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# On the Relation between 2 and $\infty$ in Galois Cohomology of Number Fields

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**Abstract.** We remove the assumption '$p \neq 2$ or $k$ is totally imaginary' from several well-known theorems on Galois groups with restricted ramification of number fields. For example, we show that the Galois group of the maximal extension of a number field $k$ which is unramified outside 2 has finite cohomological 2-dimension (also if $k$ has real places).

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**Key words.** Galois cohomology, restricted ramification, real places, cohomological dimension.

## 1. Introduction

Number theorist's nightmare, the prime number 2, frequently causes technical problems and requires additional efforts. In Galois cohomology, the problems with $p = 2$ are essentially due to the fact that the decomposition groups of the real places are 2-groups and so the case of a totally imaginary number field is comparatively easier to deal with.

A classical object of study in number theory is Galois groups with restricted ramification. For a number field $k$, a set $S$ of primes of $k$ and a prime number $p$, one is interested in the Galois group $G_S(p) = G(k_S(p)|k)$ of the maximal $p$-extension $k_S(p)$ of $k$ which is unramified outside $S$. If $S$ is empty, then $G_S(p)$ is the Galois group of the so-called $p$-class field tower of $k$ and, besides the fact that it can be infinite (Golod-Šafarevič), not much is known about this group. The situation is easier in the case that $S$ contains the set $S_p$ of primes dividing $p$, where the cohomological dimension of $G_S(p)$ is known to be less than or equal to two (cf. [9], (8.3.17), (10.4.9)). However, there is an exception: if $p = 2$ and $k$ has at least one real place. If, in this exceptional case, $S$ contains all real places, then these places become complex in $k_S(2)$ and therefore $G_S(2)$, containing involutions, has infinite cohomological dimension. Furthermore, the virtual cohomological dimension $\mathrm{vcd}\, G_S(2)$ is less than or equal to two in this case, i.e. $G_S(2)$ has an open subgroup $U$ with $\mathrm{cd}\, U \leqslant 2$. The case when not all real places are in $S$ has been open so far and is the subject of this paper.

THEOREM 1. *Let $k$ be a number field and let $S$ be a set of primes of $k$ which contains all primes dividing 2. If no real prime is in $S$, then* $\operatorname{cd} G_S(2) \leqslant 2$. *If $S$ contains real primes, then they become complex in $k_S(2)$ and* $\operatorname{cd} G_S(2) = \infty$, $\operatorname{vcd} G_S(2) \leqslant 2$.

*If $S$ is finite, then $H^i(G_S(2)) := H^i(G_S(2), \mathbb{Z}/2\mathbb{Z})$ is finite for all $i$ and $\chi_2(G_S(2)) = -r_2$, where $\chi_2(G_S(2)) = \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{F}_2} H^i(G_S(2))$ is the second partial Euler characteristic and $r_2$ is the number of complex places of $k$.*

The key for the proof of Theorem 1 is the following Theorem 2 in the case $p = 2$ and $T = S \cup S_{\mathbb{R}}$, where $S_{\mathbb{R}}$ is the set of real places of $k$. Theorem 2 is the number theoretical analogue of Riemann's existence theorem and was previously known under the assumption that $p$ is odd or that $S$ contains $S_{\mathbb{R}}$ (see [9], (10.5.1)).

THEOREM 2. *Let $k$ be a number field, $p$ a prime number and $T \supset S \supseteq S_p$ sets of primes of $k$. Then the canonical homomorphism*

$$\mathop{*}_{\mathfrak{p} \in T \setminus S(k_S(p))} T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \longrightarrow G(k_T(p)|k_S(p))$$

*is an isomorphism. Here $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \subset G(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$ is the inertia group and $*$ denotes the free pro-$p$-product.*

Since the cyclotomic $\mathbb{Z}_2$-extension $k_\infty(2)$ of $k$ is contained in $k_{S_2}(2)$, the group $G_{S_2}(2)$ is infinite, in particular, it is nontrivial. Hence, for $S \supseteq S_2$ and $S \cap S_{\mathbb{R}} = \varnothing$, the group $G_S(2)$ is of cohomological dimension 1 or 2. The next theorem gives a criterion for which case occurs. In condition (3) below, $\mathrm{Cl}_S^0(k)(2)$ denotes the 2-torsion part of the $S$-ideal class group in the narrow sense of $k$.

THEOREM 3. *Assume that $S \supseteq S_2$ and $S \cap S_{\mathbb{R}} = \varnothing$. Then* $\operatorname{cd} G_S(2) = 1$ *if and only if the following conditions* (1)–(3) *hold.*

(1) $S_2 = \{\mathfrak{p}_0\}$, *i.e. there exist exactly one prime dividing 2 in $k$.*
(2) $S = \{\mathfrak{p}_0\} \cup \{$*complex places*$\}$.
(3) $\mathrm{Cl}_S^0(k)(2) = 0$.

*In this case, $G_S(2)$ is a free pro-2-group of rank $r_2 + 1$ and $\mathfrak{p}_0$ does not split in $k_{S \cup S_{\mathbb{R}}}(2)$. In particular, if $k$ is totally real and $G_S(2)$ is free, then $k_S(2) = k_\infty(2)$.*

Let $k$ be a number field, $p$ a prime number and $S \supseteq S_p$ a set of places of $k$. A (necessarily infinite) extension $K|k$ is called $p$-$S$-closed if it has no $p$-extension which is unramified outside $S$. If $p$ is odd and $K$ is $p$-$S$-closed, then the group $\mathrm{Cl}_S(K(\mu_p))(p)(j)^{G(K(\mu_p)|K)}$ is trivial for $j = 0, -1$, where $\mu_p$ is the group of $p$th roots of unity, $(p)$ denotes the $p$-torsion part and $(j)$ the $j$th Tate-twist (see [9], (10.4.7)). The corresponding result for $p = 2$ is the following

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THEOREM 4. *Let $k$ be a number field, $S \supseteq S_2$ a set of primes of $k$ and $K$ a 2-$S$-closed extension of $k$. Then the following holds.*

(i) $\mathrm{Cl}_S(K(\mu_4))(2) = 0$,
(ii) $\mathrm{Cl}^0_S(K)(2) = 0$.

*Remarks*. (1) The triviality of $\mathrm{Cl}(K)(2)$ and, hence, also that of $\mathrm{Cl}_S(K)(2)$, follows easily from the principal ideal theorem; assertions (i) and (ii) do not.

(2) In (i) one can replace $K(\mu_4)$ by any totally imaginary extension of degree 2 of $K$ in $K_S(2)$.

Finally, we consider the full extension $k_S$, i.e. the maximal extension of $k$ which is unramified outside $S$, and its Galois group $G_S = G(k_S|k)$.

THEOREM 5. *Let $k$ be a number field and $S$ a set of primes of $k$ containing all primes dividing 2. Then $\mathrm{vcd}_2 G_S \leqslant 2$ and $\mathrm{cd}_2 G_S \leqslant 2$ if and only if $S$ contains no real primes. For every discrete $G_S(2)$-module $A$ the inflation maps*

$$\inf\colon H^i(G_S(2), A) \longrightarrow H^i(G_S, A)(2)$$

*are isomorphisms for all $i \geqslant 1$.*

*Remark*. If $\mathrm{cd}\, G_S(K)(2) = 2$ (e.g. if $K$ contains at least two primes dividing 2) for some finite subextension $K$ of $k$ in $k_S$, then $\mathrm{vcd}_2 G_S = 2$. This is always the case if $S \supset S_{\mathbb{R}}$ because the class numbers of the cyclotomic fields $\mathbb{Q}(\mu_{2^n})$ are nontrivial for $n \gg 0$. But, for example, we do not know whether $\mathrm{cd}_2 G(\mathbb{Q}_{S_2}|\mathbb{Q})$ equals 1 or 2. The answer would be '2' if at least one of the real cyclotomic fields $\mathbb{Q}(\mu_{2^n})^+$, $n = 2, 3, \dots$, would have a nontrivial class number. But this is unknown.

In Section 5 we investigate the relation between the cohomology of the group $G_S(k)$ and the modified étale cohomology of the scheme $\mathrm{Spec}(\mathcal{O}_{k,S})$. A discrete $G_S(k)$-module $A$ induces a locally constant sheaf on $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$, which we will denote by the same letter. We show the following theorem which is well known if $S$ contains all real primes (and also for odd $p$).

THEOREM 6. *Let $k$ be a number field and $S$ a finite set of primes of $k$ containing all primes dividing 2. Then for every 2-primary discrete $G_S(k)$-module $A$ the natural comparison maps $H^i(G_S(k), A) \longrightarrow H^i_{\mathrm{et,mod}}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$ are isomorphisms for all $i \geqslant 0$.*

For finite $A$ it is not difficult to show that the modified étale cohomology groups on the right-hand side of the comparison map are finite and that they vanish for $i \geqslant 3$ if $S$ contains no real primes. Therefore one could deduce Theorem 1 (with $G_S(k)(2)$ replaced by $G_S(k)$) from Theorem 6. However, in order to prove Theorem 6, one needs information on the interaction between the decomposition groups of the real primes and so Theorems 1 and 6 are both consequences of Theorem 2.

