion $L \hookrightarrow Y$, and use the fact that $J^1(L) = 0$.

Hence we may assume $f(P) \neq f(Q)$. We have an isomorphism of mixed Hodge structures

$$H^{2n-1}(X - \{f(P), f(Q)\}, \mathbb{Z}) \cong H^{2n-1}(Y - f^{-1}(\{f(P), f(Q)\}), \mathbb{Z}),$$

and a morphism of mixed Hodge structures

$$\psi: H^{2n-1}(Y - \{P, Q\}, \mathbb{Z}) \to H^{2n-1}(Y - f^{-1}(\{f(P), f(Q)\}), \mathbb{Z}).$$

So we will be done if we prove that ψ is an isomorphism. This is obvious if neither of $f(P), f(Q)$ is the point x blown up. If $f(P) = x$, on the other hand

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(as we may assume), and E is the exceptional divisor of f, then the natural map $\eta\colon H^{2n}_{\{P\}}(Y, \mathbb{Z}) \to H^{2n}_{E}(Y, \mathbb{Z})$ is an isomorphism, since it fits into an exact sequence

$$H^{2n-1}_{E-\{P\}}(Y - \{P\}, \mathbb{Z}) \to H^{2n}_{\{P\}}(Y, \mathbb{Z}) \to H^{2n}_{E}(Y, \mathbb{Z}) \to H^{2n}_{E-\{P\}}(Y - \{P\}, \mathbb{Z}),$$

where the extreme terms are 0 (Thom isomorphism, and $E \cong \mathbb{P}^{n-1}$). Since η is an isomorphism, it follows easily that ψ is one as well, by an easy diagram chase. □

LEMMA 3.5. *$AJ^n_X\colon Z^n(X)_{\deg 0} \to J^n(X)$ factors through $CH^n(X)_{\deg 0}$.*

Proof. We first consider the case $n = 1$. Now we must show that if $f \in R(X, X)$, and $\delta = (f)_X$, then $AJ^1_X(\delta) = 0$. From the definition of $R(X, X)$, we may regard f as a morphism $f\colon X \to \mathbb{P}^1$, such that $f^*([0] - [\infty]) = (f)_X$ as cycles. Since $J^1(\mathbb{P}^1) = 0$, the preceeding lemma finishes the proof.

Now we consider the general case. By Lemma 2.1, we are reduced to showing that if $C \subset X$ is a reduced Cartier curve, $f \in R(C, X)$, and $\delta = (f)_C$, then $AJ^n_X(\delta) = 0$.

From Lemma 3.2, there is a commutative diagram

$$\begin{array}{ccc} Z^1(C)_{\deg 0} & \xrightarrow{AJ^1_C} & J^1(C) \\ \downarrow & & \downarrow \\ Z^n(X)_{\deg 0} & \xrightarrow{AJ^n_X} & J^n(X) \end{array}$$

If $R(C, X) = R(C, C)$, we would be done, by the case $n = 1$ already considered. However, if C has singular points which are smooth points of X, then in general $R(C, C) \subsetneq R(C, X)$. We then proceed as follows.

Let $\widetilde{C} \to C$ be the partial normalization of C, such that

(i) $\pi\colon \widetilde{C} \to C$ is an isomorphism over a neighbourhood of $C \cap X_{\text{sing}}$, and
(ii) $\pi^{-1}(C - X_{\text{sing}})$ is smooth.

These conditions uniquely define $\widetilde{C}$. Since $\widetilde{C} \to C$ is an isomorphism over the smooth locus of C, we may regard $Z^1(C)$ as a subgroup of $Z^1(\widetilde{C})$. Since smooth points of $\widetilde{C}$ map to smooth points of X, there is a natural map $\pi_*\colon Z^1(\widetilde{C}) \to Z^n(X)$, yielding $Z^1(\widetilde{C})_{\deg 0} \to Z^n(X)_{\deg 0}$, such that $Z^1(C) \to Z^n(X)$ factors through $Z^1(\widetilde{C})$. From the definitions (see Sect. 1), we verify easily that $R(\widetilde{C}, \widetilde{C}) = R(C, X)$ (where we have used that $C, \widetilde{C}$ have the same generic points), and for any $g \in R(C, X)$, we have $(g)_C = \pi_*((g)_{\widetilde{C}})$. In particular, taking $g = f$, we have that $\delta = \pi_*((f)_{\widetilde{C}})$, where by the case $n = 1$, we know that $AJ^1_{\widetilde{C}}((f)_{\widetilde{C}}) = 0 \in J^1(\widetilde{C})$.

