Large deviations and fairness for a betting game with a constant ratio of capital

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1. *Introduction*

This Article is a follow-up to a recent *Gazette* Article about a probabilistic betting game studied by Abdin et al. [1]. We examine the speed of convergence of the probability needed to investigate this game by giving concrete examples, using the *large deviation*, which is a valuable tool for estimating probabilities of repeated trials (see [2], [3, Chapter 6], [4, Section 5.11]). Moreover, to get a deep understanding of the game, we study fairness when it is repeated infinite times. Let us call it *fairness in the sense of infinity*, whose exact definition will be given in the final section.

We know that casino games are unfair to bettors by considering expectations. In contrast, there are some games that are difficult to interpret as fair or not. For example, we have the *St Petersburg game*, whose fairness has been discussed by many authors (see [5]). For this game, Feller [6, Sections X.3 and X.4] provided an important study of the fairness using the weak law of large numbers, and Stoica [7] efficiently obtained large deviation estimates to investigate some properties of the game. In addition, the *Feller games* and the *super-Petersburg games*, which are games derived from the St Petersburg game, were investigated by [8, 9] and [10, 11], respectively.

Abdin et al. [1] also investigated another interesting probabilistic betting game with a constant ratio of capital. Let us explain the game by using some notation.

Game 1 (b-betting games): Letting $\mathbb{N} = \{1, 2, \dots\}$, we consider a bettor who repeats a bet at each time $n \in \mathbb{N}$. Let ξ_n be the net gain of the nth bet, namely $(\xi_n)_{n \in \mathbb{N}}$ are $\mathbb{R}$-valued independent identically distributed (i.i.d.) random variables with

$$\#\{a \in \mathbb{R} : \xi_1 = a\} < \infty \text{ and } P(\xi_1 < 0) > 0 \text{ and } P(\xi_1 > 0) > 0, \tag{1}$$

where $\mathbb{R} = (-\infty, \infty)$. In other words, the number of outcomes for the bet is finite, and the winning probability and the losing probability are both positive. Let M_n be the bettor's capital at $n \in \mathbb{N} \cup \{0\}$, in particular, $M_0 > 0$ is the initial capital, which is a constant. Given a constant ratio $b \in (0, 1)$, she bets bM_{n-1} at the nth bet for each $n \in \mathbb{N}$. Then since she gets $(bM_{n-1})\xi_n$, it follows that

$$M_n = M_{n-1} + (bM_{n-1})\xi_n = (1 + b\xi_n)M_{n-1} = \dots = \prod_{i=1}^{n}(1 + b\xi_n)M_0. \tag{2}$$

Note that we assume

$$1 + b\xi_1 > 0 \tag{3}$$

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to satisfy $M_n > 0$. In other words, she can continue to bet without getting into debt. For simplicity, we call this game the *betting game* or the *b-betting game* if we need to emphasise the dependence on b.

We say that the bet with (1) or the betting game is *fair*, *favourable* and *unfavourable* if $\mathrm{E}(\xi_1) = 0$, $\mathrm{E}(\xi_1) > 0$ and $\mathrm{E}(\xi_1) < 0$, respectively. For these betting games, we would like to know the relation between the fairness of the bets and $\mathrm{P}(M_n > M_0)$ for large n which is the probability that she will be in a winning position after a large number of bets. Abdin et al. [1] define

$$\beta = \beta(b) = \mathrm{E}(\log(1 + b\xi_1)), \tag{4}$$

which is called the *betting index*, and showed the following.

Theorem 1 ([1]): For arbitrary betting games, the limit of the probability of being in a winning position is

$$\lim_{n \to \infty} \mathrm{P}(M_n > M_0) = \begin{cases} 0, & \text{if } \beta < 0, \\ \frac{1}{2}, & \text{if } \beta = 0, \\ 1, & \text{if } \beta > 0. \end{cases} \tag{5}$$

In particular, if the betting game is fair, namely $\mathrm{E}(\xi_1) = 0$, then

$$\mathrm{E}(M_n) = M_0 \qquad \text{for } n \in \mathbb{N}, \tag{6}$$

but

$$\lim_{n \to \infty} \mathrm{P}(M_n > M_0) = 0. \tag{7}$$

In fact, the fairness criterion yields $\mathrm{E}(1 + b\xi_1) = 1$. Therefore (6) follows from (2). Since $\log(1 + x) < x$ for $-1 < x \neq 0$,

$$\beta = \mathrm{E}(\log(1 + \xi_1)) < \mathrm{E}(b\xi_1) = b\,\mathrm{E}(\xi_1) = 0.$$

Hence (5) implies (7).

