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STABILITY CRITERIA FOR CONTRACTIVE SEMIGROUPS VIA MAXIMALITY PROCEDURES

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An abstract metrical version of the well-known Ekeland and Brøndsted maximality principle is used to derive a number of stability criteria for a class of (function) contractive semigroups on (complete) metric spaces, extending a number of classical contributions due to Brézis and Browder.

0. Introduction

One of the most important problems concerning a wide class of evolution processes acting on a (complete) metric space is that of finding sufficient "local" conditions in order that a "global" stability property - involving a certain class of subsets of the ambient metric space - be obtained. To attack this problem, the basic instrument is the differential inequalities technique; see, as a classical reference, the excellent 1970 Bhatia and Szegö survey [1]. Recently, a second way of investigating the same topic has been founded by the 1976 Brézis and Browder paper [6] in which a general ordering principle - discovered by the authors - is used to derive a number of important results. The present note may be considered as being included in this last category; more exactly, our main aim is to state and prove a number of stability criteria for a class of (nonempty) subsets of a metric space, with respect to a certain family of contractive semigroups acting on that space, the basic tool of our approaches being a

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maximality principle on ordered metric structures extending - from an abstract metric viewpoint - a similar one to Ekeland and Brøndsted's [16], [7]. It must be observed at this moment that this maximality principle also contains the above quoted differential inequalities procedure as a common basis for studying such a class of problems; we refer to the author's paper [32] for more details concerning these aspects.

1. A maximality principle

Let X be an abstract nonempty set and let $\leq$ be a (*partial*) *ordering* on X. For any subset Y of X and any couple x, y of Y with $x \leq y$, let $Y(x, \leq)$ (respectively, $Y(x, y)$) denote the subset of all z in Y with $x \leq z$ $(x \leq z \leq y)$. A subset Y of X will be called a *chain* if and only if, for any $x, y \in Y$, either $x \leq y$ or $y \leq x$; also, an enumerable subset (or, in other words, a sequence) $Y = (y_n ; n \in N)$ of X will be called (*strict*) *monotone* if and only if $(y_i < y_j)$ $y_i \leq y_j$, whenever $i < j$, $i, j \in N$ (here $<$ denotes the *strict ordering* on X induced by $\leq$). Let d be a metric on X. Let us denote (for any subset Y of X and any element x in X) by $d(x, Y)$ the usual distance between x and Y (the infimum of all $d(x, y)$, $y \in Y$). At the same time, a subset Y of X will be termed *strong order-compact* if for any monotone sequence $(y_n ; n \in N)$ of Y there exists a subsequence $(y_{p(n)} ; n \in N)$ of Y and an element z of Y with $y_{p(n)} \to z$ as $n \to \infty$ and $y_n \leq z$, all $n \in N$. The other notational conventions are more or less standard.

Now, with these preliminaries, the following maximality principle may be stated and proved.

THEOREM 1. *Suppose the ordered metric space* $(X, d, \leq)$ *and the subset* Y *of* X *are such that*

(i) Y *is strong order-compact.*

Then, for every x *in* Y *there is a* Y*-maximal element* z *in* Y *with* $x \leq z$ *(that is, for every* $x \in Y$ *there is an element* $z \in Y$ *with* $x \leq z$ *and, in addition, for any* $y \in Y$*, the relation* $z < y$ *does not hold).*

Proof. Let C be a (nonempty) chain in Y. We claim that the following property holds:

(1) *for any* $\varepsilon > 0$ *there exists* $x = x(\varepsilon)$ *in* $\bar{C}$ *such that* y, z *in* C *and* $x \le y \le z$ *imply* $d(y, C(z, \le)) < \varepsilon$.

