Canad. Math. Bull. Vol. **27** (1), 1984

LIE IDEALS IN ASSOCIATIVE ALGEBRAS

BY

G. J. MURPHY

ABSTRACT. It is shown that in a certain extensive class of algebras one can associate with each Lie ideal a corresponding associative ideal which facilitates the study of Lie ideals, especially for simple algebras. We apply this construction to obtain new, simpler proofs of some known results of Herstein [10] and others on the Lie structure of associative rings.

Introduction. Let B be the algebra of all bounded operators on a separable Hilbert space of infinite dimension, and for J a subset of B let $[B, J]$ denote the set of all finite sums of elements $[T, X] = TX - XT$ where $T \in B$ and $X \in J$. It is shown in Fong–Meiers–Sorrour [8] that a linear manifold L in B is a Lie ideal in B if and only if there is an associative two-sided ideal J in B such that $[B, J] \subseteq L \subseteq J + \mathbb{C}1$. Their proof uses some deep results in Operator Theory. In this paper we show that analogous results hold in a much more general context, and our proofs are simpler and more algebraic. We also give a partial generalization of a result of de la Harpe [4] (see also Murphy–Radjavi [11]). In certain simple algebras we are able to completely characterize the Lie ideals (see [10], where these results were originally obtained by different methods).

Terminology. The terms *algebra* and *ideal* when not qualified will always mean associative algebra and associative two-sided ideal. Every algebra B over a field F is a Lie algebra with respect to the commutator product $[x, y] = xy - yx$.

Let B be an algebra over a field F and suppose B has a set of 2×2 matrix units e_{11}, e_{12}, e_{21}, e_{22}. It turns out that we can say a lot about the Lie ideals of such algebras. If A is the centralizer of the matrix units, i.e. $A = \{x \in B : xe_{ij} = e_{ij}x, 1 \leq i, j \leq 2\}$, then A is a subalgebra of B, and it's well known that B is isomorphic to the algebra $M_2(A)$ of all 2×2 matrices with entries in A (see [5] p. 134).

The class of algebras over $\mathbb{C}$ having a set of 2×2 matrix units is extensive: it includes the properly infinite von Neumann algebras, and the hyperfinite type II_1-factors [13] pp. 48–49. Of course if H is a Hilbert space of infinite

Received by the editors January 26, 1982 and, in revised form, October 5 and December 31, 1982.

AMS Subject Classification: 16A68, 17B60

dimension and $B(H)$, $K(H)$ denote the algebra of bounded operations, and the ideal of compact operators respectively, then $B(H)$ and $B(H)/K(H)$ admit sets of 2×2 matrix units. More generally, any 2-homogeneous von-Neumann algebra admits a set of 2×2 matrix units [13]. In fact, if B is a Banach $*$-algebra and there exists $P, V\in B$ with P a projection and $V^*V=P$ and $VV^*=1-P$, then the set $e_{11}=P$, $e_{12}=V^*$, $e_{21}=V$, $e_{22}=1-P$, forms a set of 2×2 matrix units for B.

Let B be a unital algebra over a field F not of characteristic 2. We shall use the following notation. If S, T are subsets of B we let $[S, T]$ denote the set of all sums $[x, y]$ where $x\in S$ and $y\in T$. For I an ideal in B, we let $I^\sim=\{x\in B:[b, x]\in I(b\in B)\}$. Clearly $I+F1\subseteq I^\sim$. In some important examples (see below) we have equality, $I+F1=I$.

Before proving the following theorem a short observation will be useful: if L is a Lie ideal in B, and $u\in B$ such that $u^2=1$, then $uLu\subseteq L$. This follows from the elementary calculation $uxu=x-1/2[u,[u, x]]$. (This calculation appears in [8].)

THEOREM 1. *Let B be an algebra, over a field F not of characteristic 2, which contains a set of 2×2 matrix units. If L is a Lie ideal in B, then there is an ideal I in B such that $[B, I]\subseteq L\subseteq I^\sim$.*

Proof. We assume w.l.o.g. that $B=M_2(A)$ for some algebra A over F. Define

$$J=\left\{x\in A:\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\in L\right\}.$$

Now if $x\in J$ and $a\in A$, then

$$\left[\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix}0 & ax\\ 0 & 0\end{pmatrix}\in L \quad\text{and}\quad \left[\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}\right]=\begin{pmatrix}0 & xa\\ 0 & 0\end{pmatrix}\in L.$$

Thus $ax, xa\in J$. Since it's obvious J is a linear manifold, we conclude that J is an ideal in A. Now define $I=\left\{\begin{pmatrix}x & y\\ z & t\end{pmatrix}: x, y, z, t\in J\right\}$. Then I is an ideal in B. First we show $L\subseteq I^\sim$. Let $\begin{pmatrix}x & y\\ z & t\end{pmatrix}\in L$ and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in B$. We have to show $\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}x & y\\ z & t\end{pmatrix}\right]\in I$.