The main ingredients in the proofs of Theorems 1–5 are Poitou–Tate duality, the validity of the weak Leopoldt conjecture for the cyclotomic $\mathbb{Z}_p$-extension and, most essential, the systematic use of free products of bundles of profinite groups over a topological base. The reason that the above theorems had not been proven earlier seems to be a psychological one. At least the author always thought that one has to prove Theorem 1 first, before showing the other assertions. For example, Theorem 2 for $p=2$, $T=S_2\cup S_{\mathbb{R}}$ and $S=S_2$ was known if $k_{S_2}(2)=k_\infty(2)$ (see [12], § 4.2, for the case $k=\mathbb{Q}$ and [15], Satz 1.4, for the general case). But now it is Theorem 2 which is used in the proof of Theorem 1. Finally, we should mention that Theorem 1 was formulated as a conjecture in O. Neumann's article [10].

## 2. Free Products of Inertia Groups

In this section we briefly collect some facts on free products of profinite groups and how they naturally occur in number theory. For a more detailed presentation and for proofs of the facts cited below we refer the reader to [9], chap. IV and chap. X, §1.

A profinite space is a topological space which is compact and totally disconnected. Equivalently, a profinite space is a topological inverse limit of finite discrete spaces. A profinite group is a group object in the category of profinite spaces. It can be shown that a profinite group is the inverse limit of finite groups. A full class of finite groups $\mathfrak{c}$ is a full subcategory of the category of all finite groups which is closed under taking subgroups, quotients and extensions. A pro-$\mathfrak{c}$-group is a profinite group which is the inverse limit of groups in $\mathfrak{c}$.

Let $T$ be a profinite space. A *bundle of profinite groups* $\mathcal{G}$ over $T$ is a group object in the category of profinite spaces over $T$. We say that $\mathcal{G}$ is a bundle of pro-$\mathfrak{c}$-groups if the fibre $\mathcal{G}_t$ of $\mathcal{G}$ over every point $t\in T$ is a pro-$\mathfrak{c}$-group. The functor 'constant bundle', which assigns to a pro-$\mathfrak{c}$-group $G$ the bundle $\mathrm{pr}_2\colon G\times T\to T$ has a left adjoint

$$\begin{array}{ccc}\{\text{bundles of pro-}\mathfrak{c}\text{-groups over } T\} & \longrightarrow & \{\text{pro-}\mathfrak{c}\text{-groups}\}\\ \mathcal{G} & \longmapsto & \mathop{*}\limits_{T}\mathcal{G}.\end{array}$$

The image $*_T\mathcal{G}$ of a bundle $\mathcal{G}$ under this functor is called its free pro-$\mathfrak{c}$-product. It satisfies a universal property which is determined by the functor adjunction. Bundles of pro-$\mathfrak{c}$-groups often arise in the following way:

Let $G$ be a pro-$\mathfrak{c}$-group and assume we are given a continuous family of closed subgroups of $G$, i.e. a family of closed subgroups $\{G_t\}_{t\in T}$ indexed by the points of a profinite space $T$ which has the property that for every open subgroup $U\subset G$ the set $T(U)=\{t\in T\mid G_t\subseteq U\}$ is open in $T$. Then

$$\mathcal{G}=\{(g,t)\in G\times T\mid g\in G_t\}$$

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is in a natural way a bundle of pro-$\mathfrak{c}$-groups over $T$. We have a canonical homomorphism $\phi : \ast_T \mathcal{G} \longrightarrow G$ and we say that $G$ is the free product of the family $\{G_t\}_{t\in T}$ if $\phi$ is an isomorphism.

The usual free pro-$\mathfrak{c}$-product of a discrete family of pro-$\mathfrak{c}$-groups as defined in various places in the literature (e.g. [8]) fits into the picture as follows. For a family $\{G_i\}_{i\in I}$ we consider the disjoint union $(\dot\cup_i G_i)\,\dot\cup\,\{\ast\}$ of the $G_i$ and one external point $\ast$. Equipped with a suitable topology, this is a bundle of pro-$\mathfrak{c}$-groups over the one-point compactification $\bar I = I \,\dot\cup\, \{\ast\}$ of $I$ and the free pro-$\mathfrak{c}$-product of the family $\{G_i\}_{i\in I}$ coincides with that of the bundle (cf. [9], chap. IV, §3, examples 2 and 4). For the free product of a discrete family of pro-$\mathfrak{c}$-groups we have the following profinite version of Kurosh's subgroup theorem (see [2] or [9], (4.2.1)).

THEOREM 2.1. *Let $G = \ast_{i\in I} G_i$ be the free pro-$\mathfrak{c}$-product of the discrete family $G_i$ and let $H$ be an open subgroup of $G$. Then there exist systems $S_i$ of representatives $s_i$ of the double coset decomposition $G = \bigcup_{s_i\in S_i} Hs_iG_i$ for all $i$ and a free pro-$\mathfrak{c}$-group $F \subseteq G$ of finite rank*

$$\mathrm{rk}(F) = \sum_{i\in I} [(G:H) - \#S_i] - (G:H) + 1,$$

*such that the natural inclusions induce a free product decomposition*

$$H = \ast_{i,s_i} (G_i^{s_i} \cap H) \ast F,$$

*where $G_i^{s_i}$ $(= s_iG_is_i^{-1})$ denotes the conjugate subgroup.*

In number theory, continuous families of pro-$\mathfrak{c}$-groups occur in the following way. For a number field $k$ we denote the one-point compactification of the set of all places of $k$ by $\mathrm{Sp}(k)$. The compactifying point will be denoted by $\eta_k$ and should be thought as the generic point of the scheme $\mathrm{Spec}(\mathcal{O}_k)$ in the sense of algebraic geometry or as the trivial valuation of $k$ from the point of view of valuation theory. For an infinite extension $K|k$, we set

$$\mathrm{Sp}(K) = \varprojlim_{k'} \mathrm{Sp}(k'),$$

where $k'$ runs through all finite subextensions of $k$ in $K$. The complement of the (closed and open) subset of all Archimedean places of $K$ in $\mathrm{Sp}(K)$ is naturally isomorphic to $\mathrm{Spec}(\mathcal{O}_K)$ endowed with the constructible topology (see [6], chap. I, §7, (7.2.11) for the definition of the constructible topology of a scheme). Let $S$ be a set of primes of $k$ and $\bar S$ its closure in $\mathrm{Sp}(k)$ ($\bar S = S$ if $S$ is finite, $\bar S = S \cup \{\eta_k\}$ if $S$ is infinite). The pre-image $\bar S(K)$ of $\bar S$ under the natural projection $\mathrm{Sp}(K) \to \mathrm{Sp}(k)$ is the closure of the set $S(K)$ of all prolongations of primes in $S$ to $K$ in $\mathrm{Sp}(K)$.

Now assume that $M \supset K \supset k$ are possibly infinite extensions of $k$ such that $M|K$ is Galois and $G(M|K)$ is a pro-$\mathfrak{c}$-group. The natural projection $\bar S(M) \to \bar S(K)$ has a section (in fact, there are many of them). For a fixed section $s$: $\bar S(K) \to \bar S(M)$ we consider the family of inertia groups $\{T_{s(\mathfrak{p})}(M|K)\}_{\mathfrak{p}\in\bar S(K)}$, where by convention

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$T_{\eta_M} = \{1\}$. Since a finite extension of number fields is ramified only at finitely many primes, this is a continuous family of subgroups of $G(M|K)$ indexed by $\bar{S}(K)$. We obtain a natural homomorphism

$$\phi\colon \mathop{\ast}_{\bar{S}(K)} T_{s(\mathfrak{p})}(M|K) \longrightarrow G(M|K),$$

which we also write in the form

$$\phi\colon \mathop{\ast}_{\mathfrak{p}\in S(K)} T_{\mathfrak{p}}(M|K) \longrightarrow G(M|K).$$

The cohomology groups of the free product on the left-hand side with coefficients in a trivial module do not depend on the particularly chosen section $s$. The question, however, whether the homomorphism $\phi$ is an isomorphism *does* depend on $s$. Moreover, if $s$ is a section for which $\phi$ is an isomorphism, we always find a section $s'$ for which it is not, at least if $\mathfrak{c}$ is not the class of $p$-groups, where $p$ is a prime number. In the case of pro-$p$-groups this pathology does not occur because of the following easy and well-known.

LEMMA 2.2. *Let $p$ be a prime number and let $\phi$: $G' \longrightarrow G$ be a (continuous) homomorphism of pro-$p$-groups. Let $A$ be $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial action. Then $\phi$ is an isomorphism if and only if the induced homomorphism*

$$H^i(\phi, A)\colon\ H^i(G, A) \longrightarrow H^i(G', A)$$

*is an isomorphism for $i = 1$ and injective for $i = 2$.*

In the number theoretical situation above, we have the following formula for the cohomology of the free product with values in a torsion group $A$ (considered as a module with trivial action) and for $i \geqslant 1$:

$$H^i\left(\mathop{\ast}_{\mathfrak{p}\in S(K)} T_{\mathfrak{p}}(M|K), A\right) = \varinjlim_{k'} \bigoplus_{\mathfrak{p}\in S(k')} H^i(T_{\mathfrak{p}}(M'|k'), A),$$

where $k'$ runs through all finite subextensions of $k$ in $K$ and $M'$ is the maximal pro-$\mathfrak{c}$ Galois subextension of $M|k'$ (so $M = \varinjlim M'$). The limit on the right-hand side depends on $K$ and not on $k$ and we denote it by $\bigoplus'_{\mathfrak{p}\in S(K)} H^i(T_{\mathfrak{p}}(M|K), A)$. If $K|k$ is Galois, then this limit is the maximal discrete $G(K|k)$-submodule of the product $\prod_{\mathfrak{p}\in S(K)} H^i(T_{\mathfrak{p}}(M|K), A)$.