Remark 1:

(i) If the betting game is fair then the bettor cannot be in any winning position in the long run, no matter how she adjusts $b \in (0, 1)$.

(ii) From the proof of Theorem 1 of [1], not only (7) but also $\lim_{n \to \infty} \mathrm{P}(M_n \geqslant M_0) = 0$ is true. However, this equation seems to be paradoxical when compared with (6). We study it in the last section.

Note that even a favourable game may have (7) if the bettor does not choose b carefully. We confirm it using the following simple setting.

Game 2 (b-binary games): Let us consider a b-betting game with

$$\mathrm{P}(\xi = 1) = p \qquad \text{and} \qquad \mathrm{P}(\xi = -1) = 1 - p \qquad \text{for } p \in (0, 1), \tag{8}$$

which is investigated in [1, Section 5]. For simplicity, we call it the *binary* *game* or the *b-binary game*. In particular, the original betting game discussed in [12, 13] is the $\frac{1}{2}$-binary one. In this case, we have $\beta = p \log 3 - \log 2$. Therefore (5) implies that if $\frac{1}{2} < p < \frac{\log 2}{\log 3}$ then (7) holds.

The organisation of this Article is as follows. In Section 2, we estimate the probability $\mathrm{P}(M_n > M_0)$ using large deviations. In Section 3, we elaborate on the results of Section 2 for b-binary games, and study them with varying $b \in (0, 1)$. In Section 4, numerical examples are given. Finally, in Section 5, we discuss the fairness in the sense of infinity to resolve the paradox of Remark 1 (ii).

2. *Large deviations with the rate index*

In order to extend Theorem 1, we make some preparations. For $b \in (0, 1)$ and $(\xi_n)_{n \in \mathbb{N}}$ let us put

$$X_i = \log(1 + b\xi_i) \qquad \text{for } i \in \mathbb{N}, \tag{9}$$

which are well-defined by (3). Then $(X_i)_{i \in \mathbb{N}}$ are i.i.d. with

$$\mathrm{P}(X_1 < 0) > 0 \qquad \text{and} \qquad \mathrm{P}(X_1 > 0) > 0 \tag{10}$$

because

$$\mathrm{P}(X_1 < 0) = \mathrm{P}(\log(1 + b\xi_1) < 0) = \mathrm{P}(b\xi_1 < 0) = \mathrm{P}(\xi_1 < 0) > 0,$$

from (1), and similarly $\mathrm{P}(X_1 < 0) > 0$. The betting index is simply expressed by $\beta = \mathrm{E}(X_1)$. The moment generating function of X_1 is defined as $\varphi(t) = \varphi_{X_1}(t) = \mathrm{E}(e^{tX_1})$ for $t \in \mathbb{R}$. Let us put

$$\rho = \inf_{t \in \mathbb{R}} \varphi(t), \tag{11}$$

and say that ρ is a *rate index*. We see that $\varphi(t) > 0$ for $t \in \mathbb{R}$, and

$$t \to \varphi(t) \text{ is smooth and strictly convex.} \tag{12}$$

In fact, the smoothness follows from the fact that X_1 takes only a finite number of values. Therefore, interchanging differentiation inside the expectation yields $\varphi''(t) = \mathrm{E}(X_1^2 e^{tX_1})$ (see for example [4, 5.1 (12), p. 170]). Hence (10) implies $\varphi''(t) > 0$ for $t \in \mathbb{R}$, which gives (12). The rate index ρ is calculated by the following.

(I) If $\beta < 0$ then there exists a unique $\tau > 0$ satisfying $\rho = \phi(t) \in (0, 1)$.