Now $\left[\begin{pmatrix}x & y\\ z & t\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix}0 & -y\\ z & 0\end{pmatrix}$, and $\left[\begin{pmatrix}0 & -y\\ z & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix}0 & y\\ z & 0\end{pmatrix}$ both lie in L. Hence $\begin{pmatrix}0 & y\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ z & 0\end{pmatrix}\in L$. But if $u=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ then $u^2=1$ in B, so $uLu\subseteq L$ by the remark preceding this theorem. Hence $\begin{pmatrix}0 & z\\ 0 & 0\end{pmatrix}=u\begin{pmatrix}0 & 0\\ z & 0\end{pmatrix}u\in L$.

https://doi.org/10.4153/CMB-1984-002-0 Published online by Cambridge University Press

Thus $y, z \in J$, and so $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \in I$. Also $\begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix} \in L$. Now $\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} 0 & t-x \\ 0 & 0 \end{pmatrix} \in L$, so $t-x \in J$. Thus $\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} 0 & y \\ z & t-x \end{pmatrix}$, and the second term is in I. Thus to show $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & t \end{pmatrix}\right] \in I$ we need now only show $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right] = \begin{pmatrix} [a, x] & [b, x] \\ [c, x] & [d, x] \end{pmatrix} \in I$, i.e. we need only show $[a, x] \in J$ for all a in A. But $\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} [a, x] & 0 \\ 0 & 0 \end{pmatrix} \in L$. Hence $\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} [a, x] & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -[a, x] \\ 0 & 0 \end{pmatrix} \in L$, implying $[a, x] \in J$. Thus $L \subseteq I^{\sim}$.

Now we show $[B, I] \subseteq L$. Let $x \in J$ and $a \in A$. Then $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in L$, and so $\begin{pmatrix} [a, x] & 0 \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\right] \in L$. Also if $x, y \in J$ then $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are in L, hence $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in L$. Thus if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$ and $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in I$, then

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & t \end{pmatrix}\right] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & t \end{pmatrix}\right] + \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} [a, x] & 0 \\ 0 & [d, t] \end{pmatrix} + \begin{pmatrix} 0 & bt-xb \\ cx-tc & 0 \end{pmatrix} + \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\right] \in L$$

since $bt-xb$ and $cx-tc \in J$, $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \in L$, and $\begin{pmatrix} 0 & 0 \\ 0 & [d, t] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} [d, t] & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in L$.

COROLLARY 2. *If the Lie ideal L in the above theorem is of finite codimension in B one can choose the ideal I to be of finite codimension also.*

Proof. Let I and J be as in the proof of the above theorem. Now if $y_1, \ldots, y_n$ are linearly independent elements of A such that $J \cap [y_1, \ldots, y_n] = 0$ where $[y_1, \ldots, y_n]$ denotes the linear span, then $\begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & y_n \\ 0 & 0 \end{pmatrix}$ are linearly independent in B and $L \cap \left[\begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & y_n \\ 0 & 0 \end{pmatrix}\right] = 0$. Thus $n \leq \dim(B/L)$, and so $\dim(A/J) < \infty$.

Now suppose $J \oplus [y_1, \ldots, y_n] = A$. Then if M denotes the linear span in B of the elements $\begin{pmatrix} y_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix}$ $(1 \le i \le n)$, it is easily seen that $I \oplus M = B$. Thus $\dim(B/I) < \infty$.

REMARK. If $\mathcal{H}$ is a separable Hilbert space of infinite dimensions and $\mathcal{J}$ is an ideal in $\mathcal{B}(\mathcal{H})$, then it was shown by Calkin [3] that $\mathcal{J}^{\sim} = I + \mathbb{C}1$. We use this interesting fact to deduce the following theorem from Theorem 1.