Let $S \subset C$ be the finite subset where $\widetilde{C} \to C$ is not an isomorphism. By the embedded resolution of singularities for curves, we can find a projective morphism $h\colon \widetilde{X} \to X$, which is a composition of blow-ups at smooth points lying over points of S, such that the strict transform of C in $\widetilde{X}$ is $\widetilde{C}$ (that is, $\overline{h^{-1}(C - S)} = \widetilde{C}$). Now for the given $f \in R(\widetilde{C}, \widetilde{C}) = R(\widetilde{C}, \widetilde{X})$, if we set $\widetilde{\delta} = (f)_{\widetilde{C}}$, then we have that (i) $h_*(\widetilde{\delta}) = \delta$, and (ii) $AJ^1_{\widetilde{C}}(\widetilde{\delta}) = 0 \in J^1(\widetilde{C})$. Hence $AJ^n_{\widetilde{X}}(\widetilde{\delta}) = 0 \in J^n(\widetilde{X})$, by Lemma 3.2. By Lemma 3.4, we deduce that

$$AJ^n_X(\delta) = AJ^n_X(h_*(\widetilde{\delta})) = 0 \in J^n(X). \qquad \Box$$

COROLLARY 3.6. *With the hypotheses of Lemma* 3.4, *there is a commutative diagram*

$$\begin{array}{ccc} CH^n(Y) & \xrightarrow{AJ^n_Y} & J^n(Y) \\ f_* \downarrow \cong & & \cong \uparrow f^* \\ CH^n(X) & \xrightarrow{AJ^n_X} & J^n(X) \end{array}$$

whose vertical arrows are isomorphisms.

We sum up our progress at this stage: we have constructed $AJ^n_X : CH^n(X)_{\deg 0} \to J^n(X)$, and proved its functoriality for the Gysin maps associated to inclusions of suitable reduced subvarieties $Y \subset X$. This proves (c) of the Main Theorem (modulo the surjectivity of AJ^m_Y, as remarked above after Lemma 3.2). We have also proved the invariance of the Chow group and J^n under blow ups of smooth points.

We now further analyze the map AJ^1_C for a curve C.

LEMMA 3.7. *Let C be a reduced curve. Then $CH^1(C) \cong \operatorname{Pic} C$. Further, $AJ^1_C : CH^1(C)_{\deg 0} \to J^1(C)$ is surjective, and is an isomorphism on torsion subgroups. If C is seminormal, then AJ^1_C is an isomorphism.*

Proof. It is easy to see from the definitions that $CH^1(C)$ is naturally isomorphic to the group of linear equivalence classes of Cartier divisors on C, which (as is well known) is naturally isomorphic to $\operatorname{Pic} C$. Under the isomorphism $CH^1(C) \to \operatorname{Pic} C$, the class of a point $x \in C - C_{\text{sing}}$ is associated to the invertible sheaf $\mathcal{O}_C(x)$.

If $f : \widetilde{C} \to C$ is the semi-normalization, then f is bijective on points, and on the singular loci, so $f^* : H^1(C, \mathbb{Z}(1)) \to H^1(\widetilde{C}, \mathbb{Z}(1))$ and $f^* : J^1(C) \to J^1(\widetilde{C})$, as well as $f^* : Z^1(C) \to Z^1(\widetilde{C})$, are isomorphisms. On the other hand, $CH^1(C) \cong \operatorname{Pic} C$, $CH^1(\widetilde{C}) \cong \operatorname{Pic} \widetilde{C}$, and $f^* : Z^1(C) \to Z^1(\widetilde{C})$ induces $f^* : CH^1(C) \to CH^1(\widetilde{C})$, which corresponds to the map $f^* : \operatorname{Pic} C \to \operatorname{Pic} \widetilde{C}$ obtained by pulling back line bundles.