Indeed, suppose (1) is not valid. Then, there must be a number $\varepsilon > 0$ such that, for every x in C , a couple y, z in C may be found with $x \le y \le z$ and $d(y, C(z, \le)) \ge \varepsilon$ (of course, in such a situation $y < z$). Let x_1 in C ; by this assumption, we get a couple y_1, z_1 in C with $x_1 \le y_1 < z_1$ and $d(y_1, C(z_1, \le)) \ge \varepsilon$. Put $z_1 = x_2$; by the same procedure, a couple y_2, z_2 in C may be chosen with $x_2 \le y_2 < z_2$ and $d(y_2, C(z_2, \le)) \ge \varepsilon$, and so on. Consequently, we inductively get a strict monotone sequence $(y_n;\, n \in N)$ in C with

$$d(y_n, y_m) \ge \varepsilon \text{ , all } n, m \in N \text{ , } n < m \text{ ,}$$

contradicting *(i)* and (1) is proved. In such a case, it is not hard to build up a monotone sequence $(x_n;\, n \in N)$ in C satisfying the condition

(2) $n \in N$, $y, z \in C$ *and* $x_n \le y \le z$ *imply* $d(y, C(z, \le)) < (\frac{1}{2})^n$.

Now, let $y_1 \in C(x_1, x_2)$ be given; by (2), an element $y_2 \in C(x_2, \le)$ may be found with $d(y_1, y_2) < \frac{1}{2}$. Without any loss of generality one may suppose $y_2 \in C(x_2, x_3)$ in which case, again by (2), an element $y_3 \in C(x_3, \le)$ may be chosen with $d(y_2, y_3) < (\frac{1}{2})^2$ and so on. By induction, we get a monotone sequence $(y_n;\, n \in N)$ in C satisfying

$$d(y_n, y_{n+1}) < (\tfrac{1}{2})^n \text{ , all } n \in N \text{ ,}$$

so that it appears as a monotone Cauchy sequence in Y and therefore, by *(i)*, $y_n \to c$ as $n \to \infty$ and $y_n \le c$, all $n \in N$, for some c in Y . Now, let z in C be arbitrary and fixed. Without loss of generality one may suppose $y_n \le z$, for all $n \in N$ in which case, by (2), a monotone sequence $(z_n;\, n \in N)$ in $C(z, \le)$ may be constructed with

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$$d(y_n, z_n) < (\tfrac{1}{2})^n \text{ , all } n \in N \text{ ,}$$

and therefore $z_n \to c$ as $n \to \infty$ which implies, again by *(i)*, $z_n \le c$ all $n \in N$, and, in particular, $z \le c$, proving C is bounded above in Y. By the classical Zorn theorem [21, p. 33] corresponding to any x in Y there is a Y-maximal element z in Y with $x \le z$, and this completes the proof. //

Now X, d and $\le$ being as before, let us call a subset Y of X *order-admissible* if for any y of Y there exists z in X with $y < z$. Then, as an immediate - and important - consequence of the above result, we have

THEOREM 2. *Suppose the ordered metric space* $(X, d, \le)$ *and the nonempty proper subset* Y *of* X *are such that condition (i) plus*

(ii) Y *is order-admissible*

hold. In this case, for every $x \in Y$ *there is an element* $z \in X\backslash Y$ *with* $x < z$.

Proof. As Theorem 1 applies, for the arbitrary fixed element x of Y a Y-maximal element y in Y may be found with $x \le y$. Furthermore, given y in Y there is, by *(ii)*, an element z in X with $y < z$ and this will complete the proof because, evidently (by the Y-maximality of y in Y) the relation $z \in Y$ is impossible. //

A partial indication about the power of this maximality principle follows from the considerations below. Again let X, d and $\le$ be endowed with their usual meaning. The considered ordering $\le$ will be termed *self-closed* if, for any monotone sequence $(x_n;\, n \in N)$ in X and any element x of X with $x_n \to x$ as $n \to \infty$ we have $x_n \le x$, all $n \in N$; correspondingly, a subset Y of X will be called *order-compact* if any monotone sequence in Y has a (monotone) subsequence converging to some element of Y. In such a case, if we suppose condition *(i)* is replaced by the pair of stronger conditions

(iii) $\le$ *is a self-closed ordering,*

(iv) Y *is order-compact,*

the corresponding variant of Theorem 1 appears as a direct extension of a

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similar one of the author's [32] and is proved by an ordinary induction argument. Moreover, it was shown the above quoted author's result may be considered as an abstract metric variant of Ekeland's as well as Brøndsted's approaches [15, 16, 17, 7, 9] (see also Brézis and Browder [6] and Turinici [31]) or, equivalently - under a pattern developed by Bourbaki [4] and refined by Brøndsted [8] - of Caristi's fixed point theorem [12] (see in this direction Browder [11], Kasahara [20], Kirk [22], Pasicki [26], Siegel [29], Turinici [33], Wong [39] for a number of related view-points concerning this topic) so that, the maximality principle we presented before also extends all these contributions. On the other hand, a sufficient condition assuring the validity of *(iv)* is evidently