THEOREM 3 (Fong–Miers–Sourour [8]). *If $\mathcal{H}$ is a separable infinite dimensional Hilbert space and $\mathcal{L}$ is a linear manifold in $\mathcal{B}(\mathcal{H})$ then $\mathcal{L}$ is a Lie ideal in $\mathcal{B}(\mathcal{H})$ if and only if there is an ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ such that $[\mathcal{B}(\mathcal{H}), \mathcal{J}] \subseteq \mathcal{L} \subseteq \mathcal{J} + \mathbb{C}1$.*

REMARK. It was shown by de la Harpe [4] that if $\mathcal{H}$ is a Hilbert space, $\dim H = \infty$, and $\mathcal{L}$ a Lie ideal in $K(\mathcal{H})$ of finite codimension, then necessarily $L = K(\mathcal{H})$ (see also Murphy–Radjavi [11] for related results). Using our results above we can give a quick proof in the case where $\mathcal{H}$ is separable of a weaker version of de la Harpe's theorem, viz if $\mathcal{L}$ is a Lie ideal of $\mathcal{B}(\mathcal{H})$ which is of finite codimension in $K(\mathcal{H})$, then $\mathcal{L} = K(\mathcal{H})$. For, by Theorem 3, there is an ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ such that $[\mathcal{B}(\mathcal{H}), \mathcal{J}] \subseteq \mathcal{L} \subseteq I + \mathbb{C}1$. Hence $\mathcal{J} \neq \mathcal{B}(\mathcal{H})$, since $[\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})] = \mathcal{B}(\mathcal{H})$ [2]. Thus $\mathcal{J} \subseteq K(\mathcal{H})$. Hence $\mathcal{J}$ is of finite codimension in $K(\mathcal{H})$, since $\mathcal{L} \subseteq \mathcal{J} + \mathbb{C}1$. But this implies $\mathcal{J}$ is closed [11]. Hence $\mathcal{J} = \mathcal{K}(\mathcal{H})$. Now $\mathcal{K}(\mathcal{H}) = [\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})]$ [1]. Hence $\mathcal{L} = \mathcal{K}(\mathcal{H})$ since $\mathcal{L} \supseteq [\mathcal{B}(\mathcal{H}), \mathcal{J}]$.

Recall that the centre $Z(B)$ of an algebra B is the set $\{x \in B : xy = yx (y \in B)\}$. It is clear that if L is a linear manifold in B such that $L \subseteq Z(B)$ or $[B, B] \subseteq L$, then L is a Lie ideal in B. The following theorem gives a class of algebras for which every Lie ideal is got in this manner, and is a weaker version of a theorem of Herstein.

THEOREM 4 (Herstein [10]). *Let B be a simple algebra, over a field F not of characteristic 2, which has a set of 2×2 matrix units. Then the Lie ideals of B are precisely the linear manifolds L of B such that $L \subseteq Z(B)$ or $[B, B] \subseteq L$.*

Proof. Let L be a Lie ideal of B. By Theorem 1, there is an ideal I in B such that $[B, I] \subseteq L \subseteq I^{\sim}$. Now by the simplicity of B, $I = 0$ or B. If $I = 0$, then $L \subseteq I^{\sim} = Z(B)$. If $I = B$, the $[B, B] \subseteq L$.

REMARK. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $B = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then B is simple and has a set of 2×2 matrix units. Also $[B, B] = B$ [2], and $Z(B) = \mathbb{C}1$. Hence by Theorem 4 the only Lie ideals of B are 0, $\mathbb{C}1$, and B itself. This observation is apparently originally due to Topping.

If an algebra B satisfies $B = [B, B]$ and has a set of 2×2 matrix units, then we are able to characterize its Lie ideals in terms of its ideals. Some examples of such algebras are: $\mathscr{B}(\mathscr{H})$ where $\mathscr{H}$ is an infinite dimensional Hilbert space; $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ if $\mathscr{H}$ is also separable; also every properly infinite von Neumann algebra [12].

THEOREM 5. *Let Be be an algebra, over a field F not of characteristic* 2, *with a set of* 2×2 *matrix units, and suppose* $B=[B,B]$. *Then a linear manifold L in B is a Lie ideal if and only if there is an ideal I in B such that* $[B,I]\subseteq L\subseteq I^{\sim}$.