We claim, from the formulas $H^1(C, \mathcal{O}^*_C) = \operatorname{Pic} C$ and $H^1(\widetilde{C}, \mathcal{O}^*_{\widetilde{C}}) = \operatorname{Pic} \widetilde{C}$, that $f^* : \operatorname{Pic} C \to \operatorname{Pic} \widetilde{C}$ is surjective, and is an isomorphism on torsion. To see this, consider the exact sheaf sequence $0 \to \mathcal{O}^*_C \to \mathcal{O}^*_{\widetilde{C}} \to \mathcal{F} \to 0$, where $\mathcal{F}$ is a finite direct sum of skyscraper sheaves whose stalks are of the form $\widetilde{R}^*/R^*$ for a one-dimensional local ring R and its semi-normalization $\widetilde{R}$ (which is also one-dimensional local with the same residue field, so that $\widetilde{R}^*/R^* \cong \widetilde{R}/R$ is a $\mathbb{C}$-vector space via the logarithm map). The exact cohomology sequence of this sequence

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of sheaves yields an exact sequence $0 \to H^0(C, \mathcal{F}) \to \operatorname{Pic} C \to \operatorname{Pic} \widetilde{C} \to 0$ since $H^0(C, \mathcal{O}^*_C) = H^0(\widetilde{C}, \mathcal{O}^*_{\widetilde{C}}) = (\mathbb{C}^*)^{\oplus r}$ where r is the number of connected components of C.

Clearly, by (an easy case of) Lemma 3.3, the map $AJ^1_C: CH^1(C)_{\deg 0} \to J^1(C) = J^1(\widetilde{C})$ factors through the surjection $CH^1(C)_{\deg 0} \twoheadrightarrow CH^1(\widetilde{C})_{\deg 0}$.

So we are reduced to proving that for seminormal C, the map AJ^1_C is an isomorphism. But this is just the content of [D] (10.3.8), in the light of (10.1.3). □

We end this section with a proof of the algebricity of the map $U \to J^n(X)$, $x \mapsto AJ^n_X([x] - [x_0])$ (where $x_0 \in U$ is a base point). As remarked in the introduction, the 'correct' approach (in our opinion) is to deduce it from a general algebraic construction of the 1-motive associated to $H^{2n-1}(Y, \mathbb{Z}(n))$, for any n-dimensional complex variety Y. This will be done in more detail in [BS].

We now sketch another proof that $U \to J^n(X)$ is a morphism, using the fact that when X is a seminormal curve, this is known to be a morphism, since in that case, it is the natural map from the smooth points of a singular curve (with appropriate base points) to its generalized Jacobian variety (by Lemma 3.7). This implies (by Lemma 3.7 again) that when $\dim X = 1$, the map $U \to J^1(X)$ is a morphism. Hence in general, the map $U \to J^n(X)$, which is clearly holomorphic, is an algebraic morphism when restricted to any curve in U. Since (as seen above), the composite $U \to J^n(X) \to J^n(\widetilde{X})$ is the (restriction to U of the) Albanese mapping of $\widetilde{X}$, it is an algebraic morphism; hence we must show that any holomorphic section of the algebraic principal torus-bundle $f^*J^n(X) \mid_U \to U$, which restricts to an algebraic section on any curve, is in fact an algebraic section.

The algebraic torus bundle $f^*J^n(X) \mid_U \to U$ is Zariski locally trivial, since $J^n(X) \to J^n(\widetilde{X})$ is. We then reduce to showing that if V is a nonsingular affine variety (an open subset of U in our case), and $h: V \to \mathbb{C}$ is a holomorphic function which restricts to an algebraic (regular) function on every curve in V, then h is a regular function on V.