(iv)' Y is compact

proving Theorem 1 also may be interpreted as a partial metric extension of Theorem 2.2 established by the author [30] through a direct "intersection" technique (see also Wallace [37] as well as Ward [38]). Finally, Theorem 2 may be considered as a "threshold" theorem, largely used in the formulation of the main results, as we shall see below.

2. The main results

In what follows, a precise statement of the results discussed in the introduction will be given. Let $f : R_+ \to R_+$ be a given function: it will be called *normal*, provided that

$$f(t)f(s) \geq f(t+s) \ , \ all \quad t, s \in R_+ \ , \tag{3}$$

and the *associated function* $f^* : R_+ \to R_+$ defined by

$$f^*(t) = \sup\big(f(s);\ 0 \leq s \leq t\big) \ , \quad t \in R_+ \ , \tag{4}$$

exists. In the same context, a function $h : R_+ \to R_+$ will be termed *f-admissible*, if it satisfies

$$f(s)h(t-s) \leq h(t) - h(s) \ , \quad t, s \in R_+ \ , \quad t \geq s \ , \tag{5}$$

$$h(t) > 0 = h(0) \ , \ all \quad t > 0 \ , \tag{6}$$

and the initial function f is said to be *proper* if the class A_f of all admissible function $h : R_+ \to R_+$ is not empty. Concerning these notions, it should be noted that, in many practical situations, a normal function is also a proper one: for example, if we suppose the normal function f

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satisfies, instead of (3), the stronger condition

(3)' $$f(t)f(s) = f(t+s) \ , \text{ all } \ t, s \in R_+ \ ,$$

as well as the supplementary hypothesis

(7) $$f(t) > 1 = f(0) \ \big(< 1 = f(0)\big) \ , \text{ all } \ t > 0 \ ,$$

then, for every $\lambda < 0$ (> 0) the function $h : R_+ \to R_+$ defined by $h(t) = \lambda\big(1-f(t)\big)$, $t \in R_+$, is a f-admissible one, while in the case of an arbitrary normal function f satisfying the second part of (7), a standard f-admissible function is that defined by the above procedure with $\lambda = 1$.

In what follows, (V, d) is a complete metric space and $f : R_+ \to R_+$ a normal and proper function. By a *f-contractive semigroup* (in Brezis and Browder's terminology) or, equivalently, a *f-contractive semidynamical system* (in Bhatia and Szegö's terminology) on V we mean a mapping $(t, v) \mapsto S(t, v) = S(t)v$ from $R_+ \times V$ into V satisfying

(8) $$S(0)v = v \ , \text{ all } \ v \in V \ ,$$

(9) $$d\big(S(t)u, S(s)v\big) \le f(s)d\big(S(t-s)u, v\big) \ , \quad t \ge s \ge 0 \ , \quad u, v \in V \ ,$$

and, in the same context, given a function $h : R_+ \to R_+$, a nonempty subset W of V will be called *h-stable* with respect to S provided that

(10) $$d\big(S(t)u, W\big) \le h(t) \ , \text{ all } \ t \in R_+ \ , \quad u \in W \ .$$

A satisfactory motivation for introducing these notions will be offered later; for the moment, we are only interested to state and prove an useful stability criterion involving these elements (in fact, the first main result of the present note), a criterion that may be formulated as follows.