(II) If $\beta > 0$ then there exists a unique $\tau < 0$ satisfying $\rho = \phi(t) \in (0, 1)$.

Because X_1 can take both positive and negative values with non-zero probabilities as seen in (10), $\varphi(t) \to \infty$ for both $t \to +\infty$ and $t \to -\infty$. Therefore (12) implies that $\varphi(t)$ has a unique minimum at $t = \tau \in \mathbb{R}$. If $\beta < 0$ then $\tau > 0$ because $\varphi(t) > 0$, $\varphi(0) = 1$ and $\varphi'(0) = \beta < 0$. Thus we obtain (I). (II) is also proved in the same fashion.

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Theorem 2 (Large deviations for betting games): For arbitrary betting games, we have for $n \in \mathbb{N}$

$$\mathrm{P}\left(M_n > M_0\right)\begin{cases} \leqslant \rho^n & \text{if } \beta < 0, \\ \to \frac{1}{2} \text{ as } n \to \infty & \text{if } \beta = 0, \\ \geqslant 1 - \rho^n & \text{if } \beta > 0, \end{cases} \tag{13}$$

where $\rho \in (0, 1)$ is calculated by (I) and (II). In particular, for $\varepsilon \in (0, 1)$, putting

$$n_* = \left\lfloor \frac{\log \varepsilon}{\log \rho} \right\rfloor + 1, \tag{14}$$

where $\lfloor x \rfloor$ denotes the integer part of $x > 0$, we have for $n \geqslant n_*$

$$\mathrm{P}\left(M_n > M_0\right)\begin{cases} \leqslant \varepsilon & \text{if } \beta < 0, \\ \geqslant 1 - \varepsilon & \text{if } \beta > 0. \end{cases} \tag{15}$$

Proof: When $\beta = 0$, the proof is similar to Theorem 1. Defining $S_n = \sum_{i=1}^{n} X_i$, we have $\left\{M_n > M_0\right\} = \left\{S_n > 0\right\}$. If $\beta < 0$ then $\tau > 0$ by (I). This assures us that

$$\mathrm{P}\left(M_n > M_0\right) = \mathrm{P}\left(\tau S_n > 0\right) = \mathrm{P}\left(e^{\tau S_n} > 1\right) \leqslant \mathrm{E}\left(e^{\tau S_n}\right) = \left(\varphi(\tau)\right)^n = \rho^n.$$

Similarly, if $\beta > 0$ then $\tau < 0$. Therefore we also obtain

$$\begin{aligned} \mathrm{P}\left(M_n > M_0\right) &= \mathrm{P}\left(\tau S_n < 0\right) = 1 - \mathrm{P}\left(e^{\tau S_n} \geqslant 1\right) \geqslant 1 - \mathrm{E}\left(e^{\tau S_n}\right) \\ &= 1 - \left(\varphi(\tau)\right)^n = 1 - \rho^n. \end{aligned}$$

Hence (13) follows. Finally, combining (13) and (14) gives (15), which completes the proof.

Remark 2: Not only (13) but also the following statements hold.

- If $\beta < 0$ then $\lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}\left(M_n > M_0\right) = \log \rho$.
- If $\beta > 0$ then $\lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}\left(M_n \leqslant M_0\right) = \log \rho$.

Proofs of these results are interesting, as the *exponential change distribution* technique is used (see [4, Theorem 5.11.4, p.226]). However, they are not so useful for obtaining explicit bounds for $P\left(M_n > M_0\right)$. In Section 4, we give them numerically.

In general it is not easy to find the rate index algebraically, but we explicitly present it for binary games in the next section.