By Noether normalization and the standard argument with elementary symmetric functions, we are further reduced to proving the same assertion for the case $V = \mathbb{C}^n$. Now the holomorphic function h has a globally convergent power series expansion; since for almost all substitutions of values of $n-1$ variables, h becomes a polynomial in the last variable, the Baire category theorem implies easily that h is a polynomial. (R. R. Simha has pointed out to us that this argument goes back to work of E. E. Levi (see [Si], (1.1))).

4. Lefschetz Theorem

In this section we prove a Lefschetz Theorem, which implies part (d) of our main theorem, *i.e.*, if Y is a general complete intersection of very ample divisors in X, then $J^m(Y) \to J^n(X)$ is an isomorphism if $\dim Y = m \geqslant 2$, and is surjective

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on N-torsion for each N, if $m = 1$ (in particular, $J^1(Y) \twoheadrightarrow J^n(X)$ in this case).

Note that in view of Lemma 3.7, this also implies the surjectivity statement in (b) of the Main Theorem; the surjectivity in turn implies that in Lemma 3.2, the map $j: J^m(Y) \to J^n(X)$ is independent of the chain $\{Y = Y_m, Y_{m+1}, \ldots\}$ of reduced subvarieties of X.

It is enough by induction to prove the above statement for Y a general hyperplane section of X. Recall (proof of Lemma 3.2) that the map $J^{n-1}(Y) \to J^n(X)$ is induced by the Gysin map $H^{2n-3}(Y, \mathbb{Z}(n-1)) \to H^{2n-1}(X, \mathbb{Z}(n))$. This last map is a composite of the maps

$$H^{2n-3}(Y, \mathbb{Z}(n-1)) \to H^{2n-1}_Y(X, \mathbb{Z}(n)) \to H^{2n-1}(X, \mathbb{Z}(n)),$$

where the first map is obtained by cupping with the class of Y in $H^2_Y(X, \mathbb{Z}(1))$. The second map is the usual map from cohomology with supports to cohomology without supports.

More generally, we have the following variant of the Lefschetz Hyperplane Theorem, valid for projective varieties with arbitrary singularities.

THEOREM 4.1.

(a) *Let X be a reduced projective variety over $\mathbb{C}$ of dimension n and let Y be a general hyperplane section. Then the the Gysin map $H^i(Y, \mathbb{Z}(j)) \to H^{i+2}(X, \mathbb{Z}(j+1))$ is an isomorphism for $i < \dim Y$ and surjective for $i = \dim Y$.*

(b) *Let X be as above, $\pi: \widetilde{X} \to X$ the normalization, and $A \subset X$ a closed subvariety such that if $U = X - A$ and $V = \widetilde{X} - \pi^{-1}(A)$, the following are satisfied:*

 (i) *V is nonsingular of dimension n*
 (ii) *$V \to U$ is the normalisation of U*
 (iii) *if $W = \pi^{-1}(U_{\mathrm{sing}})$, then U_{sing} and W are nonsingular of dimension $n-1$ and $W \to U_{\mathrm{sing}}$ is an analytic covering space.*

 Let $C \subset X$ be a reduced curve such that

 (i′) *C is a local complete intersection in X*
 (ii′) *$C \cap X_{\mathrm{sing}}$ is reduced, and supported at smooth points of X_{sing}*
 (iii′) *$C \cap (X - X_{\mathrm{sing}})$ has only plane curve singularities*
 (iv′) *$C \cap A = \emptyset$.*

 Let Y be a general hypersurface section of X of sufficiently large degree, and which contains C. Then the Gysin map $H^i(Y, \mathbb{Z}(j)) \to H^{i+2}(X, \mathbb{Z}(j+1))$ is an isomorphism for $i < \dim Y$ and surjective for $i = \dim Y$.

Proof. We first prove (a), and then indicate how the proof is to be modified in order to obtain (b).