THEOREM 3. *Suppose the elements f, S, h and W defined by the above conventions are such that $h \in A_f$ and*

(v) any sequence $(v_n; n \in N)$ in W for which there is a monotone sequence $(t_n; n \in N)$ in R_+ with $t_n \to t$ as $n \to \infty$ for some $t \in R_+$ and $d\big(S(t_m-t_n)v_n, v_m\big) \le \gamma h(t_m-t_n)$, $n, m \in N$, $n < m$, for some $\gamma > 0$ has a subsequence converging to some v of W satisfying in addition

$$d\left(S\left(t-t_n\right)v_n,\ v\right) \le \gamma h\left(t-t_n\right)\ ,\ all\quad n \in N\ .$$

Then, under the specific supplementary hypothesis

$$(11)\qquad \liminf_{t \to 0}\ \left(1/h(t)\right)d\left(S(t)v,\ W\right) \le 1\ ,\ all\quad v \in W\ ,$$

the considered subset W *of* V *appears as a* h*-stable one with respect to* S *; that is,* (10) *holds.*

Proof. Suppose $\gamma > 1$ is an arbitrary fixed number and let X denote the cartesian product $R_+ \times W$ endowed with the "product" metric

$$e\left((t,\ u),\ (s,\ v)\right) = |t-s| + d(u,\ v)\ ,\quad (t,\ u),\ (s,\ v) \in X\ ,$$

and the ordering $\le$ defined by

$$(t,\ u) \le (s,\ v)\quad \text{if}\quad t \le s\quad \text{and}\quad d\left(S(s-t)u,\ v\right) \le \gamma h(s-t)$$

(the fact that $\le$ is indeed an ordering on X follows directly from (5), (8) and (9)). Now suppose $a \in R_+$ is arbitrary fixed and let Y denote the cartesian product $[0,\ a] \times W$; it immediately follows by *(v)* coupled with our conventions that Y appears as a strong order-compact subset of X . In such a case, Theorem 1 being applicable, given the element $(0,\ u)$ in Y , a Y-maximal element $(s,\ v)$ in Y may be found with $(0,\ u) \le (s,\ v)$. Suppose $s < a$; for any $r \in (s,\ a]$ and $w \in W$, the relation $(s,\ v) \le (r,\ w)$ does not hold so that we must have

$$d\left(S(r-s)v,\ w\right) > \gamma h(r-s)\ ,\quad s < r \le a\ ,\quad w \in W\ ,$$

and this gives (denoting $t = r - s$ and taking the infimum with respect to $w \in W$)

$$\left(1/h(t)\right)d\left(S(t)v,\ W\right) \ge \gamma\ ,\quad 0 < t \le a-s\ ,$$

a contradiction to (11). Therefore, $s = a$, or, in other words, given $u \in W$ there is a $v \in W$ satisfying $(0,\ u) \le (a,\ v)$, that is, $d\left(S(a)u,\ v\right) \le \gamma h(a)$, a relation equivalent in fact with (10), because $\gamma > 1$ and $a \in R_+$ were arbitrary. //

Again let $f : R_+ \to R_+$ be a normal and proper function, S a f-contractive semigroup on V , $h : R_+ \to R_+$ a function satisfying (6) and W a nonempty subset of V . Let $\alpha(\beta)$ denote the function from V into R_+ $(R_+ \cup \{+\infty\})$ defined by

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$$(12)\qquad \alpha(u) = \inf_{t>0} \bigl(1/h(t)\bigr)d\bigl(S(t)u,\ W\bigr)\ ,\quad u \in V\ ,$$

$$(13)\qquad \beta(u) = \liminf_{t\to\infty} \bigl(1/h(t)\bigr)d\bigl(S(t)u,\ W\bigr)\ ,\quad u \in V\ ,$$

and put also

$$(14)\qquad \alpha = \sup\bigl(\alpha(u);\ u \in W\bigr)\ ,\quad \beta = \sup\bigl(\beta(u);\ u \in W\bigr)\ .$$

Evidently, $\alpha(u) \le \beta(u)$ for all $u \in V$, and this immediately gives $\alpha \le \beta$. A natural question appearing in these circumstances is that of finding sufficient conditions in order that the reverse inequality be also valid. In this direction, as a complement to Theorem 3, the second main result of the present note is

THEOREM 4. *Suppose the elements f, S, h and W are such that $h \in A_f$ and condition (v) holds. Then the reverse inequality $\beta \le \alpha$ will also hold so that, necessarily, $\alpha = \beta$.*