3. *Large deviations for the binary game*

Throughout this section, we focus on b-binary games with (8). By definition, the moment generating function of X_1 is written by

$$\varphi(t) = \mathrm{E}\left(e^{tX_1}\right) = pe^{tB} + (1 - p)\, e^{-tA}, \tag{16}$$

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where

$$A = A(b) = -\log(1 - b) \text{ and } B = B(b) = \log(1 + b). \tag{17}$$

Noting that $A > 0$ and $B > 0$, we define

$$a = \frac{A}{A + B} = -\frac{\log(1 - b)}{\log\frac{1+b}{1-b}}. \tag{18}$$

For arbitrary binary games, the rate index can be represented by

$$\rho = \left(\frac{p}{a}\right)^{a}\left(\frac{1 - p}{1 - a}\right)^{1-a}. \tag{19}$$

In fact, solving

$$\varphi'(y) = pBt^{tB} - (1 - p)Ae^{-tA} = 0 \tag{20}$$

and using (I) and (II), we obtain

$$\tau = \frac{\log\left(\frac{1-p}{p}\frac{A}{B}\right)}{A + B}. \tag{21}$$

It follows that

$$e^{\tau B} = \left(\frac{1 - p}{p}\frac{A}{B}\right)^{\frac{B}{A+B}} = \left(\frac{1 - p}{p}\right)^{1-a}\left(\frac{a}{1 - a}\right)^{1-a} \text{ and}$$

$$e^{-\tau A} = \left(\frac{p}{1 - p}\frac{B}{A}\right)^{\frac{A}{A+B}} = \left(\frac{p}{1 - p}\right)^{a}\left(\frac{1 - a}{a}\right)^{a}.$$

Since $\rho = \varphi(\tau)$, substituting then into (16) gives (19).

Remark 3: From (19) we have

$$\log\rho = a\log\frac{p}{a} + (1 - a)\log\frac{1 - p}{1 - a} = -H(a, p),$$

where

$$H(a, p) = a\log\frac{a}{p} + (1 - a)\log\frac{1 - a}{1 - p}, \tag{22}$$

which is called the *Kulback-Leibler distance*. Indeed it is known that $H(a, p) \geqslant 0$ and

$$\rho < 1 \Leftrightarrow H(a, p) > 0 \Leftrightarrow p \neq a$$

(see [2, Equation (1)]).

For b-binary games with (8), the betting index is

$$\beta = pB - (1 - p)A = p\log(1 + b) + (1 - p)\log(1 - b), \tag{23}$$

and it follows that

$$\begin{cases} \beta < 0 \Leftrightarrow p < a, \\ \beta = 0 \Leftrightarrow p = a, \\ \beta > 0 \Leftrightarrow p > a. \end{cases} \tag{24}$$

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Actually (23) follows from $\beta = \mathrm{E}(X_1) = \varphi'(0)$ with (17) and (20). If $\beta = 0$ in (24) then $p = \frac{A}{A+B} = a$ by (23), and vice versa. The other two are also proved in the same manner, hence (24) holds. Combining (15) and (24), we have

$$\mathrm{P}(M_n > M_0) \begin{cases} \leqslant \rho^n & \text{if } p < a, \\ \geqslant 1 - \rho^n & \text{if } p > a, \end{cases} \tag{25}$$

where a is defined by (18). From (25), for an arbitrary $\varepsilon \in (0, 1)$ we have for any $n \geqslant n_*$ defined by (14),

$$\mathrm{P}(M_n > M_0) \begin{cases} \leqslant \varepsilon & \text{if } p < a, \\ \geqslant 1 - \varepsilon & \text{if } p > a. \end{cases} \tag{26}$$

Remark 4:

(i) Equation (25) is an extension of [1, p.37, line 9] because $a\left(\frac{1}{2}\right) = \frac{\log 2}{\log 3}$.

(ii) For arbitrary binary games, it follows that $\mathrm{P}(M_n > M_0) = \mathrm{P}(B(n,p) > na)$, where $B(n, p)$ denotes the binomial random variable with parameters n and p. From this point of view, the estimate (25) is well known (see for example [2, Theorem 1] and [3, p.24, Theorem 6.1 (1)]).

We study the probability of the winning position as a function of b. Let us consider a defined in (18) as a function $b \to a(b)$ for $b \in (0, 1)$. Then

$$a : (0, 1) \to \left(\tfrac{1}{2}, 1\right) \text{ is strictly increasing and bijective.} \tag{27}$$

In fact, since a simple calculation provides $\frac{d}{db}a(b) > 0$, $\lim_{b \to 0+0} a(b) = \frac{1}{2}$ and $\lim_{b \to 1-0} a(b) = 1$, we obtain (27).