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Let $\mathbb{P}$ be the variety of all hyperplanes in the ambient projective space. Let

$$\mathcal{Y} = \{(x, v) \in X \times \mathbb{P} \mid x \in Y_v\} \subseteq X \times \mathbb{P}$$

be the incidence locus. Here Y_v denotes the hyperplane section of X corresponding to the point $v \in \mathbb{P}$. We have a diagram,

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & X \times \mathbb{P} \\ & {\scriptstyle \pi} \searrow \quad \swarrow & \\ & \mathbb{P} & \end{array}$$

By Corollary (5.1) of Verdier[V], π is locally trivial (for the complex topology) over a Zariski open, dense set $U \subset \mathbb{P}$. Choose any point $v \in U$ and let $Y = Y_v$ be the corresponding hyperplane section. Let $x \in Y_v$ be any point. Choose a line $\mathbb{P}^1 \subset \mathbb{P}$ such that $v \in \mathbb{P}^1$ but $x \notin Y_w$ for some $w \in \mathbb{P}^1$. Then this pencil of hyperplanes does not have x as a base point and $\mathcal{Y}_{\mathbb{P}^1} := \pi^{-1}(\mathbb{P}^1) \to \mathbb{P}^1$ is the blow up of X along the base locus of this pencil. Since $v \in U$, the local triviality of π implies that there exists a fundamental system of neighbourhoods V of x (on the blow-up of X along the base locus of the above pencil), such that there are homeomorphisms of pairs $(V, V \cap Y_v) \cong (Y_v \cap V) \times (D^2, 0)$ where D^2 is the disc (i.e., $D^2 \cong \{z \in \mathbb{C} \mid \ |z| \leqslant 1\}$), and $0 \in D^2$ is the origin. Further, the homeomorphisms are compatible with restriction (if $W \subset V$ are neighbourhoods in the fundamental system, the homeomorphism for V restricts to that for W).

Since x is not a base point of the linear system $\mathcal{Y}_{\mathbb{P}^1} \to \mathbb{P}^1$, it follows that there exists a fundamental system of neighbourhoods of x in X itself such that there are homeomorphisms $(V, V \cap Y_v) \cong (V \cap Y_v) \times (D^2, 0)$, compatible with restriction.

We deduce that any point x on a general hyperplane section $Y = Y_v$ of X has a fundamental system of neighbourhoods V of x in X, such that $(V, V \cap Y_v) \cong (Y_v \cap V) \times (D^2, 0)$.

Let $\mathcal{H}^r_Y(X)(j+1)$ be the sheaf on X associated to the presheaf

$$V \mapsto H^r_{V \cap Y}(V, \mathbb{Z}(j+1)),$$

where $V \subseteq X$ is open. We claim that

(i) $\mathcal{H}^r_Y(X)(n) = 0$ for $r \neq 2$, and
(ii) $\mathcal{H}^2_Y(X)(j+1) \cong \mathbb{Z}(j)_Y$.

This is because, using the Kunneth decomposition for any given x and for any $V \ni x$ as above,

$$\begin{aligned} H^r_{Y \cap V}(V, \mathbb{Z}(j+1)) &\cong H^r_{Y \cap V}((Y \cap V) \times D^2, \mathbb{Z}(j+1)) \\ &\cong H^{r-2}(Y \cap V, \mathbb{Z}(j)) \otimes H^2(D^2, S^1, \mathbb{Z}(1)) \cong H^{r-2}(Y \cap V, \mathbb{Z}(j)). \end{aligned}$$

Since any algebraic variety can be triangulated, so that any point on it has a basis of neighbourhoods consisting of contractible sets, we see that $\mathcal{H}_Y^r(X)(j+1)$ is zero for $r \neq 2$. There exists a map of sheaves $\mathbb{Z}(j)_Y \to \mathcal{H}_Y^2(X)(j+1)$ using the cohomology class of Y in $H_Y^2(X, \mathbb{Z}(1))$. The above analysis gives that this map is an isomorphism on stalks, i.e., we have a sheaf isomorphism $\mathcal{H}_Y^2(X)(j+1) \cong \mathbb{Z}(j)_Y$. This proves (i) and (ii) as claimed.