Proof. The expected relation will be trivially satisfied when α is infinite so, without loss of generality, one may suppose α is finite. In this case, $\gamma > \alpha$ being arbitrary fixed, let again X denote the cartesian product $R_+ \times W$ metrized and ordered as in the previous result; given a (t, u) in X there is, by the choice of γ, a $r > 0$ and a $v \in W$ satisfying $d\bigl(S(r)u, v\bigr) \le \gamma h(r)$ so that (putting $s = t + r$) we may find a (s, v) in X with $(t, u) < (s, v)$ (here $<$ indicates the strict ordering on X induced by $\le$). Furthermore, given $a > 0$ and defining Y as the cartesian product $[0, a] \times W$ it follows by this argument Y is order-admissible and Theorem 2 applies; in other words, given $u \in W$ there is, by that result, a (s, v) in $X\setminus Y$ with $(0, u) < (s, v)$ (note that, in such a case, $s > a$ since, otherwise, $s \le a$ would imply $(s, v) \in Y$, a contradiction). In other words, to every $u \in W$ and $a > 0$ there corresponds a $v \in W$ and a $s > a$ satisfying $d\bigl(S(s)u, v\bigr) \le \gamma h(s)$; from this relation we immediately derive $\bigl(1/h(s)\bigr)d\bigl(S(s)u, W\bigr) \le \gamma$ and therefore $\beta \le \gamma$. As $\gamma > \alpha$ was arbitrary, we get $\beta \le \alpha$ and the proof is complete.

3. Some particular cases

The importance of the results we presented above follows, among other things, from the fact that, in certain standard contexts, they are

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particularly involved in a number of classical questions pertaining to *stability theory* (mainly *flow-invariance theory*) of dynamical systems and, in a series of problems belonging to *convex* as well as *nonconvex analysis*. We start our considerations by observing that, evidently, condition *(v)* is the key of all arguments we developed in the preceding section so it seems to be natural to express it in a more practical form. To this end, (V, d) being a complete metric space, suppose the elements f, S, h and W introduced in the general conventions adopted before are such that

(vi) $h \in A_f$ and $h(t) \to 0$ as $t \to 0$,

(vii) $S(t)v \to v$ as $t \to 0$, all $v \in V$,

(viii) W is a closed subset of V ,

then we claim condition *(v)* will be satisfied. Indeed, let the sequence $(v_n;\, n \in N)$ in W and the monotone sequence $(t_n;\, n \in N)$ in R_+ with $t_n \to t$ as $n \to \infty$ for some $t \in R_+$, satisfy

$$d\big(S(t_m-t_n)v_n,\, v_m\big) \le \gamma h(t_m-t_n) \ , \quad n, m \in N \ , \quad n < m \ , \tag{15}$$

for some $\gamma > 0$; by (8) and (9) we derive (remembering the definition (4) of the associated function)

$$\begin{aligned} d(v_{n+p},\, v_{n+q}) &\le d\big(S(t_{n+p}-t_n)v_n,\, v_{n+p}\big) + d\big(S(t_{n+q}-t_n)v_n,\, v_{n+q}\big) \\ &\qquad + d\big(S(t_{n+p}-t_n)v_n,\, S(t_{n+q}-t_n)v_n\big) \\ &\le \gamma\big(h(t_{n+p}-t_n)+h(t_{n+q}-t_n)\big) + f(t_{n+q}-t_n)d\big(S(t_{n+p}-t_{n+q})v_n,\, v_n\big) \\ &\le \gamma\big(h(t_{n+p}-t_n)+h(t_{n+q}-t_n)\big) + f^*(t)d\big(S(t_{n+p}-t_{n+q})v_n,\, v_n\big) \ , \\ &\qquad\qquad n, p, q \in N \ , \quad p > q \ , \end{aligned}$$

and therefore, a standard argument (see, for example, Lemma 1 of Brézis and Browder [6]) assures us by *(vi)* and *(vii)* that $(v_n;\ n \in N)$ appears as a Cauchy sequence in W and by *(viii)* that $v_n \to v$ as $n \to \infty$ for some $v \in W$, showing the first part of *(v)* holds. Now, again by (8) and (9), the relations (15) also give

$$\begin{aligned} d\big(S(t-t_n)v_n,\ v_m\big) &\le d\big(S(t-t_n)v_n,\ S(t_m-t_n)v_n\big) + d\big(S(t_m-t_n)v_n,\ v_m\big) \\ &\le f(t_m-t_n)\,d\big(S(t-t_m)v_n,\ v_n\big) + \gamma h(t_m-t_n) \\ &\le f^*(t)\,d\big(S(t-t_m)v_n,\ v_n\big) + \gamma h(t-t_n) \ , \end{aligned}$$