## 3. Proof of Theorem 2

Let us first remark that for $\mathfrak{p} \in T \setminus S(k)$ the inertia group has the following structure:

- if $\mathfrak{p}$ is non-Archimedean and $N(\mathfrak{p}) \equiv 1 \bmod p$ (i.e. if there is a primitive $p$th root of unity in $k_{\mathfrak{p}}$), then $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$ is a free pro-$p$-group of rank 1, i.e. isomorphic to $\mathbb{Z}_p$.
- if $\mathfrak{p}$ is non-Archimedean and $N(\mathfrak{p}) \not\equiv 1 \bmod p$, then $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) = \{1\}$.

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- if $\mathfrak{p}$ is real and $p=2$, then $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \cong \mathbb{Z}/2\mathbb{Z}$.
- if $\mathfrak{p}$ is real and $p \neq 2$ or if $\mathfrak{p}$ is complex, then $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) = \{1\}$.

If $p$ is odd or if $p=2$ and $S \supset S_{\mathbb{R}}$, then Theorem 2 is known (see [9], (10.5.1)). So we assume that $p=2$ and $S \not\supset S_{\mathbb{R}}$. For a pro-2-group $G$ we use the notation $H^i(G)$ for $H^i(G, \mathbb{Z}/2\mathbb{Z})$. We start with the following lemma:

LEMMA 3.1. *Let $G$ and $G'$ be pro-2-groups which are generated by involutions and assume that $H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0 = H^2(G', \mathbb{Q}_2/\mathbb{Z}_2)$. Let $\phi\colon G' \to G$ be a (continuous) homomorphism. Then the following assertions are equivalent.*

(i) *$\phi$ is an isomorphism.*
(ii) *$H^1(\phi) : H^1(G) \to H^1(G')$ is an isomorphism.*
(iii) *$H^2(\phi) : H^2(G) \to H^2(G')$ is an isomorphism.*

*Proof*. Clearly, (i) implies (ii) and (iii) and, by Lemma 2.2, (ii) and (iii) together imply (i). So it remains to show that (ii) and (iii) are equivalent. Since $H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0$, the exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}_2/\mathbb{Z}_2 \to \mathbb{Q}_2/\mathbb{Z}_2 \to 0$ induces the four term exact sequence

$$0 \to H^1(G) \xrightarrow{\alpha} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\beta} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\gamma} H^2(G) \to 0.$$

Since $G$ is generated by involutions, $\alpha$ is an isomorphism. Hence $\beta$ is zero and $\gamma$ is an isomorphism. The same argument also applies to $G'$ and therefore (ii) and (iii) are both equivalent to

(iv) $H^1(\phi, \mathbb{Q}_2/\mathbb{Z}_2)\colon H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \to H^1(G', \mathbb{Q}_2/\mathbb{Z}_2)$ is an isomorphism.

This concludes the proof. $\square$

We show Theorem 2 first in the special case $T = S_2 \cup S_{\mathbb{R}}$, $S = S_2$. The groups $*_{\mathfrak{p} \in S_{\mathbb{R}}(k_{S_2}(2))} T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ and $G(k_{S_2 \cup S_{\mathbb{R}}}(2)|k_{S_2}(2))$ are both generated by involutions. Since $H^2(T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ for every $\mathfrak{p} \in S_{\mathbb{R}}(k_{S_2}(2))$, we have

$$H^2\left(\mathop{*}_{\mathfrak{p} \in S_{\mathbb{R}}(k_{S_2}(2))} T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2\right) = 0.$$

By [9], (10.4.8), the inflation map

$$H^2(G(k_{S_2 \cup S_{\mathbb{R}}}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow H^2(G(k_{S_2 \cup S_{\mathbb{R}}}|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2)$$

is an isomorphism and, since $k_{S_2}(2)$ contains the cyclotomic $\mathbb{Z}_2$-extension $k_\infty(2)$ of $k$, the validity of the weak Leopoldt conjecture for the cyclotomic $\mathbb{Z}_p$-extension (see [9], (10.3.25)) implies (by [9], (10.3.22)) that

$$H^2(G(k_{S_2 \cup S_{\mathbb{R}}}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

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By Lemma 3.1 and the calculation of the cohomology of free products (see Section 1), it therefore suffices to show that the natural map

$$H^2(\phi): H^2(G(k_{S_2\cup S_{\mathbb{R}}}(2)|k_{S_2}(2)) \to \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(k_{S_2}(2))}{}' H^2(T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}))$$

is an isomorphism. Now let $K$ be a finite extension of $k$ inside $k_S(2)$. The 9-term exact sequence of Poitou–Tate induces the exact sequence

$$\begin{aligned}&0 \to Ш^2(K_{S_2\cup S_{\mathbb{R}}}, \mathbb{Z}/2\mathbb{Z}) \to H^2(G(k_{S_2\cup S_{\mathbb{R}}}|K), \mathbb{Z}/2\mathbb{Z}) \to \\ &\bigoplus_{\mathfrak{p}\in S_2\cup S_{\mathbb{R}}(K)} H^2(G(\bar{k}_{\mathfrak{p}}|K_{\mathfrak{p}}), \mathbb{Z}/2\mathbb{Z}) \to H^0(G(k_{S_2\cup S_{\mathbb{R}}}|K), \mu_2)^{\vee} \to 0,\end{aligned}$$

where $\vee$ denotes the Pontryagin dual. Furthermore, we have

$$Ш^2(K_{S_2\cup S_{\mathbb{R}}}, \mathbb{Z}/2\mathbb{Z}) \cong Ш^1(K_{S_2\cup S_{\mathbb{R}}}, \mu_2)^{\vee} = Ш^1(K_{S_2\cup S_{\mathbb{R}}}, \mathbb{Z}/2\mathbb{Z})^{\vee} = \mathrm{Cl}_{S_2}(K)/2.$$

For a finite, nontrivial extension $K'$ of $K$ inside $k_{S_2}(2)$ the corresponding homomorphism $H^0(G(k_{S_2\cup S_{\mathbb{R}}}|K), \mu_2)^{\vee} \to H^0(G(k_{S_2\cup S_{\mathbb{R}}}|K'), \mu_2)^{\vee}$ is the dual of the norm map, hence trivial. Furthermore, $H^2\big(G(\bar{k}_{\mathfrak{p}}|(k_{S_2}(2))_{\mathfrak{p}}), \mathbb{Z}/2\mathbb{Z}\big) = 0$ for $\mathfrak{p} \in S_2(k_{S_2}(2))$ (see [9], (7.1.8)(i)). Therefore, we obtain the following exact sequence in the limit over all finite subextensions $K|k$ in $k_{S_2}(2)|k$ (the omitted coefficients are $\mathbb{Z}/2\mathbb{Z}$):

$$\mathrm{Cl}_{S_2}(k_{S_2}(2))/2 \hookrightarrow H^2\big(G(k_{S_2\cup S_{\mathbb{R}}}|k_{S_2}(2))\big) \twoheadrightarrow \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(k_{S_2}(2))}{}' H^2\big(G(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}})\big).$$

The principal ideal theorem implies that $\mathrm{Cl}(k_{S_2}(2))(2) = 0$, and therefore also $\mathrm{Cl}_{S_2}(k_{S_2}(2))/2 = 0$. Furthermore, $G(\bar{k}_{\mathfrak{p}}|k_{\mathfrak{p}}) = T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ for $\mathfrak{p} \in S_{\mathbb{R}}(k_{S_2}(2))$ and the inflation map

$$H^2\big(G(k_{S_2\cup S_{\mathbb{R}}}(2)|k_{S_2}(2))\big) \longrightarrow H^2\big(G(k_{S_2\cup S_{\mathbb{R}}}|k_{S_2}(2))\big)$$

is an isomorphism (see [9], (10.4.8)). This concludes the proof of Theorem 2 in the case $T = S_2 \cup S_{\mathbb{R}}$, $S = S_2$. For the proof in the general case we need the

PROPOSITION 3.2. *Let $k$ be a number field, $p$ a prime number and $T \supset S \supseteq S_p$ sets of primes in $k$. Let $K$ be a $p$-$S_p$-closed extension of $k$. Then the following assertions are equivalent.*

(i) *The natural homomorphism*

$$\phi_{T,S_p}: \mathop{\ast}_{\mathfrak{p}\in T\smallsetminus S_p(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \to G(K_T(p)|K)$$

*is an isomorphism.*

(ii) *The natural homomorphisms*

$$\phi_{T,S}: \mathop{\ast}_{\mathfrak{p}\in T\smallsetminus S(K_S(p))} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \to G(K_T(p)|K_S(p))$$

*and*

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$$\phi_{S,S_p}\colon \mathop{*}_{\mathfrak{p}\in S\setminus S_p(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \to G(K_S(p)|K)$$