We have a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{H}_Y^q(X)(j+1)) \Rightarrow H_Y^{p+q}(X, \mathbb{Z}(j+1)).$$

Since $\mathcal{H}_Y^q(X)(j+1) = 0$ for $q \neq 2$ and $\mathcal{H}_Y^2(X)(j+1) \cong \mathbb{Z}(j)_Y$, we easily deduce that $E_2^{i,2} = H^i(Y, \mathbb{Z}(j)) \to H_Y^{i+2}(X, \mathbb{Z}(j+1))$ is an isomorphism. This isomorphism is just the cup product with the class of Y.

The Gysin map $H^i(Y, \mathbb{Z}(j)) \to H^{i+2}(X, \mathbb{Z}(j+1))$ is the composite

$$H^i(Y, \mathbb{Z}(j)) \to H_Y^{i+2}(X, \mathbb{Z}(j+1)) \xrightarrow{\alpha} H^{i+2}(X, \mathbb{Z}(j+1)),$$

where α forms part of the long exact sequence of cohomology groups

$\ldots \to H^{i+1}(X-Y, \mathbb{Z}(j+1)) \to H_Y^{i+2}(X, \mathbb{Z}(j+1)) \xrightarrow{\alpha} H^{i+2}(X, \mathbb{Z}(j+1)) \to H^{i+2}(X-Y, \mathbb{Z}(j+1)) \to \ldots$

Since $X - Y$ is affine, the first and the last terms vanish by the Weak Lefschetz theorem when $i < \dim Y$ and the last term vanishes by the same result even if $i = \dim Y$. Hence the Gysin map is an isomorphism for $i < \dim Y$ and is surjective for $i = \dim Y$.

Proof of (b). Let $\mathbb{P}$ be the projective space parametrizing all divisors (hypersurface sections of X) in the linear system $|\ Y\ |$, which contain C. We have, as before, the following diagram

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & X \times \mathbb{P} \\ & {\scriptstyle p}\searrow \qquad \swarrow & \\ & \mathbb{P} & \end{array}$$

Since Y is general, we may assume it corresponds to a point of $\mathbb{P}$ lying in the Zariski open set given by Corollary 5.1 of [V], applied to the morphism $p: \mathcal{Y} \to \mathbb{P}$.

Consider as before the spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{H}_Y^q(X)(n)) \Rightarrow H_Y^{p+q}(X, \mathbb{Z}(n)).$$

As in the proof of Theorem 4.1(a), this reduces us to proving that for any $x \in Y$, the stalk $\mathcal{H}_Y^i(\mathbb{Z}(n))_x = 0$ for $i \neq 2$, and that $\mathcal{H}_Y^2(n)_x \cong \mathbb{Z}(n-1)$ is generated by the class of Y.

Since we only consider divisors in X containing C, our previous analysis about the stalks of the sheaves $\mathcal{H}_Y^i(X)(n)$ is valid except at points of C. But at a point

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of $x \in C - X_{\rm sing}$, C is either nonsingular or has a plane curve singularity; hence Bertini's theorem implies that Y is a smooth Cartier divisor on X near x (see [B], Chap. 5), so that the stalks again have the form claimed.

To finish the proof, we need to consider stalks at a point $x \in C \cap X_{\rm sing} = C \cap U_{\rm sing}$. It suffices to show that there exists a fundamental system of neighbourhoods N of x in U such that $(N, N \cap Y) \cong (N \cap Y) \times (D^2, 0)$. Since Y is general, and $U_{\rm sing}$ is nonsingular of dimension $n-1$, we may assume (Bertini's theorem) that $Y \cap U_{\rm sing}$ is nonsingular of dimension $n-2$.

Let $\widetilde{Y} = \pi^{-1}(Y)$. Since $W \to U_{\rm sing}$ is an analytic covering space, $\widetilde{Y} \cap W$ is a smooth Cartier divisor on W, in a neighbourhood of $\pi^{-1}(x)$. Hence $\widetilde{Y}$ is smooth and meets W transversally in a neighbourhood of $\pi^{-1}(x)$. Let $\pi^{-1}(x) = \{x_1, \ldots, x_k\}$.