Proof. The forward implication has been shown in Theorem 1. So let's suppose L is a linear manifold and I is an ideal such that $[B,I]\subseteq L\subseteq I^{\sim}$. Let $x\in L$ and $a,b\in B$. Then $[[a,b],x]=[a,[b,x]]-[b,[a,x]]$ by the Jacobi identity. But $x\in I^{\sim}$, so $[a,x],[b,x]\in I$. Now $[B,I]\subseteq L$, hence $[a,[b,x]]$ and $[b,[a,x]]\in L$. Thus $[[a,b],x]\in L$. But since $[B,B]=B$, this implies $[c,x]\in L$ for all c in B. Thus L is a Lie ideal.

THEOREM 6. *Let B be an infinite dimensional algebra over a field F not of characteristic* 2 *and suppose* $B=[B,B]$ *and B has a set of* 2×2 *matrix units. Then B has proper finite codimensional Lie ideals if and only if B has proper finite codimensional ideals.*

Proof. The backward implication is clearly trivial. Suppose L is a proper Lie ideal with $\text{Dim}\, B/L<\infty$. Then by Corollary 2 there is an ideal I in B with $[B,I]\subseteq L\subseteq I^{\sim}$ and $\dim B/I<\infty$. Hence $I\neq 0$ since $\dim B=\infty$. Also $I\neq B$ since $B=[B,B]\subseteq L\neq B$.

REMARK. We finish this paper with a generalization of a result in [8]. It's shown there (Theorem 1) that a linear manifold $\mathscr{L}$ in $\mathscr{B}(\mathscr{H})$ ($\mathscr{H}$ a separable infinite dimensional Hilbert space) is a Lie ideal in $\mathscr{B}(\mathscr{H})$ iff $U^*\mathscr{L}U\subseteq\mathscr{L}$ for every unitary U in $\mathscr{B}(\mathscr{H})$. An inspection of the proof shows that it only uses the following two facts about $\mathscr{B}(\mathscr{H})$.

(a) Every unitary in $\mathscr{B}(\mathscr{H})$ is a product of (four) symmetries.

(b) Every operator in $\mathscr{B}(\mathscr{H})$ is a finite linear combination of projections. The first result is due to Halmos–Kakutani [9], and in [7] it is shown that this result can be extended to any properly infinite von Neumann algebra B (i.e. every unitary in B is a product of symmetries in B.) The second result is due to Fillmore [6], and in [12] it is generalized to a properly infinite von Neumann algebra B (i.e. every element of B is a linear combination of projections in B.) Hence, we conclude: A linear manifold L in a properly infinite von Neumann algebra B is a Lie ideal in B if and only if $u^*Lu\subseteq L$ for every unitary u in B.

REFERENCES

1. J. Anderson, *Commutators of compact operators*. Journal für die reine und angewandte Mathematik, Band **291** (1977) 128–132.

2. A. Brown and C. Pearcy, *Structure of commutators of operators*. Ann of Math. (2) **82** (1965) 112–127.

https://doi.org/10.4153/CMB-1984-002-0 Published online by Cambridge University Press

3. J. N. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*. Ann of Math, **42** (1941) 839–873.

4. P. de la Harpe, *The algebra of compact operators does not have any finite-codimensional ideal*, Studia Math **66** (1979), 33–36.

5. C. Faith, *Algebra: Rings, Modules and Categories I*. Springer, 1973.

6. P. Fillmore, *Sum of operators with square zero*. Acta Sci. Math. **28** (1967) 285–288.

7. P. Fillmore, *On products of symmetries*. Canad. J. Math. **18** (1966) 897–900.

8. C. K. Fong, C. R. Meiers and A. R. Sourour, *Lie and Jordan ideals of operators on Hilbert space* (preprint).

9. P. Halmos and S. Kakutani, *Products of symmetries*. Bull. Amer. Math. Soc. **64** (1958) 77–78.

10. I. N. Herstein, *Topics in Ring Theory*. University of Chicago, 1969.

11. G. J. Murphy and H. Radjavi, *Associative and Lie subalgebras of finite codimension* (to appear in Studia Math.).

12. C. Pearcy and D. Topping, *Sums of small numbers of idempotents*. Michigan Math. J. **14** (1967) 453–465.

13. D. Topping, *Lectures on von Neumann Algebras*. Van Nostrand, 1971.

University of New Hampshire
Durham, NH 03824
U.S.A.