*are isomorphisms.*

*Here $*$ denotes the free pro-$p$-product.*

*Proof.* If $\phi_{T,S_p}$ is an isomorphism, then also $\phi_{S,S_p}$ is an isomorphism. Furthermore, a straightforward application of Theorem 2.1 shows that also $\phi_{T,S}$ is an isomorphism in this case. Let us show the converse statement. Assume that $\phi_{T,S}$ and $\phi_{S,S_p}$ are isomorphisms. Note that all primes in $S\setminus S_p(K_S(p))$ split completely in $K_T(p)|K_S(p)$. Therefore the extension of pro-$p$-groups

$$1 \to G(K_T(p)|K_S(p)) \to G(K_T(p)|K) \to G(K_S(p)|K) \to 1 \tag{1}$$

splits. By Lemma 2.2, we have to show that the induced homomorphism

$$H^i(\phi_{T,S_p}) : H^i\big(G(K_T(p)|K)\big) \longrightarrow \bigoplus_{\mathfrak{p}\in T\setminus S_p(K)}{}' H^i(T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}))$$

is an isomorphism for $i = 1$ and injective for $i = 2$ (coefficients $\mathbb{Z}/p\mathbb{Z}$). This follows easily from the Hochschild–Serre spectral sequence associated to the split exact sequence (1):

$$E_2^{ij} = H^i\big(G(K_S(p)|K), H^j(G(K_T(p)|K_S(p)))\big) \Longrightarrow H^{i+j}(G(K_T(p)|K)).$$

First of all, the differentials $d_2$ are zero ($-d_2$ is the cup-product with the extension class, see [9], (2.1.8)). Furthermore, every prime in $T\setminus S(K)$ splits completely in $K_S(p)|K$ because these primes are unramified in $K_S(p)|K$ and $K$ contains $K_\infty(p)$. Since $\phi_{T,S}$ is an isomorphism, the $G(K_S(p)|K)$-module ($j \geqslant 1$)

$$\begin{aligned} H^j(G(K_T(p)|K_S(p))) &= \bigoplus_{\mathfrak{p}\in T\setminus S(K_S(p))}{}' H^j(T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})) \\ &= \mathrm{Ind}_{G(K_S(p)|K)} \bigoplus_{\mathfrak{p}\in T\setminus S(K)}{}' H^j(T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})) \end{aligned}$$

is cohomologically trivial. Therefore we obtain short exact sequences

$$0 \to H^i(K_S(p)|K) \to H^i(K_T(p)|K) \to \bigoplus_{\mathfrak{p}\in T\setminus S(K)}{}' H^i(T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})) \to 0$$

for $i = 1, 2$, and the result follows from the five-lemma. $\square$

Now we can prove Theorem 2 in the general case. It is true for odd $p$ and for $p = 2$ in the special cases $T = S_2 \cup S_{\mathbb{R}}$, $S = S_2$ and $T = \{\text{all primes}\}$, $S = S_2 \cup S_{\mathbb{R}}$. Applying Proposition 3.2 in the situation $p = 2$, $T = \{\text{all primes}\}$, $S = S_2 \cup S_{\mathbb{R}}$ and $K = k_{S_2}(2)$, we obtain Theorem 2 in the 'extremal' case $T = \{\text{all primes}\}$, $S = S_2$. Applying Proposition 3.2 again, we obtain the case $T = \{\text{all primes}\}$ and $S$ arbitrary and then the general case. This concludes the proof of Theorem 2. $\square$

A straightforward limit process shows the following variant of Theorem 2.

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THEOREM 2$'$. *Let $k$ be a number field, $p$ a prime number and $T \supset S \supseteq S_p$ sets of primes of $k$. Let $K$ be a $p$-$S$-closed extension field of $k$. Then the canonical homomorphism*

$$\mathop{*}_{\mathfrak{p} \in T \setminus S(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \longrightarrow G(K_T(p)|K)$$

*is an isomorphism.*

## 4. Proofs of the Remaining Statements

In order to prove Theorem 1, we may assume that $S \not\supset S_{\mathbb{R}}$ and we investigate the Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(G_S(2), H^j(G(k_{S\cup S_{\mathbb{R}}}(2)|k_S(2))) \Longrightarrow H^{i+j}(G_{S\cup S_{\mathbb{R}}}(2)),$$

where the omitted coefficient are $\mathbb{Z}/2\mathbb{Z} = \mu_2$. By Theorem 2, we have complete control over the $G_S(2)$-modules $H^j(G(k_{S\cup S_{\mathbb{R}}}(2)|k_S(2))$, which are for $j \geqslant 1$ isomorphic to

$$\mathrm{Ind}_{G_S(2)} \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}} \setminus S(k)} H^j(G(\mathbb{C}|\mathbb{R})).$$

In particular, $E_2^{ij} = 0$ for $ij \neq 0$. Therefore the spectral sequence induces an exact sequence

$$\begin{aligned} 0 \to H^1(G_S(2)) \to H^1(G_{S\cup S_{\mathbb{R}}}(2)) \to \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}} \setminus S(k)} H^1(G(\mathbb{C}|\mathbb{R})) \to \\ H^2(G_S(2)) \to H^2(G_{S\cup S_{\mathbb{R}}}(2)) \to \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}} \setminus S(k)} H^2(G(\mathbb{C}|\mathbb{R})) \to 0 \end{aligned} \tag{2}$$

and exact sequences

$$0 \to H^i(G_S(2)) \to H^i(G_{S\cup S_{\mathbb{R}}}(2)) \to \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}} \setminus S(k)} H^i(G(\mathbb{C}|\mathbb{R})) \to 0 \tag{3}$$

for $i \geqslant 3$. If $S$ is finite, this shows the finiteness statement on the cohomology of $G_S(2)$ and that $\chi_2(G_S(2)) = \chi_2(G_{S\cup S_{\mathbb{R}}}(2))$. But $\chi_2(G_{S\cup S_{\mathbb{R}}}(2)) = \chi_2(G_{S\cup S_{\mathbb{R}}}) = -r_2$ (see [9], (8.6.16) and (10.4.8)).

For arbitrary $S$ and $i \geqslant 3$ the restriction map

$$H^i(G_{S\cup S_{\mathbb{R}}}(2)) \to \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(k)} H^i(G(\mathbb{C}|\mathbb{R}))$$

is an isomorphism (see [9], (8.6.13)(ii) and (10.4.8)). This together with (3) shows that the natural homomorphism

$$H^i(G_S(2)) \to \bigoplus_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(k)} H^i(G(\mathbb{C}|\mathbb{R}))$$

is an isomorphism for $i \geqslant 3$. Therefore cd $G_S(2) \leqslant 2$ if $S \cap S_{\mathbb{R}} = \varnothing$. For later use we formulate the last result as a proposition.

PROPOSITION 4.1. *Let $k$ be a number field and $S \supset S_2$ a set of primes. Then the natural homomorphism*

$$H^i(G_S(2), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(k)} H^i(G(\mathbb{C}|\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

*is an isomorphism for $i \geqslant 3$.*

In order to conclude the proof of Theorem 1, it remains to show that every real prime in $S$ ramifies in $k_S(2)$. Let $S^f$ be the subset of non-Archimedean primes in $S$. Then Theorem 2 yields an isomorphism

$$\mathop{*}_{\mathfrak{p}\in S_{\mathbb{R}}(k_{S^f}(2))} T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \cong G(k_S(2)|k_{S^f}(2))$$

which shows the required assertion. This finishes the proof of Theorem 1. □

Now we prove Theorem 3. To fix conventions, we recall the following definitions. For a set $S$ of primes of $k$ the group $\mathcal{O}_{k,S}^{\times}$ of $S$-units is defined as the subgroup in $k^{\times}$ of those elements which are units at every finite prime not in $S$ and positive at every real prime not in $S$. The $S$-ideal class group $\mathrm{Cl}_S^0(k)$ in the narrow sense of $k$ is the quotient of the group of fractional ideals of $k$ by the subgroup generated by the non-Archimedean primes in $S$ and the principal ideals $(a)$ with $a$ positive at every real place of $k$ not contained in $S$. In particular, $\mathrm{Cl}_{\emptyset}^0(k) = \mathrm{Cl}^0(k)$ is the ideal class group in the narrow sense and $\mathrm{Cl}_{S\cup S_{\mathbb{R}}}^0(k) = \mathrm{Cl}_S(k)$ is the usual $S$-ideal class group. By class field theory, $\mathrm{Cl}_S^0(k)$ is isomorphic to the Galois group of the maximal Abelian extension of $k$ which is unramified outside $S_{\mathbb{R}}$ and in which every prime in $S$ splits completely. By Kummer theory, we can replace condition (3) of Theorem 3 by the following condition

$$\{x \in k^{\times} \mid x \in k_{\mathfrak{p}_0}^{\times 2} \text{ and } 2 \mid v_{\mathfrak{p}}(x) \text{ for every finite prime } \mathfrak{p}\} = k^{\times 2}. \tag{3}$$

LEMMA 4.2. *If $S \supseteq S_2$ and* $\mathrm{cd}\, G_{S_2}(2) = 1$, *then $S = S_2$.*

*Proof*. By Theorem 2, we have an isomorphism

$$\mathop{*}_{\mathfrak{p}\in S\setminus S_2(k_{S_2}(2))} T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \xrightarrow{\sim} G(k_S(2)|k_{S_2}(2))$$

Since for non-Archimedean primes $\mathfrak{p} \notin S_2$ the maximal unramified 2-extension of $k_{\mathfrak{p}}$ is realized by $k_{\infty}(2) \subset k_{S_2}(2)$, this shows that for $\mathfrak{p} \in S \setminus S_2$ the maximal 2-extension of the local field $k_{\mathfrak{p}}$ is realized by $k_S(2)$ or, in other words, the natural homomorphism

$$G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \longrightarrow G_S(2)$$

is injective. But for these primes we have $\mathrm{cd}\, G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \geqslant 2$ which shows that $S \setminus S_2 = \emptyset$. □