Choose local analytic coordinate functions $z_1, \ldots, z_{n-1} \in \mathcal{O}_{U_{\rm sing},x}$ so that $\widetilde{z_i} = z_i \circ \pi$ give local analytic coordinates on W at each x_i, and $\widetilde{Y} \cap W$ is defined on W by $\widetilde{z_1} = 0$. There exists an analytic function $\widetilde{z_n}$ defined in a neighbourhood of $\pi^{-1}(x)$ in V such that $\widetilde{z_n} = 0$ defines W in V near $\pi^{-1}(x)$, and for $i = 1, \ldots, n-1$ the functions $\widetilde{z_i}$ extend (holomorphically) to a neighbourhood of $\pi^{-1}(x)$ in V such that $\widetilde{Y}$ is now defined locally in V by $\widetilde{z_1} = 0$. Define $N = \pi(\{|\widetilde{z_i}| < \epsilon | 1 \leqslant i \leqslant n\})$ for sufficiently small ϵ. Note that $\pi^{-1}(N)$ is the disjoint union of ϵ polydisc neighbourhoods around x_i and N is obtained by identifying these polydiscs along $\widetilde{z_n} = 0$. It is now clear that the pair $(N, N \cap Y)$ is homeomorphic to $(N \cap Y) \times (D^2, 0)$. Shrinking ϵ gives a fundamental system of such neighbourhoods N. □

It follows from the above discussion that, for Y a general complete intersection of very ample divisors in X with $\dim Y = m \geqslant 2$, we have $H^{2m-1}(Y, \mathbb{Z}(m)) \cong H^{2n-1}(X, \mathbb{Z}(n))$, which implies a corresponding isomorphism on tensoring with $\mathbb{C}$. Since by [D], and Lemma 3.1, all the morphisms involved are morphisms of mixed Hodge structures, $F^0H^{2m-1}(Y, \mathbb{C}(m))$ maps onto $F^0H^{2n-1}(X, \mathbb{C}(m))$ under the above isomorphism. Hence the natural map $J^m(Y) \to J^n(X)$ is an isomorphism if $m \geqslant 2$. In case $m = 1$, the map $J^1(Y) \to J^n(X)$ is a surjection on N-torsion for each N, because the map on lattices is surjective. This proves (d) of the Main Theorem.

5. Roitman Theorem

We now prove the Roitman theorem, i.e., part (e) of the main theorem. In the case when $\dim X = 2$, i.e., X is a reduced singular surface, this is the Roitman Theorem proved by Barbieri, Pedrini and Weibel in [BPW]. (We have a different, shorter proof of this case which, however, we omit in order to save space.) The higher dimensional case will be reduced to the surface case.

From now on, let X be a reduced projective variety (over $\mathbb{C}$) of dimension $n \geqslant 3$.

We first choose an algebraic subset $A \subset X_{\rm sing}$, as follows. Let $X^{(n)}$ denote the union of the n-dimensional components of X, and X' the union of the smaller dimensional components. Let $\pi: \widetilde{X} \to X$ be the normalisation. Then there exists a closed set $A \subset X$, with $X' \subset A$, codim $_{X^{(n)}}(A \cap X^{(n)}) \geqslant 2$, such that if $U = X - A$ and $V = \widetilde{X} - \pi^{-1}(A)$, the hypotheses (i)–(iii) of Theorem 4.1(b) are satisfied.

Now let δ be a 0-cycle with $[\delta] \in {}_kCH^n(X)_{\deg 0}$, and $AJ^n_X(\delta) = 0$. Then by Lemma 2.1, the cycle $k\delta = (f)_C \in R^n(X)$, where C is a reduced Cartier curve on X, and $f \in R(C, X)$. Further, we may assume C meets X_{sing} only at smooth points of X_{sing}, and $C \cap X_{\text{sing}}$ is reduced. Finally, we can choose C to be disjoint from A (apply Lemma 2.1, taking A in that lemma to be the above $(A \cap X^{(n)})$.) Thus with the above notation, $C \subset U$, $C \cap X_{\text{sing}} = C \cap U_{\text{sing}}$, and $C \cap U_{\text{sing}}$ is a reduced finite set of points.