$$\text{all} \quad n, m \in N \ , \quad n < m \ ,$$

so that, passing to limit as $m \to \infty$, the second part of *(v)* follows and our claim is proved. Now, as a second step of our considerations, it must be emphasized that a fundamental choice of the generating normal and proper function f is that expressed by

$$f(t) = e^{\omega t} \ , \quad t \in R_+ \ , \textit{ for some } \ \omega \in R \ , \tag{16}$$

in which case the associated function f^* is

$$f^*(t) = e^{\sigma t} \ , \quad t \in R_+ \ , \textit{ where } \ \sigma = \max(\omega, 0) \ . \tag{17}$$

As a notational simplification, we shall adopt the convention that any f-notion be indicated as a ω-notion; with this convention, it is now evident that a standard ω-admissible function is

$$\begin{aligned} h(t) &= (\delta/\omega)\big(e^{\omega t}-1\big) \ , \quad t \in R_+ \ , \quad \omega \neq 0 \ , \\ &= \delta t \qquad\qquad\quad\ , \quad t \in R_+ \ , \quad \omega = 0 \ , \end{aligned} \tag{18}$$

$\delta > 0$ being a positive number (let us remark at this moment the supplementary condition *(vi)* also holds). Suppose further S is a ω-contractive semigroup on V satisfying *(vii)*. In this case, as an important application of the first main result, we have

THEOREM 5. *In the above circumstances, let the closed subset* W *of* V *and the number* $\delta > 0$ *be such that*

$$\liminf_{t \to 0} \ (1/t)\,d\big(S(t)v,\ W\big) \le \delta \ , \textit{ all } \ v \in W \ ; \tag{19}$$

then W *appears as a* h*-stable subset with respect to* S *(the function* h *being expressed by* (18)*).*

Proof. By the arguments we indicated before, condition *(v)* will be satisfied. On the other hand, evidently,

$$h(t)/t \to \delta \quad \text{as} \quad t \to 0 \ ,$$

and this immediately implies condition (11) is in fact equivalent to (19), completing the proof. //

At this point we mention that, as important special cases of *(vii)*, one may consider

(ix) $t \mapsto S(t)v$ *is continuous on* R_+ *for all* $v \in V$

as well as

(x) $S(t)v \to v$ *as* $t \to 0$, *uniformly with respect to* $v \in W$

in which case, Theorem 5 reduces to Brézis and Browder's result [6, Theorem 2] (see also Ekeland [17]) and respectively, to the author's result [34] (note that, in a close connection with these references it must be quoted in addition Bhatia and Szegö's contribution [1, Chapter IX] as well as the classical Nemytskii and Stepanov one [25, Chapter V]). Moreover, it was shown by the author that in the case V is a Banach space and the ω-contractive semigroup S possesses a strong infinitesimal generator $A : V \to V$; that is, for any $v \in V$,

$$Av = \lim_{t \to 0} (1/t)\big(S(t)v-v\big) \quad \text{exists,}$$

then (19) may be written in the form

$$\liminf_{t \to 0} (1/t)d(v+tAv,\, W) \le \delta \ , \ all \quad v \in W \ , \tag{20}$$

and in such a case Theorem 5 appears as an abstract metric variant of Martin's invariance result [23] (see also in this direction, Brezis [5], Nagumo [24], Pavel [27], Yorke [40]) being also interpreted either from a "geometric" viewpoint (Bony [3], Crandall [13], Redheffer [28]) or from an "inwardness" one (Caristi [12], Halpern [18]).

Now, passing to the second part of our considerations, assume that, under the general hypotheses *(vi)* and *(vii)*, the function f satisfies the second part of (7) and the functions h and f are such that

(xi) $f(t)/h(t) \to 0$ *as* $t \to \infty$.