Now assume that $G_{S_2}(2)$ is free. For a prime $\mathfrak{p}$ we denote the local group $G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ by $\mathcal{G}_{\mathfrak{p}}$ and the inertia group $T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ by $\mathcal{T}_{\mathfrak{p}}$. By Čebotarev's density theorem, we find a finite set of non-Archimedean primes $T \supset S_2$ such that the natural homomorphism

$$H^1(G_{S_2}) \longrightarrow \bigoplus_{\mathfrak{p}\in T\smallsetminus S} H^1(\mathcal{G}_\mathfrak{p}/\mathcal{T}_\mathfrak{p})$$

is an isomorphism. It is then an easy exercise using Lemma 2.2 to show that the natural homomorphism

$$\mathop{*}_{\mathfrak{p}\in T\smallsetminus S_2} \mathcal{G}_\mathfrak{p}/\mathcal{T}_\mathfrak{p} \longrightarrow G_{S_2}(2)$$

is an isomorphism. Theorem 2 for $T = S_2 \cup S_\mathbb{R}$ and $S = S_2$ and the same arguments as in the proof of Proposition 3.2 show that the natural homomorphism

$$\mathop{*}_{\mathfrak{p}\in T\smallsetminus S_2} \mathcal{G}_\mathfrak{p}/\mathcal{T}_\mathfrak{p} \; * \mathop{*}_{\mathfrak{p}\in S_\mathbb{R}} \mathcal{G}_\mathfrak{p} \longrightarrow G_{S_2\cup S_\mathbb{R}}(2)$$

is an isomorphism. Then, by ([16], Theorem 6) or ([9], (10.7.2)), we obtain the conditions (1)–(3) and that the unique prime $\mathfrak{p}_0$ dividing 2 in $k$ does not split in $k_{S_2\cup S_\mathbb{R}}$. If, on the other hand, conditions (1)–(3) of Theorem 3 are satisfied, then we obtain (loc. cit.) the above isomorphism and deduce that $G_{S_2}(2)$ is free. The statement on the rank of $G_{S_2}(2)$ follows from $\chi_2(G_{S_2}(2)) = -r_2$. If $k$ is totally real, then the homomorphism

$$G_{S_2}(2) \longrightarrow G(k_\infty(2)|k)$$

is a surjection of free pro-2-groups of rank 1 and hence an isomorphism. This concludes the proof of Theorem 3.

Next we show Theorem 4. Let $S$ be a set of finite primes of $k$ and $\Sigma = S \cup S_\mathbb{R}$. If $S$ is finite, then the image of the group of $\Sigma$-units of $k$ under the logarithm map $\mathrm{Log}\colon \mathcal{O}_{k,\Sigma}^\times \longrightarrow \bigoplus_{v\in\Sigma}\mathbb{R}$, $a \mapsto (\log|a|_v)_{v\in S}$ is a lattice of rank equal to $\#S + r_1 + r_2 - 1$ (Dirichlet's unit theorem). Complementary to this map is the signature map (which is also defined for infinite $S$)

$$\mathrm{Sign}_{k,S}\colon \mathcal{O}_{k,\Sigma}^\times \longrightarrow \bigoplus_{v\in S_\mathbb{R}} \mathbb{R}^\times/\mathbb{R}^{\times 2}.$$

More or less by definition, there exists a five-term exact sequence

$$0 \to \mathcal{O}_{k,S}^\times \to \mathcal{O}_{k,\Sigma}^\times \to \bigoplus_{v\in S_\mathbb{R}(k)} \mathbb{R}^\times/\mathbb{R}^{\times 2} \to \mathrm{Cl}_S^0(k) \to \mathrm{Cl}_\Sigma^0(k) \to 0,$$

and so the cokernel of $\mathrm{Sign}_{k,S}$ measures the difference between the usual $S$-ideal class group $\mathrm{Cl}_S(k) = \mathrm{Cl}_\Sigma^0(k)$ and that in the narrow sense. Of course this discussion is void if $k$ is totally imaginary. If $K$ is an infinite extension of $k$, we define the signature map

$$\mathrm{Sign}_{K,S}\colon \mathcal{O}_{K,\Sigma}^\times \longrightarrow \varinjlim_{k'} \bigoplus_{v\in S_\mathbb{R}(k')} \mathbb{R}^\times/\mathbb{R}^\times$$

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as the limit over the signature maps $\mathrm{Sign}_{k',S}$, where $k'$ runs through all finite sub-extension $k'|k$ of $K|k$. If $K$ is 2-$S$-closed, then $\mathrm{Cl}_S(K)(2) = 0$ and so statement (ii) of Theorem 4 is equivalent to the statement that $\mathrm{Sign}_K$ is surjective.

Now assume that $k$, $S$, $K$ are as in Theorem 4. By Theorem 1, all real places in $S$ become complex in $K$. By the principal ideal theorem, $\mathrm{Cl}(K)(2) = 0$ and so statement (i) and (ii) are trivial if $K$ is totally imaginary (note that $K = K(\mu_4)$ in this case). So we may assume that $S_{\mathbb{R}}(K) \neq \emptyset$ and, by Theorem 1, we may suppose $S \cap S_{\mathbb{R}} = \emptyset$.

Let $K' = K(\mu_4)$. Then $K'$ is totally imaginary and $G = G(K'|K)$ is cyclic of order 2. Let $\Sigma = S \cup S_{\mathbb{R}}$ and let $K_\Sigma$ be the maximal (not just the pro-2) extension of $K$ which is unramified outside $\Sigma$. Inspecting the Hochschild–Serre spectral sequence associated to $K_\Sigma|K_\Sigma(2)|K$ and using the well-known calculation of $H^i(G(K_\Sigma|K), \mathcal{O}^\times_{K_\Sigma,\Sigma})$ (cf. [9], (10.4.8)) we see that

$$H^1(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) = H^1(G(K_\Sigma|K), \mathcal{O}^\times_{K_\Sigma,\Sigma})(2) \qquad (4)$$
$$= \mathrm{Cl}_S(K)(2) = 0$$

and the same argument shows that

$$H^1(G(K_\Sigma(2)|K'), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \cong \mathrm{Cl}_S(K')(2). \qquad (5)$$

Next we consider the Hochschild–Serre spectral sequence for the extension $K_\Sigma(2)|K'|K$ and the module $\mathcal{O}^\times_{K_\Sigma(2),\Sigma}$. By (4) and (5), we obtain an exact sequence

$$0 \to \mathrm{Cl}_S(K')(2)^G \to H^2(G, \mathcal{O}^\times_{K',\Sigma}) \xrightarrow{\phi} H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}).$$

Since $G$ is a 2-group, in order to prove assertion (i), it suffices to show that $\phi$ is injective. Let $c$ be a generator of the cyclic group $H^2(G, \mathbb{Z})$. For each prime $\mathfrak{p} \in S_{\mathbb{R}}(K)$ (respectively for the chosen prolongation of $\mathfrak{p}$ to $K_\Sigma(2)$, cf. the discussion in § 1), the composition $T_{\mathfrak{p}}(K_\Sigma(2)|K) \to G(K_\Sigma(2)|K) \to G$ is an isomorphism and we denote the image of $c$ in $H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K), \mathbb{Z})$ by $c_{\mathfrak{p}}$. As is well known, the cup-product with $c$ induces an isomorphism $\hat{H}^0(G, \mathcal{O}^\times_{K',\Sigma}) \xrightarrow{\sim} H^2(G, \mathcal{O}^\times_{K',\Sigma})$ and the similar statement holds for each $c_{\mathfrak{p}}$, $\mathfrak{p} \in S_{\mathbb{R}}(K)$.

The quotient $\mathcal{O}^\times_{K_\Sigma(2),\Sigma}/\mu_{2^\infty}$ is uniquely 2-divisible, and so we obtain a natural isomorphism

$$H^2(G(K_\Sigma(2)|K), \mu_{2^\infty}) \xrightarrow{\sim} H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}).$$

Furthermore, for each $\mathfrak{p} \in S_{\mathbb{R}} \smallsetminus S$ we obtain an isomorphism

$$H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K), \mu_{2^\infty}) \xrightarrow{\sim} H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma})$$
$$\cong H^2(G(\bar{K}_{\mathfrak{p}}|K_{\mathfrak{p}}), \bar{K}^\times_{\mathfrak{p}}).$$

Therefore, the calculation of the cohomology in dimension $i \geqslant 2$ of free products with values in torsion modules (see [10], Satz 4.1 or [9], (4.1.4)) and Theorem 2 for the pair $\Sigma$, $S$ show that we have a natural isomorphism

$$H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(K)}{}' H^2(G(\bar{K}_\mathfrak{p}|K_\mathfrak{p}), \bar{K}^\times_\mathfrak{p}).$$