Fixing $p \in (0, 1)$ for (19), we regard the rate index ρ calculated by (19) as a function of $b \to \rho(b)$ like $a(b)$.

Theorem 3 (Varying b for b-binary games): For b-binary games, we have the following.

(i) Suppose $0 < p \leqslant \frac{1}{2}$. Then no matter how the bettor chooses $b \in (0, 1)$ it follows that $\mathrm{P}(M_n > M_0) \leqslant \rho^n$ for $n \in \mathbb{N}$. Moreover, $b \to \rho(b)$ is strictly decreasing for $b \in (0, 1)$.

(ii) Suppose $\frac{1}{2} < p < 1$. Then $b \to \beta(b)$ takes a unique maximum

$$\beta(b_{\max}) = \log 2 - h(p) > 0 \tag{28}$$

at $b = b_{\max} = 2p - 1$, and there exists a unique $b_* \in (0, 1)$ satisfying $a(b_*) = p > \frac{1}{2}$ and

$$\mathrm{P}(M_n > M_0) \begin{cases} \leqslant \rho^n & \text{if } b \in (b_*, 1), \\ \geqslant 1 - \rho^n & \text{if } b \in (0, b_*), \end{cases} \tag{29}$$

where $h(p) = -p\log p - (1-p)\log(1-p)$ is the *entropy function*.

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Moreover, it follows that

$$\rho(b) \text{ is strictly } \begin{cases} \text{decreasing} & \text{for } b \in (b_*, 1), \\ \text{increasing} & \text{for } b \in (0, b_*). \end{cases} \tag{30}$$

Proof:

(i) Since $0 < p \leqslant \frac{1}{2}$ and (27), we obtain $p \leqslant \frac{1}{2} < a$. Therefore, applying (25), we have $\mathrm{P}(M_n > M_0) \leqslant \rho^n$. Moreover, it follows that $\dfrac{\partial H(a,p)}{\partial a} = \log \dfrac{a(1-p)}{p(1-a)} > 0$ for $0 < p \leqslant \frac{1}{2}$. Hence $\rho(b) = e^{-H(a(b),p)}$ and (27) yield the desired result.

(ii) From (23) the equation $\beta'(b) = 0$ has a unique solution $b = 2p - 1 \in (0, 1)$ because of $\frac{1}{2} < p < 1$. Checking the increasing and decreasing of $b \to \beta(b)$, we have (28). From (27), there exists a unique $0 < b_* < 1$ with $b(b_*) = p > \frac{1}{2}$. Applying (25) with b_*, we have (29). The proof of (30) is similar to (i).

Remark 5: It follows from (28) that $b_{\max} \in (0, b_*)$.

4. *Examples*

In this section, we examine three examples investigated in [1]. Indeed, we numerically evaluate (13) and (15). Note that n_* defined by (14) depends on ε and b, but for all examples we set $\varepsilon = 0.05$ and write $n_* = n_*(b)$.

Example 1 (American roulette): We consider the $\frac{1}{2}$-binary game, and suppose $p = \frac{18}{38} = \frac{9}{19}$ in (8), which is the win probability for American red-or-black roulette (see [1, Section 4]). Since

$$\mathrm{E}(\xi_1) = -\frac{1}{19} = -0.0526\ldots < 0, \tag{31}$$

the bet is unfavourable. Moreover, we have

$$\beta = -0.1727\ldots < 0 \tag{32}$$

written in [1, p. 36],

$$\begin{cases} a = \frac{\log 2}{\log 3} & = 0.6309\ldots, \\ \tau = \frac{\log\left(\frac{10}{9}\frac{\log 2}{\log(3/2)}\right)}{\log 3} & = 0.5839\ldots, \\ \rho = \frac{10}{19}\frac{\log 3}{\log(3/2)}\left(\frac{9}{10}\frac{\log(3/2)}{\log 2}\right)^{\frac{\log 2}{\log 3}} & = 0.9513\ldots, \end{cases}$$

and $n_* = 61$, namely

$$\mathrm{P}(M_n > M_0) \leqslant 0.05, \qquad \text{if } n \geqslant 61. \tag{33}$$