We claim in this framework the function β defined by (13) is constant on V ; indeed, $u,\, v \in V$ being arbitrary fixed, we have, by (8) and (9),

$$\big(1/h(t)\big)d\big(S(t)u,\, W\big) \le \big(f(t)/h(t)\big)d(u,\, v) + \big(1/h(t)\big)d\big(S(t)v,\, W\big) \ , \quad t > 0 \ ,$$

and, conversely,

$$\big(1/h(t)\big)d\big(S(t)v,\, W\big) \le \big(f(t)/h(t)\big)d(u,\, v) + \big(1/h(t)\big)d\big(S(t)u,\, W\big) \ , \quad t > 0 \ ,$$

so passing to $\liminf$ as $t \to \infty$ in both sides our claim will be established. Moreover, from these conditions it also follows the considered f-contractive semigroup S possesses a unique stationary point $z \in V$ (that is, $S(t)z = z$, all $t \in R_+$); in fact, again by (8) and (9) combined with the classical Contraction Mapping Principle it is not hard to build up a mapping $t \mapsto z(t)$ from $(0, \infty)$ into V uniquely determined by the property

$$S(t)z(t) = z(t) \text{ , all } t > 0 \text{ ,}$$

in which case, taking into account (3),

$$\begin{aligned} d\big(z(t), z(t/2)\big) &= d\big(S(t)z(t), S(t/2)z(t/2)\big) \le f(t/2)d\big(S(t/2)z(t), z(t/2)\big) \\ &= f(t/2)d\big(S(t/2)z(t), S(t/2)z(t/2)\big) \le f(t)d\big(z(t), z(t/2)\big) \text{ ,} \\ &\qquad \text{all } t > 0 \text{ ,} \end{aligned}$$

proving (by a standard argument) the considered mapping is constant on $R_+^* = \{p/2^n;\ p, n \in N\}$, hence (by a limit process involving also condition *(vii)*) constant on R_+ and our assertion is proved. In such a situation, as an important application of the second main result, we derive

THEOREM 6. *Under the conventions we accepted before, there exists a unique stationary point* $z = z(S)$ *in* V *of the considered* f*-contractive semigroup* S *on* V *satisfying in addition*

$$\sup_{u \in W} \inf_{t>0} \big(1/h(t)\big)d\big(S(t)u, W\big) = \big(1/h(\infty)\big)d(z, W) \tag{21}$$

for any closed subset W *of* V.

Proof. By the above conclusions, for any $u \in V$,

$$\beta(u) = \beta(z) = \liminf_{t\to\infty} \big(1/h(t)\big)d\big(S(t)z, W\big) = \big(1/h(\infty)\big)d(z, W)$$

(remember that, by (5), the function h is monotone increasing on R_+ and therefore, $h(\infty)$ exists), so, by Theorem 4, the proof is in fact completed. //

Concerning the elements f and h involved in this result, it must be mentioned that, an important - and useful - special case of them is represented by the choices (16) (with $\omega < 0$) and (18) respectively, in which case, (21) becomes

(21)' $$\sup_{u\in W}\inf_{t>0} d(S(t)u,\ W)/(1-e^{\omega t}) = d(z,\ W)$$

and the corresponding variant of Theorem 6 is identical with a result established by Brézis and Browder [*6*, Theorem 3] and having as a direct consequence a very elegant proof of the well-known drop theorem (see Corollary 7 of the above quoted Brézis and Browder paper as well as Brøndsted [*7*], Daneš [*14*] and Turinici [*35*]) a result that has been successfully applied to *convex* (and *nonconvex*) *analysis* (Bishop and Phelps [*2*], Holmes [*19*, Chapter III]) to *normal solvability theory* (Browder [*10*], Zabreĭko and Krasnosel'skiĭ [*41*]) and to *time optimal control theory* (Vrabie [*36*]).

At the end of our exposition, f, S and h having their general significance, let us call a family $(W_t;\ t \in R_+)$ of nonempty subsets of V *h-stable* if and only if

(10)' $$d(S(t)u,\ W_{a+t}) \le h(t)\ ,\ all\ \ a,\ t \in R_+\ ,\ \ u \in W_a\ ;$$

then - by the use of an appropriate ordering introduced on the cartesian product $R_+ \times V$ - a number of "time-dependent" stability criteria in the above sense may be stated and proved, extending in this way - from a "dynamic" viewpoint - the results we presented in the last two sections; some aspects of this problem will be discussed in a forthcoming paper.

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