(Alternatively, we could have obtained this isomorphism from the calculation of the cohomology of the $\Sigma$-units, cf. ([9], (8.3.10)(iii)) by passing to the limit over all finite subextensions of $k$ in $K$.) We obtain the following commutative diagram

$$\begin{tikzcd}
\hat{H}^0(G, \mathcal{O}^\times_{K',\Sigma}) \arrow[rr, dashed, "\psi"] \arrow[d, "\cup c", "\wr"'] & & \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(K)}{}' \hat{H}^0(G(\bar{K}_\mathfrak{p}|K_\mathfrak{p}), \bar{K}^\times_\mathfrak{p}) \arrow[d, "\bigoplus' \cup c_\mathfrak{p}", "\wr"'] \\
H^2(G, \mathcal{O}^\times_{K',\Sigma}) \arrow[r, "\phi"] & H^2(G(K_\Sigma(2)|K), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \arrow[r, "\sim"] & \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(K)}{}' H^2(G(\bar{K}_\mathfrak{p}|K_\mathfrak{p}), \bar{K}^\times_\mathfrak{p}).
\end{tikzcd}$$

Hence, $\ker(\phi) \cong \ker(\psi)$ and $\operatorname{coker}(\phi) \cong \operatorname{coker}(\psi)$. Since

$$\hat{H}^0(G, \mathcal{O}^\times_{K',\Sigma}) = \mathcal{O}^\times_{K,\Sigma}/N_{K'|K}(\mathcal{O}^\times_{K',\Sigma}),$$

each element in $\ker(\psi)$ is represented by an $S$-unit in $K$ and we have to show that all these are norms of $\Sigma$-units in $K'$. Let $e \in \mathcal{O}^\times_{K,S}$. Then $K(\sqrt{e})|K$ is a 2-extension which is unramified outside $S$, hence trivial. Therefore $e$ is a square in $K$ and if $f^2 = e$, then $f \in \mathcal{O}^\times_{K,\Sigma}$ and $e = N_{K'|K}(f)$.

This concludes the proof of assertion (i).

To show assertion (ii), it remains to show that $\operatorname{coker}(\mathrm{Sign}_{K,S}) = \operatorname{coker}(\psi) \cong \operatorname{coker}(\phi)$ is trivial. Using the same spectral sequence as before, in order to see that $\operatorname{coker}(\phi) = 0$, it suffices to show that the spectral terms

- $E_2^{02} = H^0(G, H^2(G(K_\Sigma(2)|K'), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}))$ and
- $E_2^{11} = H^1(G, \mathrm{Cl}_S(K')(2))$

are trivial. The first assertion is easy, because $K'$ is totally imaginary and contains $k_\infty(2)$ and so $H^2(G(K_\Sigma(2)|K'), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) = 0$. That the second spectral term is trivial follows from (i). This completes the proof of Theorem 4. □

Finally, we prove Theorem 5. The statement on $\mathrm{cd}_2 G_S$ and $\mathrm{vcd}_2 G_S$ follows by choosing a 2-Sylow subgroup $H \subset G_S$ and applying Theorem 1 to all finite subextensions of $k$ in $(k_S)^H$. It remains to show the statement on the inflation map. It is equivalent to the statement that

$$\inf \otimes \mathbb{Z}_{(2)} : H^i(G_S(2), A) \otimes \mathbb{Z}_{(2)} \longrightarrow H^i(G_S, A) \otimes \mathbb{Z}_{(2)}$$

is an isomorphism for every discrete $G_S(2)$-module $A$ and all $i \geqslant 0$, where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (2).

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Since cohomology commutes with inductive limits, we may assume that $A$ is finitely generated (as a $\mathbb{Z}$-module). Using the exact sequences

$$0 \longrightarrow \mathrm{tor}(A) \longrightarrow A \longrightarrow A/\mathrm{tor}(A) \longrightarrow 0,$$

$$0 \longrightarrow A/\mathrm{tor}(A) \longrightarrow \big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q} \longrightarrow \big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and using the limit argument for $\big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q}/\mathbb{Z}$ again, we are reduced to the case that $A$ is finite. Every finite $G_S(2)$-module is the direct sum of its 2-part and its prime-to-2-part. The statement is obvious for the prime-to-2-part and every finite 2-primary $G_S(2)$-module has a composition series whose quotients are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore we are reduced to showing the statement on the inflation map for $A = \mathbb{Z}/2\mathbb{Z}$. But it is more convenient to work with $A = \mathbb{Q}_2/\mathbb{Z}_2$ (with trivial action) which is possible by the exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \longrightarrow 0.$$

Using the Hochschild–Serre spectral sequence for the extensions $k_S|k_S(2)|k$, we thus have to show that

$$H^i(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0$$

for $i \geqslant 1$. The case $i = 1$ is obvious by the definition of the field $k_S(2)$. By Theorem 1, every real prime in $S$ becomes complex in $k_S(2)$ and therefore $\mathrm{cd}_2 G(k_S|k_S(2)) \leqslant 2$. It remains to show that $H^2(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0$. Therefore the next proposition implies the remaining statement of Theorem 5.

PROPOSITION 4.3. *Let $k$ be a number field, $S \supseteq S_2$ a set of primes in $k$ and $K \supseteq k_\infty(2)$ an extension of $K$ in $k_S$. Then*

$$H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

*Proof.* Let $H$ be a 2-Sylow subgroup in $G(k_S|K)$ and $L = (k_S)^H$. Then the restriction map

$$H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow H^2(G(k_S|L), \mathbb{Q}_2/\mathbb{Z}_2)$$

is injective and so, replacing $K$ by $L$, we may suppose that $k_S = K_S(2)$. Applying Theorem 2$'$ to the 2-$S$-closed field $K_S(2)$, we obtain an isomorphism

$$G(K_{S\cup S_{\mathbb{R}}}(2)|K_S(2)) \cong \mathop{\ast}_{\mathfrak{p} \in S_{\mathbb{R}}(K_S(2))} T_{\mathfrak{p}}(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}).$$

Hence we have complete control over the Hochschild–Serre spectral sequence associated to $K_{S\cup S_{\mathbb{R}}}(2)|K_S(2)|K$. Furthermore, the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_2$-extension and $K \supseteq k_\infty(2)$, which implies that $H^2(G(K_{S\cup S_{\mathbb{R}}}(2)|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0$. The exact sequence (2) of Section 4 applied to all finite subextensions $k'|k$ of $K|k$ yields a surjection

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$$\bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}\setminus S(K)}{}' H^1(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow\!\!\!\!\!\rightarrow H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2).$$

and therefore, in order to prove the proposition, it suffices to show that the group $H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2)$ is 2-divisible. This is trivial if $S \cap S_{\mathbb{R}}(K) = \emptyset$ because then $\operatorname{cd} G(K_S(2)|K) \leqslant 2$. Otherwise, this follows from the commutative diagram

$$\begin{array}{ccc}
H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \hookrightarrow & H^3(G(K_S(2)|K), \mathbb{Z}/2\mathbb{Z}) \\
\downarrow & & \downarrow\wr \\
\bigoplus_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(K)}{}' H^2(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \hookrightarrow & \bigoplus_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(K)}{}' H^3(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Z}/2\mathbb{Z}).
\end{array}$$

The right-hand vertical arrow is an isomorphism by Proposition 4.1. But $H^2(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ for all $\mathfrak{p} \in S \cap S_{\mathbb{R}}(K)$ and therefore the object in the lower left corner is zero. □

## 5. Relation to Étale Cohomology

Let $k$ be a number field and $S$ a finite set of places of $k$. We think of $\mathrm{Spec}(\mathcal{O}_{k,S})$ as '{scheme-theoretic points of $\mathrm{Spec}(\mathcal{O}_{k,S})$} $\cup$ {real places of $k$ not in $S$}'. Essentially following Zink [17], we introduce the site $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$.

*Objects* of the category are pairs $\bar{U} = (U, U_{\mathrm{real}})$, where $U$ is a scheme together with an étale structural morphism $\phi_U$: $U \to \mathrm{Spec}(\mathcal{O}_{k,S})$ and $U_{\mathrm{real}}$ is a subset of the set of real valued points $U(\mathbb{R}) = \mathrm{Mor}_{\mathrm{Schemes}}(\mathrm{Spec}(\mathbb{R}), U)$ of $U$ such that $\phi_U(U_{\mathrm{real}}) \subset S_{\mathbb{R}}(k) \setminus S$.

*Morphisms* are scheme morphisms $f: U \to V$ over $\mathrm{Spec}(\mathcal{O}_{k,S})$ satisfying $f(U_{\mathrm{real}}) \subset V_{\mathrm{real}}$.

*Coverings* are families $\{\pi_i\colon \bar{U}_i \to \bar{U}\}_{i\in I}$ such that $\{\pi_i\colon U_i \to U\}_{i\in I}$ is an étale covering in the usual sense and $\bigcup_{i\in I} \pi_i(U_{i\mathrm{real}}) = U_{\mathrm{real}}$.