Now as in [B], Chapter 5, we may make a sequence of blow-ups at smooth points $\alpha: X' \to X$, and find a 0-cycle δ' on X', such that

(1) $\alpha_* \delta' = \delta$

(2) for some reduced Cartier curve C' on X' and some $f' \in R(C', X')$, we have

$$(f')_{C'} = k\delta'$$

(3) each irreducible component of C' is smooth outside X'_{sing}, and any $x \in X' - X'_{\text{sing}}$ lies on at most 2 irreducible components of C'

(4) if a point $x \in X' - X'_{\text{sing}}$ lies on 2 irreducible components of C', then both components of C' have distinct tangents at x.

By Lemma 3.4 and Corollary 3.6, the sequence of blow-ups does not change either the Chow group or J^n; hence we may assume without loss of generality that X, C, f themselves have the above properties (2), (3), (4). Replacing U, A by their inverse images in X', we may assume (i), (ii), (iii) hold as well (replace $\pi: \widetilde{X} \to X$ by its base change to X', etc.). Now the (modified) curve C satisfies (i′)–(iv′) of Theorem 4.1(b).

Applying Bertini's theorem, we can find a reduced complete intersection surface S of large degree in X such that $C \subset S$, and S is smooth outside X_{sing} (using the same arguments as given in [B]; the point is that C has local embedding dimension $\leqslant 2$ outside X_{sing}). Since C is a Cartier curve, we can also choose S such that C is a Cartier divisor on S. Now since δ is supported on S, we apply Roitman's Theorem for surfaces [BPW], as follows.

LEMMA 5.2. *Let X, C and S be as above. Then the Gysin map $H^3(S, \mathbb{Z}(2)) \to H^{2n-1}(X, \mathbb{Z}(n)$, and the induced natural map $J^2(S) \to J^n(X)$, are isomorphisms.*

Proof. Let Y be a general hypersurface section of X of sufficiently high degree containing C. Then C is a Cartier curve on Y, and Y is smooth outside X_{sing}.

By (b) of Theorem 4.1, the Gysin map $H^{2n-3}(Y, \mathbb{Z}(n-1)) \to H^{2n-1}(X, \mathbb{Z}(n))$ is an isomorphism. This implies that the corresponding map $J^{n-1}(Y) \to J^n(X)$ is an isomorphism as well. The lemma then follows by induction on $n = \dim X$. □

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Finally, we have the diagram

$$\begin{array}{ccc} {}_kCH^2(S)_{\deg 0} & \longrightarrow & {}_kCH^n(X)_{\deg 0} \\ \downarrow & & \downarrow \\ {}_kJ^2(S) & \longrightarrow & {}_kJ^n(X). \end{array}$$

Now suppose that the chosen cycle class $[\delta] \in {}_kCH^n(X)_{\deg 0}$ is such that $AJ_X^n(\delta) = 0$. Then for the Cartier curve C, the function $f \in R(C, X)$, and a surface S containing C as above, we have (i) $[\delta] \in {}_kCH^2(S)$, since $k\delta = (f)_C$, and $R(C, X) = R(C, S)$; (ii) $AJ_S^2(\delta) = 0 \in {}_kJ^2(S)$, from the injectivity of ${}_kJ^2(S) \to {}_kJ^n(X)$ (i.e., the above lemma).

Hence $[\delta] = 0$ in $CH^2(S)$, by the Roitman theorem for surfaces [BPW]. This implies $[\delta] = 0$ in $CH^n(X)$.

Hence, we have shown that the map ${}_kCH^n(X)_{\deg 0} \to {}_kJ^n(X)$ is injective for every k. Since this map has already been proved to be surjective (by part (b) of the Main Theorem), we get that it is an isomorphism. This finishes the proof of the Roitman theorem (part (e) of the Main Theorem).

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