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Example 2 (American roulette with mixed bets): For b-betting games, we suppose that the probability distribution of ξ_1 is

$$\begin{cases} P(\xi_1 = x_1) = p_1, \\ P(\xi_1 = x_2) = p_2, \\ P(\xi_1 = x_3) = p_3, \end{cases} \quad \text{where} \begin{cases} x_1 = 18 - \frac{1}{2}, & p_1 = \frac{1}{38}, \\ x_2 = \frac{1}{2} - \frac{1}{2}, & p_2 = \frac{18}{38}, \\ x_3 = -1, & p_3 = \frac{19}{38}, \end{cases}$$

whose distribution shows that the bettor mixes her bets by choosing more than one possible outcome (see [1, p. 36]). Since $E(\xi_1) = -\frac{3}{76} = -0.039\ldots < 0$, it is unfavourable, but more advantageous than (31). If $b = \frac{1}{2}$ then the betting index is $\beta = -0.2866\ldots$ from [1, p. 36], which is smaller than (32). For $b \in (0, 1)$ since

$$\varphi(t) = E\left(e^{tX_1}\right) = p_1 e^{tC_1} + p_2 e^{tC_2} + p_3 e^{tC_3},$$

we cannot solve the equation algebraically

$$\varphi'(t) = p_1 C_1 e^{tC_1} + p_2 C_2 e^{tC_2} + p_3 C_3 e^{tC_3} = 0 \text{ for } t \in \mathbb{R},$$

where $C_i = \log\left(1 + bx_i\right)$ for $i = 1, 2, 3$. When $b = \frac{1}{2}$, we obtain numerically $\tau\left(\frac{1}{2}\right) = 0.5908\ldots$, $\rho\left(\frac{1}{2}\right) = 0.4330\ldots$ and $n_*\left(\frac{1}{2}\right) = 4$, namely

$$P\left(M_n > M_0\right) \leqslant 0.05 \text{ if } n \geqslant 4,$$

which should be compared with (33).

Example 3 (Favourable b-binary games): For the b-betting game, we suppose (8) with $p = \frac{3}{5} > \frac{1}{2}$, which is studied in [1, Section 7]. Since $E\left(\xi_1\right) = \frac{1}{5} > 0$, it is favourable. In general, many people think that no favourable bet exists in the real world. However, for example, if you become the dealer rather than the bettor in Example 1 then you can consider the favourable bet.

Since b_* appeared in Theorem 3 (ii) is the unique solution of $a(b) = p = 0.6$, we get numerically $b_* = 0.38939\ldots$. Here, we investigate both the case of $b = 0.5 > b_*$ and the case of $0 < b = b_{\max} = 2p - 1 = 0.2 < b_*$ which is pointed out in Remark 5. Since the numerical calculations of (19) give $\rho(0.5) = 0.9979\ldots$ and $\rho(0.2) = 0.9949\ldots$, it turns out that

$$P\left(M_n > M_0\right) \begin{cases} \leqslant 0.998^n & \text{if } b = 0.5 > b_*, \\ \geqslant 1 - 0.995^n & \text{if } b = 0.2 < b_*. \end{cases}$$

This indicates that

$$\begin{cases} P(M_n > M_0) \leqslant 0.05 & \text{if } n \geqslant n_*(0.5) = 1490. \\ P(M_n > M_0) \geqslant 0.95 & \text{if } n \geqslant n_*(0.2) = 591. \end{cases}$$

In addition, $n_*(0.1) = 261$ follows in a similar manner. Indeed, $b \to n_*(b)$ for $0 < b < b_*$ is increasing because of (14) and (30). It suggests that if the bet is favourable then the ratio of the capital for bets

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should be as small as possible to increase the probability of being in a winning position. In American roulette, the dealer who wants to be in a winning position hopes that the bettor bets with a lower ratio of her capital.