There exists an obvious morphism of sites

$$\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et}} \longrightarrow \mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$$

and both sites coincide if $S$ contains all real places of $k$. The pair $\bar{X} = (X, X_{\mathrm{real}})$ with $X = \mathrm{Spec}(\mathcal{O}_{k,S})$ and $X_{\mathrm{real}} = S_{\mathbb{R}}(k) \setminus S$ is the terminal object of the category and the profinite group $G_S(k)$ is nothing else but the fundamental group of $\bar{X}$ with respect to this site. Let $\eta$ denote the generic point of $X$. For a sheaf $A$ of abelian groups on $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$ and for any point $v$ of $\bar{X}$ we have a specialization homomorphisms $s_v$: $A_v \to A_\eta$ from the stalk $A_v$ of $A$ in $v$ to that in $\eta$. For each point $v \in X_{\mathrm{real}}$ we consider the local cohomology $H^i_v(\bar{X}, A)$ with support in $v$. There is a long exact localization sequence (see [17])

$$\cdots \to \bigoplus_{v\in X_{\mathrm{real}}} H^i_v(\bar{X}, A) \to H^i_{\mathrm{et,mod}}(\bar{X}, A) \to H^i_{\mathrm{et}}(X, A) \to \cdots$$

and the local cohomology with support in real points is calculated as follows:

LEMMA 5.1. *For $v \in X_{\mathrm{real}}$ the following holds.*

$$H_v^0(\bar{X}, A) = \ker(s_v\colon\ A_v \to A_\eta),$$
$$H_v^1(\bar{X}, A) = \operatorname{coker}(s_v\colon\ A_v \to A_\eta),$$
$$H_v^i(\bar{X}, A) = H^{i-1}(k_v, A_v), \quad \text{for } i \geqslant 2.$$

*Here the right-hand side of the last isomorphism is the Galois cohomology of the field $k_v$.*

*Proof*. See [17], Lemma 2.3. □

*Remark*. Suppose that $S$ contains all primes dividing 2 and no real primes. Let $A$ be a locally constant constructible sheaf on $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et}}$ which is annihilated by a power of 2. We denote the push-forward of $A$ to $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$ by the same letter. By Poitou–Tate duality, the boundary map of the long exact localization sequence

$$H^i_{\mathrm{et}}(X, A) \longrightarrow \bigoplus_{v \in X_{\mathrm{real}}} H_v^{i+1}(\bar{X}, A) = \bigoplus_{v \text{ arch.}} H^i(k_v, A_v)$$

is an isomorphism for $i \geqslant 3$ and surjective for $i = 2$. Therefore, we obtain the vanishing of $H^i_{\mathrm{et,mod}}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$ for $i \geqslant 3$. In this situation the modified étale cohomology is connected to the 'positive étale cohomology' $H_2^*(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+)$ defined in [3] in the following way. There exists a natural exact sequence

$$\begin{aligned} 0 \to H^0_{\mathrm{et,mod}}&(\mathrm{Spec}(\mathcal{O}_{k,S}), A) \to \\ &\to \bigoplus_{v\,\mathrm{arch.}} H^0(k_v, A_v) \to H_2^0(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+) \to \\ &\to H^1_{\mathrm{et,mod}}(\mathrm{Spec}(\mathcal{O}_{k,S}), A) \to 0. \end{aligned}$$

and isomorphisms

$$H_2^i(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+) \xrightarrow{\sim} H^{i+1}_{\mathrm{et,mod}}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$$

for $i \geqslant 1$. This can be easily deduced from the long exact localization sequence, Lemma 5.1 and the long exact sequence (2.4) of [3].

Now let $A$ be a discrete $G_S(k)$-module. The module $A$ induces locally constant sheaves on $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$ and $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et}}$, which we will denote by the same letter. According to Lemma 5.1, we obtain for every $v \in X_{\mathrm{real}}$

$$H_v^i(\bar{X}, A) = 0 \quad \text{for } i = 0, 1.$$

Let $\widetilde{X} = (\mathrm{Spec}(\mathcal{O}_{k_S,S}), S_{\mathbb{R}}(k_S) \smallsetminus S(k_S))$ be the universal covering of $\bar{X}$. The Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(G_S(k), H^j_{\mathrm{et,mod}}(\widetilde{X}, A)) \Longrightarrow H^{i+j}_{\mathrm{et,mod}}(\bar{X}, A)$$

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induces natural comparison homomorphisms

$$H^i(G_S(k), A) \longrightarrow H^i_{\mathrm{et,mod}}(\bar{X}, A)$$

for all $i \geqslant 0$. It follows immediately from the spectral sequence that these homomorphisms are isomorphisms if $H^j_{\mathrm{et,mod}}(\widetilde{X}, A) = 0$ for all $j \geqslant 1$.

Next we are going to prove Theorem 6 of the introduction. Assume that $S$ contains all primes dividing 2 and that $A$ is 2-torsion. Both sides of the comparison homomorphism commute with direct limits, and so, in order to prove Theorem 6, we may suppose that $A$ is finite. Since $A$ is constant on $\widetilde{X}$, we can easily reduce to the case $A = \mathbb{Z}/2\mathbb{Z}$, in order to show $H^j_{\mathrm{et,mod}}(\widetilde{X}, A) = 0$ for $j \geqslant 1$. Furthermore, the assertion is trivial for $j = 1$. The theorem is well-known if $S$ contains all real primes (see [17], prop. 3.3.1 or [7], II, 2.9) and so, passing to the limit over all finite subextensions of $k$ in $k_S$, we obtain natural isomorphisms for all $j \geqslant 0$.

$$H^j(G_{S\cup S_{\mathbb{R}}}(k_S), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^j_{\mathrm{et}}(\widetilde{X} \smallsetminus S_{\mathbb{R}}(k_S), \mathbb{Z}/2\mathbb{Z}).$$

On the other hand, Theorem 2 for $T = S \cup S_{\mathbb{R}}$, $S = S$ applied to all finite subextensions of $k$ in $k_S$ in conjunction with Theorem 5 induces isomorphisms for all $j \geqslant 1$.

$$H^j(G_{S\cup S_{\mathbb{R}}}(k_S), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \bigoplus_{v\in S_{\mathbb{R}}\smallsetminus S(k_S)}{}' H^j(k_v, \mathbb{Z}/2\mathbb{Z}).$$

These two isomorphisms together with the long exact localization sequence show that $H^j_{\mathrm{et,mod}}(\widetilde{X}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $j \geqslant 1$. This completes the proof of Theorem 6. $\square$

Theorem 6 is best understood in the context of étale homotopy, namely as a vanishing statement on the 2-parts of higher homotopy groups. For a scheme $X$ we denote by $X_{\mathrm{et}}$ its étale homotopy type, i.e. a pro-simplicial set. The étale homotopy groups of $X$ are by definition the homotopy groups of $X_{\mathrm{et}}$ and, as is well known, these pro-groups are pro-finite, whenever the scheme $X$ is noetherian, connected and geometrically unibranch ([1] Theorem 11.1). If we consider the modified étale site $\mathrm{Spec}(\mathcal{O}_{k,S})_{\mathrm{et,mod}}$ as above, we obtain in exactly the same manner as for the usual étale site a pro-finite simplicial set $\bar{X}_{\mathrm{et,mod}}$. We denote the universal covering of $\bar{X}_{\mathrm{et,mod}}$ by $\widetilde{X}_{\mathrm{et,mod}}$. If $p$ is a prime number and $Y_.$ is a pro-simplicial set, we denote the pro-$p$ completion of $Y_.$ by $Y_.^{\wedge p}$. Furthermore, we write $G(p)$ for the maximal pro-$p$ factor group of a pro-group $G$.

LEMMA 5.2. *Assume that $Y_.$ is a simply connected* (*i.e.* $\pi_1(Y_.) = 0$) *pro-simplicial set such that $\pi_i(Y_.)$ is pro-finite for all $i \geqslant 2$. Then we have isomorphisms for all $i$: $\pi_i(Y_.)(p) \longrightarrow \pi_i(Y_.^{\wedge p})$.*

*Proof.* See [13], prop. 13. $\square$

For a pro-group $G$ we denote by $K(G, 1)$ the Eilenberg–MacLane pro-simplicial set associated with $G$ (cf. [1], (2.6)). If $S$ contains all real primes of $k$ the following theorem was proved in [13], prop. 14.

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THEOREM 5.3. *Let $k$ be a number field and $S$ a finite set of primes of $k$ containing all primes dividing* 2. *Let $\bar{X}$ be the pair $(X, X_{\mathrm{real}})$ with $X = \mathrm{Spec}(\mathcal{O}_{k,S})$ and $X_{\mathrm{real}} = S_{\mathbb{R}}(k) \smallsetminus S$ endowed with the modified étale topology. Then the higher homotopy groups of $\bar{X}_{\mathrm{et,mod}}$ have no* 2*-part, i.e. $\pi_i(\bar{X}_{\mathrm{et,mod}})(2) = 0$ for $i \geqslant 2$. Furthermore, the canonical morphism $(\bar{X}_{\mathrm{et,mod}})^{\wedge 2} \longrightarrow K(G_S(k)(2), 1)$ is a weak homotopy equivalence.*

*Proof.* Since $G_S(k)$ is the fundamental group of $\bar{X}_{\mathrm{et,mod}}$, Theorem 6 implies that the universal covering $\widetilde{X}_{\mathrm{et,mod}}$ of $\bar{X}_{\mathrm{et,mod}}$ has no cohomology with values in 2-primary coefficient groups. By the Hurewicz theorem ([1], (4.5)), the pro-2 completion of $\widetilde{X}_{\mathrm{et,mod}}$ is weakly contractible. Therefore, Lemma 5.2 implies

$$\pi_i(\bar{X}_{\mathrm{et,mod}})(2) \cong \pi_i(\widetilde{X}_{\mathrm{et,mod}})(2) \cong \pi_i((\widetilde{X}_{\mathrm{et,mod}})^{\wedge 2}) = 0$$

for $i \geqslant 2$, which shows the first statement of the theorem. By Theorem 5, for every finite 2-primary $G_S(k)(2)$-torsion module $A$ the inflation homomorphism $H^i(G_S(k)(2), A) \longrightarrow H^i(G_S(k), A)$ is an isomorphism for all $i$. The same arguments as above show that the universal covering of $(\bar{X}_{\mathrm{et,mod}})^{\wedge 2}$ is weakly contractible. This proves the second statement. □

## 6. Closing Remarks

### 6.1. DUALIZING MODULES

Unfortunately, we do not have (despite semi-tautological reformulations of the definition) a good description of the $p$-dualizing module $I$ of the group $G_S$, where $S$ is a finite set of finite primes containing $S_p$. If $k$ is totally imaginary, then $I$ is determined by the exact sequence

$$0 \longrightarrow \mu_{p^\infty} \xrightarrow{diag} \bigoplus_{\mathfrak{p} \in S(k_S)}{}' \mu_{p^\infty} \longrightarrow I \longrightarrow 0$$

(see [9], (10.2.1)) and the group $G_S$ is a duality group at $p$ of dimension 2 (see [13], th. 4 or [9], (10.9.1)). The general case remains unsolved (also for odd $p$).

### 6.2. FREE PROFINITE PRODUCT DECOMPOSITIONS

In this paper we used free pro-$p$-product decompositions of Galois groups of pro-$p$-extensions of global fields into Galois groups of local pro-$p$-extensions in an essential way. One might ask whether, for sets of places $T \supset S$, the natural homomorphism

$$\phi : \mathop{\ast}_{\mathfrak{p} \in T \smallsetminus S(k_S)} T(\bar{k}_{\mathfrak{p}}|k) \longrightarrow G(k_T|k_S)$$

is an isomorphism, where the free product on the left hand side is the free product of *profinite* groups. More precisely, one has to ask, whether there exists a continuous section to the natural projection $T \smallsetminus S(k_T) \to T \smallsetminus S(k_S)$ such that the above map is an isomorphism (cf. the discussion in section 2). We do not know the answers to this question in general. It is 'yes' if $S$ contains all but finitely many primes of $k$ (see below). But it seems likely that $\phi$ is never an isomorphism if $T$ and $S$ are finite. The present level of knowledge on this question is rather low. For example, we do not know whether there are infinitely many prime numbers $p$ such that $p^\infty$ divides the order of $G_T$. The best result known in this direction is that if $T$ contains all real places and all primes dividing one prime number $p$, then there exist infinitely many prime numbers $\ell$ dividing the order of $G_T$ (see [14], cor. 3 or [9], (10.9.4)).

In the case that $S$ contains all but finitely many primes of $k$, we can deduce the above statement by applying the following slightly more general result to the complement of $S$:

For a finite set $S$ of primes of $k$, let $k^S$ be the maximal extension of $k$ in which all primes in $S$ are totally decomposed. Then there exists a continuous section to the natural projection $S(\bar{k}) \to S(k^S)$ such that the natural map $*_{\mathfrak{p}\in S(k^S)} G(\bar{k}_\mathfrak{p}|k) \longrightarrow G(\bar{k}|k^S)$ is an isomorphism. This had been proved first in the special case $S = S_\mathbb{R}$ by Fried, Haran and Völklein [4] and then by Pop [11] for arbitrary finite $S$.

## 6.3. LEOPOLDT'S CONJECTURE

The Leopoldt conjecture for $k$ and a prime number $p$ holds if and only if the group $H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p)$ is trivial for one (all) finite set(s) of primes $S \supseteq S_p$. The weak Leopoldt conjecture is true for $k$, $p$ and a $\mathbb{Z}_p$-extension $k_\infty|k$ if and only if $H^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ is trivial for one (all) finite set(s) of primes $S \supseteq S_p$ (of $k$). This is well known for odd $p$ and for $p = 2$ it can be easily deduced from the above results.

## 6.4. IWASAWA THEORY

Let $k$ be a number field, $S \supseteq S_2$ a finite set of primes of $k$ and $k_\infty|k$ the cyclotomic $\mathbb{Z}_2$-extension of $k$. Let $\Gamma = G(k_\infty|k) \cong \mathbb{Z}_2$ and let $\Lambda = \mathbb{Z}_2[\![\Gamma]\!] \cong \mathbb{Z}_2[[T]]$ be the Iwasawa algebra. We consider the compact $\Lambda$-module $X_S = G(k_S(2)|k_\infty)^{ab}$. Then the following holds

(i) $X_S$ is a finitely generated $\Lambda$-module.
(ii) $\mathrm{rank}_\Lambda X_S = r_2$ (the number of complex places of $k$).
(iii) $X_S$ does not contain any nontrivial finite $\Lambda$-submodule.
(iv) the $\mu$-invariant of $X_S$ is greater than or equal to $\# S \cap S_\mathbb{R}(k)$.

Properties (i)–(iii) follow in a purely formal way (see [9], (5.6.15)) from the facts that: (a) $\chi_2(G_S(2)) = -r_2$, (b) $H^2(G_S(k_\infty)(2), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ and (c) $H^2(G_S(2), \mathbb{Q}_2/\mathbb{Z}_2)$ is 2-divisible. Assertion (iv) is trivial if $S$ contains no real places and in the general case it follows from the exact sequence

$$0 \to (\Lambda/2)^{\# S\cap S_\mathbb{R}(k)} \longrightarrow X_S \longrightarrow X_{S\smallsetminus S_\mathbb{R}} \longrightarrow 0.$$

Now let $k^+$ be a totally real number field, $k = k^+(\mu_4)$, $k^+_\infty$ the cyclotomic $\mathbb{Z}_2$-extension of $k^+$ and $k_\infty = k^+_\infty(\mu_4) = k(\mu_{2^\infty})$. Let $k_n$ be the unique subextension of degree $2^n$ in $k_\infty$ and let $J$ be the complex conjugation. We set $A_n = \mathrm{Cl}(k_n)(2)$ and

$$A_n^- := \{a \in A_n \mid aJ(a) = 1\}.$$

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Furthermore, let $A_\infty^- = \varinjlim A_n^-$, $X^+ = X_{S_2}(k^+)$, let $\vee$ denote the Pontryagin dual and $(-1)$ the Tate-twist by $-1$. Then there exists a natural homomorphism

$$\phi : (A_\infty^-)^\vee \longrightarrow X^+(-1)$$

whose kernel and cokernel are annihilated by 2. If the Iwasawa $\mu$-invariant of $k$ is zero (this is known if $k|\mathbb{Q}$ is abelian), then $\phi$ is a pseudo-isomorphism, i.e. $\phi$ has finite kernel and cokernel. This can be seen by a slight modification of the arguments given in [5], §2:

Let $M^+$ be the maximal Abelian 2-extension of $k_\infty^+$ which is unramified outside $S_2$, in particular, $M^+$ is totally real. Kummer theory shows that, for an $\alpha \in k_\infty^\times$, the field $k_\infty(\sqrt[2^n]{\alpha})$ is contained in $M^+k_\infty$ if and only if: (a) $\alpha \in k_{\infty,\mathfrak{p}}^{\times 2^n}$ for all $\mathfrak{p} \notin S(k_\infty)$ and (b) $\alpha J(\alpha) = \beta^{2^n}$ for a totally positive element $\beta \in k_\infty^+$. Let $R_n$ be the subgroup in $k_\infty^\times / k_\infty^{\times 2^n}$ generated by elements satisfying (a) and (b) and let

$$\mathfrak{M}^- := \varinjlim_n R_n \subset k_\infty^\times \otimes \mathbb{Q}_2/\mathbb{Z}_2.$$

Then we have a perfect Kummer pairing $X^+ \times \mathfrak{M}^- \to \mu_{2^\infty}$. Since all primes dividing 2 are infinitely ramified in $k_\infty|k$, for $\alpha \otimes 2^{-n} \in \mathfrak{M}^-$ there exists a unique ideal $\mathfrak{a}$ in $k_\infty$ with $\mathfrak{a}^{2^n} = (\alpha)$ and the class $[\mathfrak{a}]$ is contained in $A_\infty^-$. This yields a homomorphism $\phi^\vee : \mathfrak{M}^- \longrightarrow A_\infty^-$. A straightforward computation shows that $\operatorname{im}(\phi^\vee) \supseteq (A_\infty^-)^2$ and that $\ker(\phi^\vee)$ is the image of $\mathcal{O}_{k_\infty^+,\emptyset}^\times / \mathcal{O}_{k_\infty^+,S_\mathbb{R}}^{\times 2}$ in $\mathfrak{M}^-$ (notational conventions as in §4). Thus, if the Iwasawa $\mu$-invariant of $k$ is zero, then the cokernel of $\phi^\vee$ is finite and it remains to show the same for its kernel. Since $\mu = 0$, the $\mathbb{F}_2$-ranks of ${}_2\mathrm{Cl}^0(k_n^+)$ (the subgroup of elements annihilated by 2 in the ideal class groups in the narrow sense) are bounded independently of $n$. Thus, also the $\mathbb{F}_2$-ranks of the kernels of the signature maps

$$\mathcal{O}_{k_n^+,S_\mathbb{R}}^\times / \mathcal{O}_{k_n^+,S_\mathbb{R}}^{\times 2} \longrightarrow \bigoplus_{v \in S_\mathbb{R}(k_n^+)} \mathbb{R}^\times / \mathbb{R}^{\times 2}$$

are bounded independently of $n$. But the direct limit over $n$ of these kernels is just the group in question. Finally, we obtain the result by taking Pontryagin duals.

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