5. *Fairness in the sense of infinity for the binary game*

Throughout this section, we focus on the fair b-binary games for $b \in (0, 1)$ and (8) with $p = \frac{1}{2}$, namely $\mathrm{E}(\xi_1) = 0$. In this setting, we have

$$\lim_{n \to \infty} M_n = 0 \text{ almost surely.} \tag{34}$$

The proof of (34) is as follows. Since M_n is a product of non-negative independent random variables of mean 1 by (2) and (3), the process $(M_n)_{n \in \mathbb{N}}$ is a *martingale* relative to $(\xi_n)_{n \in \mathbb{N}}$ by [14, Section 10.4 (b)]. From Kakutani's theorem [14, Section 14.12], there exists $\lim_{n \to \infty} M_n$ almost surely, and say M_∞. Since

$$0 < \mathrm{E}\left(\sqrt{1 + b\xi_1}\right) = \frac{\sqrt{1 + b} + \sqrt{1 - b}}{2} < 1, \tag{35}$$

we have $\sum_{n=1}^{\infty}\left(1 - \mathrm{E}\left(\sqrt{1 + b\xi_n}\right)\right) = \infty$. Therefore Kakutani's theorem also yields $\mathrm{P}\left(M_\infty = 0\right) = 1$, which means (34).

Equation (34) tells us that we are not permitted in general to reverse the order of taking a limit and an expectation as follows.

$$0 = \mathrm{E}\left(M_\infty\right) \neq \lim_{m \to \infty} \mathrm{E}\left(M_n\right) = M_0 > 0, \tag{36}$$

which is similar to the calculation of the extinction probability for the *branching process* (see [14, Section 0.7 (a), p.8] and [4, Theorem 5.4.5, p.194]). The reason for (36) is that $(M_n)_{n \in \mathbb{N}}$ does not satisfy *uniform integrability* (see [14, Chapter 13]). Equation (36) causes (7), which is an explanation of Remark 1 (ii). A betting game is said to be *unfair in the sense of infinity* if (36) holds. Namely, we have the following claim.

Theorem 4: The fair b-binary games are unfair in the sense of infinity.

On the other hand, a betting game is *fair in the sense of infinity* if it satisfies

$$\mathrm{E}\left(M_\infty\right) = \lim_{n \to \infty} \mathrm{E}\left(M_n\right) = M_0. \tag{37}$$

To establish this, let us use b_n by adding the time parameter n to b.

Theorem 5: If $\sum_{n=1}^{\infty} b_n^2 < \infty$ then the fair b_n-binary games are fair in the sense of infinity.

Proof: It follows from (35) that $\mathrm{E}\left(\sqrt{1 + b_n\xi_n}\right) \sim \left(1 - \frac{1}{8}b_n^2\right)$, where $x_n \sim y_n$

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stands for $\lim_{n \to \infty} \frac{x_n}{y_n} = 1$. By assumption, we have

$$\sum_{n=1}^{\infty} \left(1 - \left(1 - \frac{b_n^2}{8}\right)\right) < \infty.$$

Consequently, $\sum_{n=1}^{\infty} \left(1 - \mathrm{E}\left(\sqrt{1 + b_n \xi_n}\right)\right) < \infty$ by the comparison test. Thus applying [14, Theorem 14.12 (v)] to $(1 + b_n \xi_n)_{n \in \mathbb{N}}$, we obtain (37), which completes the proof.

For example, if $b_n = \frac{1}{n}$ then (37) follows. Theorem 5 tells us that if we will enjoy fair betting games in the sense of infinity, we must adequately reduce the ratio of the capital for the bets.

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4. From every throat in the room there proceeded a shout, a shriek, or some other variety of cry, as the test-tube, slipping from between the victim's fingers, described a parabola through the air.

5. With cool head and tranquil judgement, imperturbably unconscious of the flight, they oscillated from asymptote to asymptote

6. STEPHEN: Here’s another for you. (*He frowns*) The reason is because the fundamental and the dominant are separated by the greatest possible interval which …

   THE CAP: Which? Finish. You can’t.
   STEPHEN: (*With an effort*) Interval which. Is the greatest possible ellipse. Consistent with. The ultimate return.